Summary of 2-D Euler Equations

- $$\frac{d}{dt}\iint_C \rho \, dA + \oint_{SC} \rho \vec{u}\cdot\vec{n}\, dS = 0$$

$$\frac{d}{dt}\iint_C \rho u \, dA + \oint_{SC} \rho u\, \vec{u}\cdot\vec{n}\, dS = -\oint_{SC} p\vec{n}\cdot\vec{i}\, dS$$

$$\frac{d}{dt}\iint_C \rho v \, dA + \oint_{SC} \rho v\, \vec{u}\cdot\vec{n}\, dS = -\oint_{SC} p\vec{n}\cdot\vec{j}\, dS$$

$$\frac{d}{dt}\iint_C \rho E \, dA + \oint_{SC} \rho E\, \vec{u}\cdot\vec{n}\, dS = -\oint_{SC} p\vec{u}\cdot\vec{n}\, dS$$

- These are often written very compactly as:

$$\frac{d}{dt}\iint_C U\, dA + \oint_{SC} \left(F\vec{i} + G\vec{j}\right)\cdot\vec{n}\, dS = 0$$

$$U \equiv \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix} \qquad F \equiv \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{bmatrix} \qquad G \equiv \begin{bmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ \rho v H \end{bmatrix}$$

conservative state vector

flux vector for x-direction

flux vector for y-direction

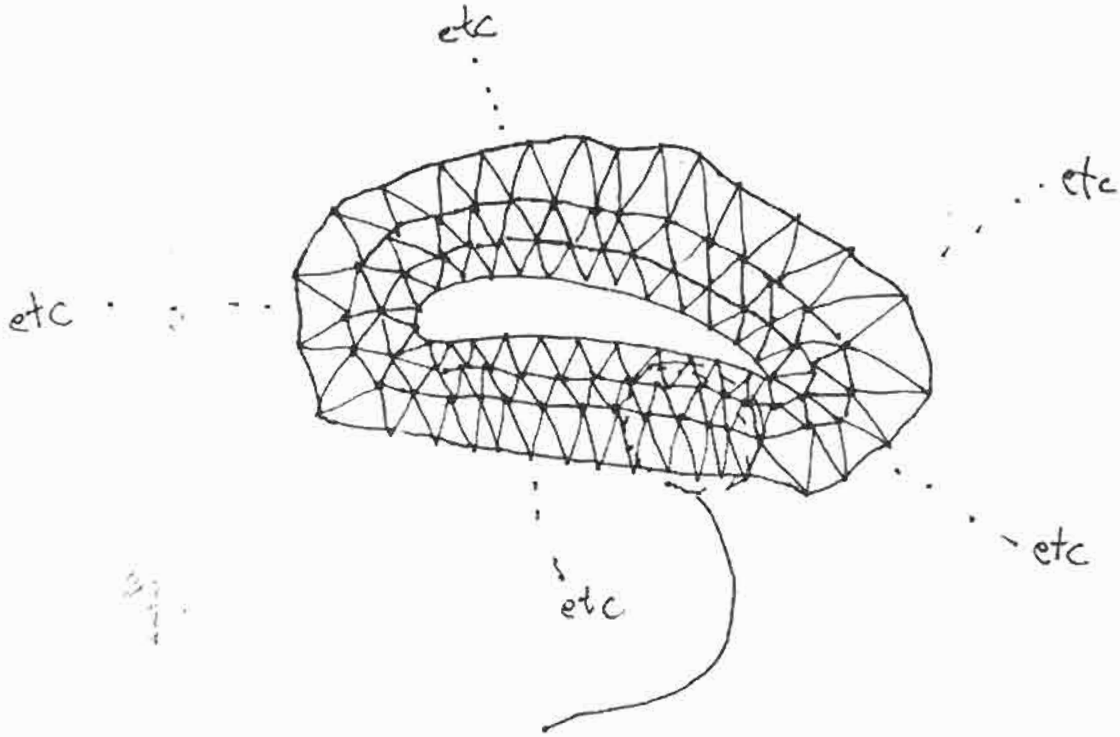
- $H \equiv$ total enthalpy $\equiv E + p/\rho$

Ideal gas: $p = \rho R T = (\gamma - 1)\left[\rho E - \frac{1}{2}\rho(u^2+v^2)\right]$

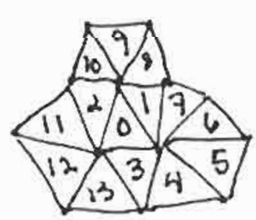
A Finite Volume Scheme for the 2-D Euler Eqns

Here's the basic idea:

(1) Divide up (i.e. discretize) the domain into simple geometric shapes (triangles & quads)



looking at this small region:



Cell 0 is surrounded by cells 1, 2 & 3.

i.e. cell 0 has 3 $\underbrace{\text{neighbors}}_{\text{nearest neighbors}}$: cell 1, 2 & 3.

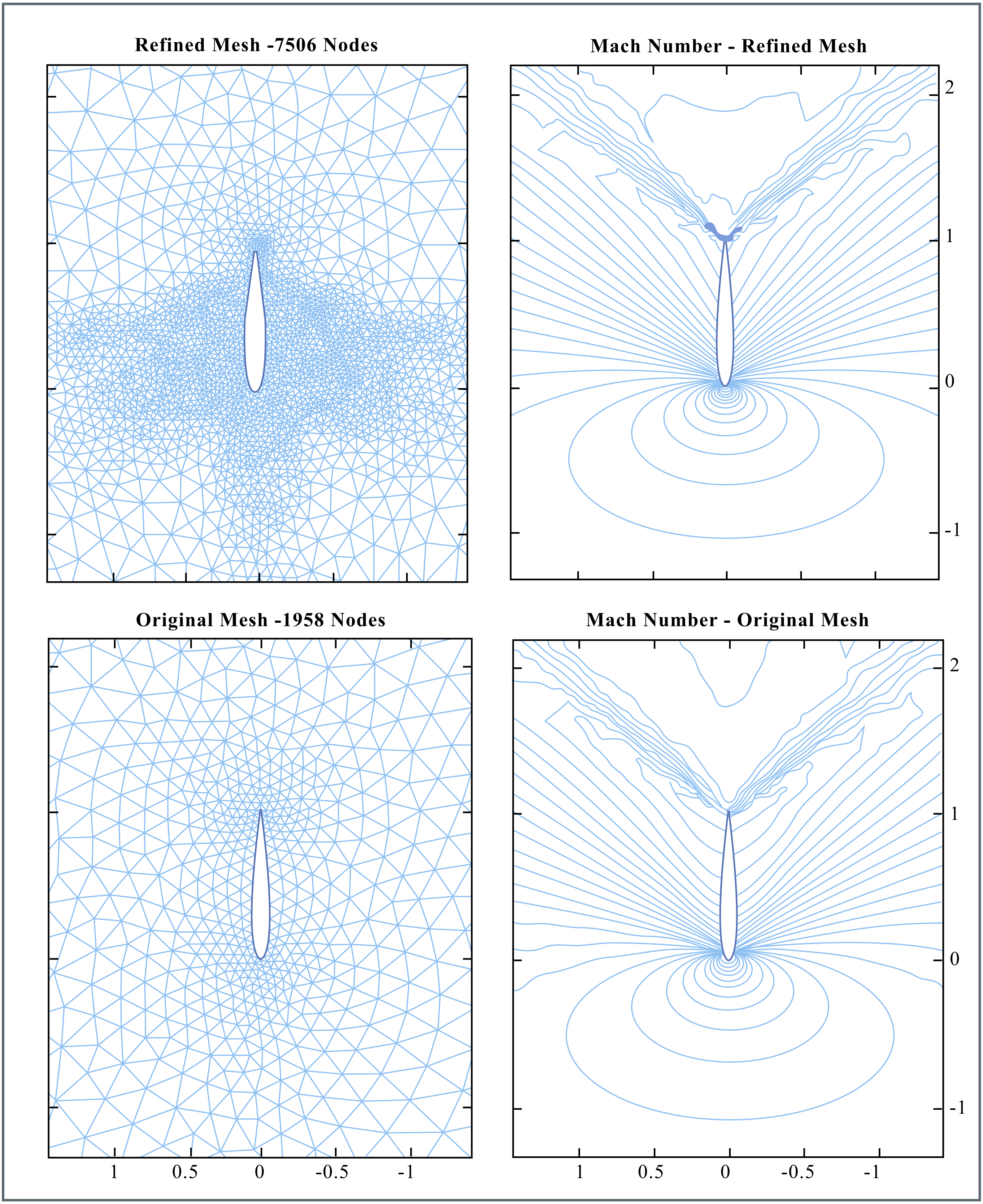


Figure by MIT OCW.

(2) Decide how to place the unknowns in the grid.

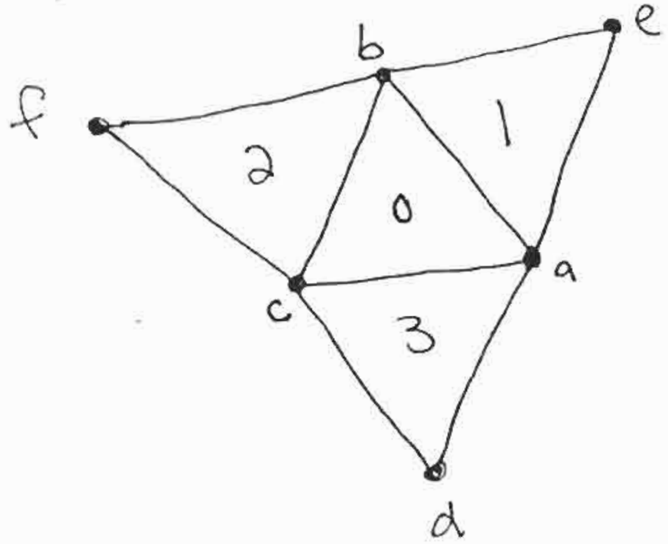
(a) <u>Cell-centered</u>: cell-average values of the conservative state vector are stored for each cell

(b) <u>Node-based</u>: point values of the conservative state vector are stored at each node.

The debate still rages about which of these options is best. We will look at cell-centered schemes because these are easiest (although not necessarily the best). Also, they are very widely used in the aerospace industry.

(3) Approximate the 2-D integral Euler equations on the grid to determine the chosen unknowns.

Let's look in detail at step 3:



cells: 0, 1, 2, 3
nodes: a, b, c, d, e, f

Cell-average unknowns:

$$U_0 = \begin{bmatrix} \rho_0 \\ (\rho u)_0 \\ (\rho v)_0 \\ (\rho E)_0 \end{bmatrix} \qquad U_1 = \begin{bmatrix} \rho_1 \\ (\rho u)_1 \\ (\rho v)_1 \\ (\rho E)_1 \end{bmatrix} \qquad U_2 = \cdots \qquad U_3 = \cdots$$

Specifically, we define U_0 as:

$$U_0 \equiv \frac{1}{A_0} \iint_{C_0} U \, dA \qquad \text{where} \begin{cases} C_0 \equiv \text{cell } 0 \\ A_0 \equiv \text{area of cell } 0 \end{cases}$$

Now, we apply conservation eqns:

$$\frac{d}{dt} \iint_{C_0} U \, dA + \oint_{\delta C_0} (F\vec{i} + G\vec{j}) \cdot \vec{n} \, dS = 0$$

The time-derivative term can be simplified a little:

$$\frac{d}{dt}\iint_{C_0} U\, dA = A_0 \frac{dU_0}{dt}$$

The surface flux integral can also be simplified a little:

$$\oint_{SC_0} (F\vec{i}+G\vec{j})\cdot\vec{n}\,dS = \int_a^b (F\vec{i}+G\vec{j})\cdot\vec{n}\,dS$$

$$+ \int_b^c (F\vec{i}+G\vec{j})\cdot\vec{n}\,dS$$

$$+ \int_c^a (F\vec{i}+G\vec{j})\cdot\vec{n}\,dS$$

Combining these expressions:

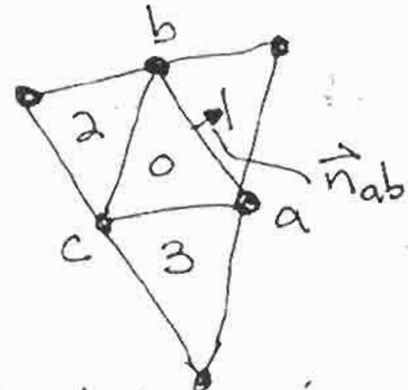
$$\boxed{A_0\frac{dU_0}{dt} + \int_a^b (F\vec{i}+G\vec{j})\cdot\vec{n}\,dS + \int_b^c (F\vec{i}+G\vec{j})\cdot\vec{n}\,dS + \int_c^a (F\vec{i}+G\vec{j})\cdot\vec{n}\,dS = 0}$$

No appro[x.] so far!

Now, we must make some approximations.

Let's look at the surface integral from a→b:

$$\int_a^b (F\vec{i} + G\vec{j}) \cdot \vec{n}\,dS$$



The normal can easily be calculated since the face is a straight line between nodes a & b.

Recall, the unknowns are stored at cell centers. So, what would be a logical approximation for:

$$\int_a^b (F\vec{i} + G\vec{j}) \cdot \vec{n}_{ab}\,dS = \;???$$

Option #1 =

Option #2 =

Note: Option #1 ≠ Option #2 in general.

There is very little difference in practice between these options. Let's stick with:

$$\tilde{\mathcal{F}}_{ab} \equiv \int_a^b (F\vec{i} + G\vec{j}) \cdot \vec{n}_{ab}\, dS = \left[\tfrac{1}{2}(F_0+F_1)\vec{i} + \tfrac{1}{2}(G_0+G_1)\vec{j}\right] \cdot \vec{n}_{AB}\, \Delta S_{ab}$$

$$\tilde{\mathcal{F}}_{bc} \equiv \int_b^c (F\vec{i} + G\vec{j}) \cdot \vec{n}_{bc}\, dS = \left[\tfrac{1}{2}(F_0+F_2)\vec{i} + \tfrac{1}{2}(G_0+G_2)\vec{j}\right] \cdot \vec{n}_{bc}\, \Delta S_{bc}$$

$$\tilde{\mathcal{F}}_{ca} \equiv \int_c^a (F\vec{i} + G\vec{j}) \cdot \vec{n}_{ca}\, dS = \left[\tfrac{1}{2}(F_0+F_3)\vec{i} + \tfrac{1}{2}(G_0+G_3)\vec{j}\right] \cdot \vec{n}_{ca}\, \Delta S_{ca}$$

where
$$F_0 \equiv F(U_0) \qquad G_0 \equiv G(U_0)$$
$$F_1 \equiv F(U_1) \qquad G_1 \equiv G(U_1)$$
$$F_2 \equiv F(U_2) \qquad G_2 \equiv G(U_2)$$
$$F_3 \equiv F(U_3) \qquad G_3 \equiv G(U_3)$$

Finally, we have to approximate $A_0 \dfrac{dU_0}{dt}$ somehow. The simplest approach is forward Euler:

$$A_0 \frac{dU_0}{dt} + \tilde{\mathcal{F}}_{ab} + \tilde{\mathcal{F}}_{bc} + \tilde{\mathcal{F}}_{ca} = 0$$

$$\boxed{A_0 \frac{U_0^{n+1} - U_0^n}{\Delta t} + \tilde{\mathcal{F}}_{ab}^{\,n} + \tilde{\mathcal{F}}_{bc}^{\,n} + \tilde{\mathcal{F}}_{ca}^{\,n} = 0}$$

where $U_0^n \equiv U_0(t^n)$ and $t^n \equiv n\Delta t$, $n \equiv$ iteration #

and $\mathcal{F}^n_{ab}$ etc are defined as:

$$\mathcal{F}^n_{ab} \equiv \left[\frac{1}{2}\left(F^n_0 + F^n_1\right)\vec{i} + \frac{1}{2}\left(G^n_0 + G^n_1\right)\vec{j}\right] \cdot \vec{n}_{ab} \Delta S_{ab}$$

$$F^n_0 \equiv F(U^n_0)$$

etc.

$$F^n_1 \equiv F(U^n_1)$$

For steady solution, basic procedure is to make a guess of U at $t=0$ and then iterate until the solution no longer changes. This is called time marching

Solution Convergence

- Recall for our triangular grid finite volume scheme, the basic iterative scheme looked like:

$$A_i \frac{U_i^{n+1}-U_i^{n}}{\Delta t} + \underbrace{\overbrace{\tilde{\mathcal{F}}^{n}_{ab_i} + \tilde{\mathcal{F}}^{n}_{bc_i} + \tilde{\mathcal{F}}^{n}_{ca_i}}^{R_i^n \,\equiv\, \text{residual of cell } i}}_{\text{approximation of } \oint_{\partial c_i} (F\vec{i}+G\vec{j})\cdot\vec{n}\,dS} = 0$$

- $\Rightarrow$ $$\boxed{U_i^{n+1} = U_i^{n} - \frac{\Delta t}{A_i} R_i^n}$$

update formula for cell i to iteration $n+1$ from iteration

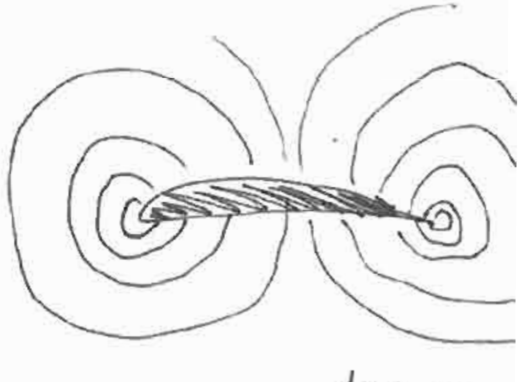
Suppose we are interested in the steady answer to our problem, i.e.

$t = 0$

$$U^0 = U^0_\infty = \begin{Bmatrix} \rho_\infty \\ \rho_\infty u_\infty \\ \rho_\infty v_\infty \\ \rho_\infty E_\infty \end{Bmatrix}$$

Initially guess of uniform flow

$\longrightarrow$ we want



answer as $t \to \infty$ when $\frac{dU}{dt} = 0$ (steady-state)

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