Lecture 21 Oct 29, 01 Monday.

Stability and Transition

A) Orr-Sommerfeld Eqn - Inviscid limit
~~B) Non-dimensionalization~~
B) Stability Boundaries

Reading: Handouts.

A) Inviscid limit:

O-S equation

$$(\alpha U - \omega)(v'' - \alpha^2 v) - \alpha U'' v + \frac{i}{Re}(v'''' - 2\alpha^2 v'' + \alpha^4 v) = 0$$

$U(y)$, α or ω inputs. Solve for α or ω, and $v(y)$

The inviscid limit, $Re \to \infty$

$$\Rightarrow \quad (\alpha U - \omega)(v'' - \alpha^2 v) - \alpha U'' v = 0 \quad \text{(Rayleigh Eqn)}$$

2^{nd} order ODE

B.Cs:

B.L $\quad y=0 \quad v=0$ $\quad$ drop v' conditions
$\quad\quad\; y=\infty \quad v=0$

Examine when instability can occur in inviscid case (what conditions are necessary in mean flow)

Assume temporal problem: $\alpha = \alpha_r$ given, $\omega = \omega_r + i\omega_i$

Rearrange

$$\left[v'' = \alpha_r^2 v + \frac{\alpha_r U'' v}{\alpha_r U - \omega} \right] v^*$$

$$-\left[\, v^{*''} = \alpha_r^2 v^* + \frac{\alpha_r U'' v^*}{\alpha_r U - \omega} \,\right] v$$

$(\,)^*$ = complex conjugate

$$\rightarrow \quad v'' v^* - v^{*''} v = \alpha_r U'' |v|^2 \left[\frac{1}{\alpha_r U - \omega} - \frac{1}{\alpha_r U - \omega^*} \right]$$

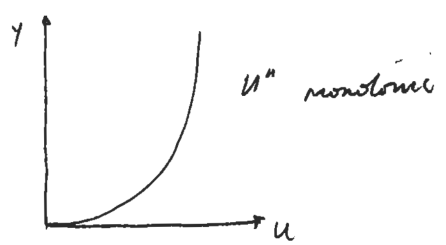
Integrate

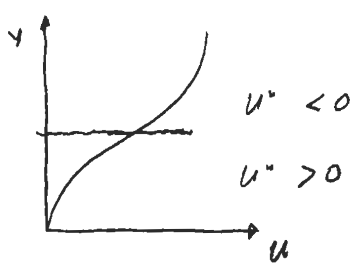
$$\int_0^\infty \frac{d}{dy}\left(v' v^* - v v^{*'} \right) dy = \int_0^\infty \frac{\alpha_r U'' |v|^2 \, 2i\omega_i}{|\alpha_r U - \omega|^2}\, dy$$

$$\cancelto{0}{v' v^* - v v^{*'}} \Big|_0^\infty$$

$$\Rightarrow \quad \int_0^\infty \frac{U'' |v|^2 \omega_i}{|U - \omega/\alpha_r|^2}\, dy = 0$$

Instability (which implies $\omega_i > 0$, $|v|^2 \neq 0$) possible only if U'' changes sign





$\int U'' |v|^2 dy = 0 \Rightarrow |v| = 0$

no growing disturbances possible

$|v| > 0$, growth possible if $\omega_i > 0$

for $H < 2.6$ (no inflection point), all α_r are stable for $Re = \infty$ ($\nu = 0$

At finite Re ($\nu > 0$), viscosity is destabilizing. (Prandtl in 1929 showed

$$V'' - \left(\alpha^2 + \frac{U''\alpha}{\alpha U - \omega}\right) V = 0 .$$

Note that $\alpha U - \omega \to 0$ some where in the boundary layer. For a neutral disturbance $\omega_i = 0$

$$\alpha_r U - \omega_r = 0 \quad \to \quad U - c_r = 0$$

This is a singular point of the inviscid stability equation. U'' must vanish for $v(y)$ to stay finite
$u' \to$ infinite
Artifact of ignoring viscosity

- Note: shear or vorticity or shear is max at inflection point.

Inflection is necessary for instability (Rayleigh, 1880)

Later Tollmein (1929) showed the inflection is also sufficient for instability.

For F-S

$$F''' + \frac{1+\beta_w}{2} F F'' + \beta_w (1 - F'^2) = 0$$

$$F''' \alpha\ U'' = -\frac{(1+\beta_w)}{2} F F'' - \beta_w (1 - F'^2)$$

If $\beta_w > 0$ then $F > 0$, $1 > F' > 0$, $F'' > 0$

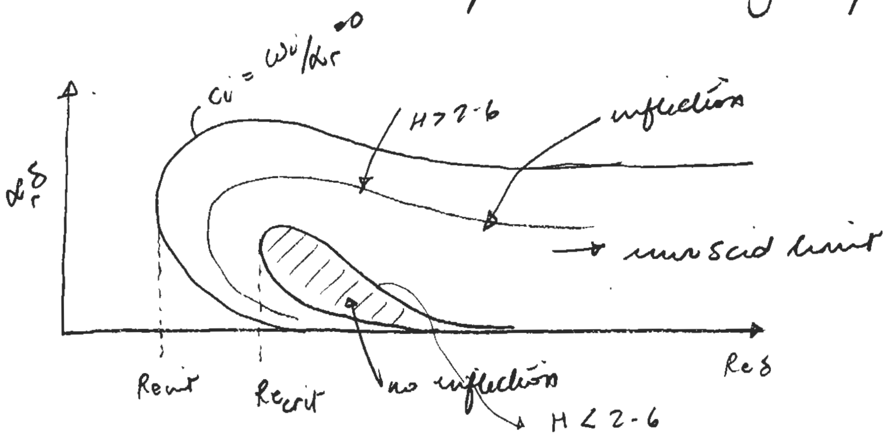
$$\Rightarrow \quad F''' (U'') < 0 \qquad \text{(no inflection)}$$

If $\beta_w < 0$ then at $y = 0$, $F''' = -\beta_w \geq 0$

and as $y \to \infty$ $\quad F''' \to -\left(\frac{1+\beta_w}{2}\right) F F'' < 0$

$\Rightarrow$ inflection

Temporal Instability Map



Non-dimensionalization: for B.L

$$L_{ref} = \delta, \delta^*, \theta$$

$$\text{so} \quad \alpha = \alpha^* L_{ref} = \alpha^* \delta^* \text{ or } \alpha^* \theta$$

$$\omega = \omega^* L_{ref}/U_{ref} = \omega^* \theta/U_e$$

$$\text{and} \quad Re \rightarrow Re_\delta \text{ or } Re_{\delta^*} \text{ or } Re_\theta$$

Stability Boundaries

Temporal Problem

$\omega_i > 0$ – growth

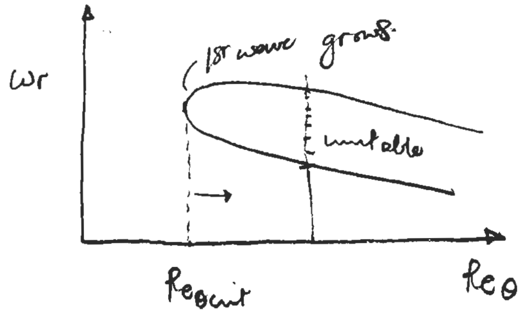
$\omega_i < 0$ – decay

Spatial Problem

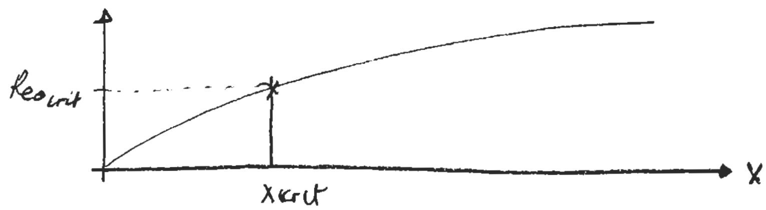
$\alpha_i < 0$ growth

$\alpha_i > 0$ decay.

Spatial

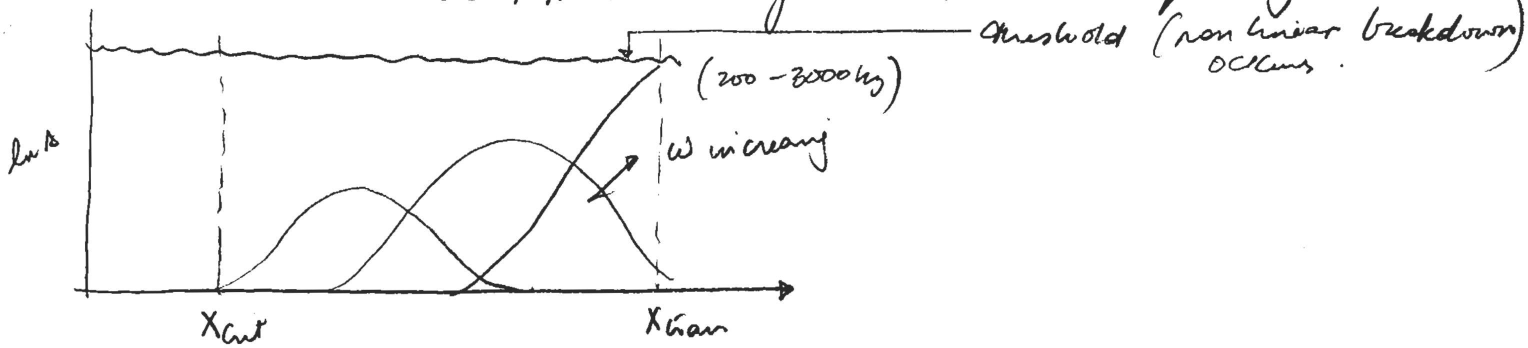


At some location in BL

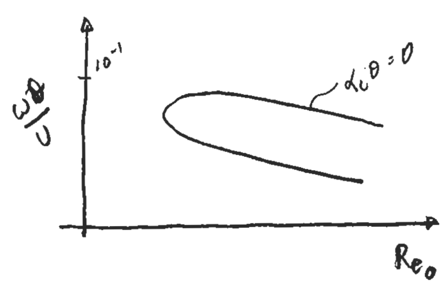


as Re_θ increases unstable region grows - more frequencies get amplified. Some will grow then decay as it moves out of unstable region. Can draw spatial growth rates

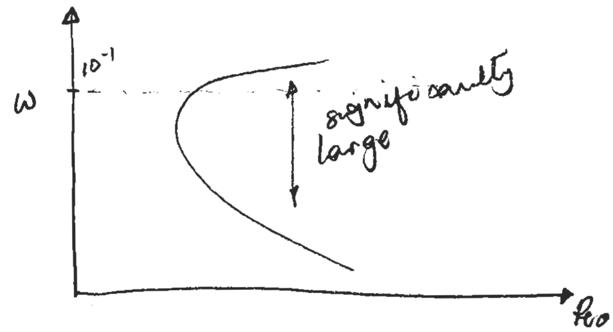
$A(x) = |v|_{max}$ or any other convenient quantity



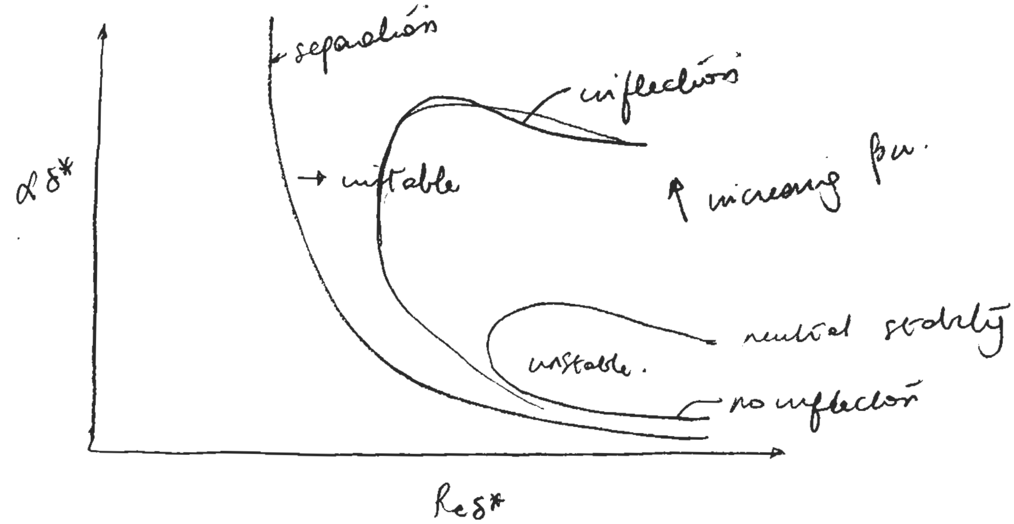
Note for $H = 2.6$



$H = 4$



Stability parameters from F-S eqn - see handout.


separation
inflection
unstable
increasing βu.
$\alpha\delta^*$
neutral stability
unstable.
no inflection
Re_{δ^*}