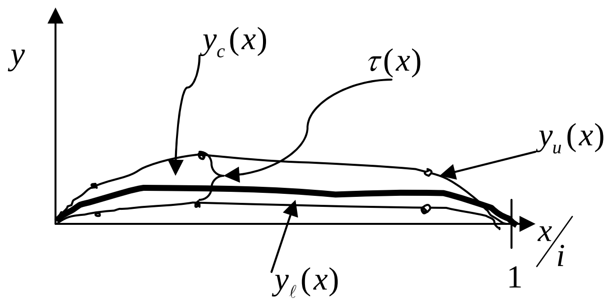
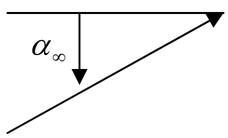
Implications of Linearized Supersonic Flow on Airfoil Lift & Drag

To begin, we will divide the airfoil geometry into camber and thickness distributions:





$\vec{U}_\infty \Rightarrow U_\infty = u_\infty \cos\alpha_\infty \vec{i} + U_\infty \sin\alpha_\infty \vec{j}$

$$y_u(x) = y_c(x) + \frac{1}{2}\tau(x)$$

$$y_\ell(x) = y_c(x) - \frac{1}{2}\tau(x)$$

Note:
The x-axis, which is the axis defined through the leading edge and trailing edge points, is used to define the freestream angle of attack. The chord line is the line connecting $i.e.$ to $t.e.$ and, the chord length is the distance between these points.

To calculate the lift and drag, we need to integrate the pressure forces around the airfoil.
$\vec{F} = -\oint_{airfoil} p\vec{n}ds$ when $\vec{n}$ points out of the airfoil surface.

Now, we begin specializing this formula for the assumptions of linearized flow, i.e.

$* \,{}^{\tau}\!/_{c} << 1$ (small thickness)

$* \left|{}^{y_c}\!/_{c}\right| << 1$ (small camber)

$* \,\alpha_\infty << 1$ (small α)

The normal on the upper surface is

$$\vec{t}_u = \frac{dx}{ds}\vec{i} + \frac{dy_u}{ds}\vec{j}$$

$$\vec{n}_u = -\frac{dy_u}{ds}\vec{i} + \frac{dx}{ds}\vec{j}$$

But, since we have thin airfoils $ds \cong dx$, thus $\vec{n}_u = -\frac{dy_\ell}{dx}\vec{i} + 1\vec{j}$

Similarly, on the lower surface $\vec{n}_\ell = \frac{dy_\ell}{dx}\vec{i} - 1\vec{j}$

Thus, the force may be written:

$$\vec{F} = -\int_0^c p_u \vec{n}_u dx - \int_0^c p_\ell \vec{n}_\ell dx$$

$$\vec{F} = -\int_0^c p_u \left(-\frac{dy_u}{dx}\vec{i} + \vec{j}\right)dx - \int_0^c p_\ell \left(\frac{dy_\ell}{dx}\vec{i} - \vec{j}\right)dy$$

$$\Rightarrow \vec{F} = \vec{i}\int_0^c \left(p_u \frac{dy_u}{dx} - p_\ell \frac{dy_\ell}{dx}\right)dx + \vec{j}\int_0^c (p_\ell - p_u)dx$$

The drag component is in the freestream direction:

$$D' = \vec{F} \bullet \left(\frac{\vec{u}_\infty}{u_\infty}\right) = \vec{F} \bullet \left(\underbrace{\cos\alpha_\infty}_{1}\vec{i} + \underbrace{\sin\alpha_\infty}_{\propto_\infty}\vec{j}\right)$$

$$D' = \vec{F} \bullet \vec{i} + \vec{F} \bullet \vec{j}\alpha_\infty$$

$$\Rightarrow \boxed{D' = \int_0^c \left(p_u \frac{dy_0}{dx} - p_\ell \frac{dy_\ell}{dx}\right)dx + \alpha_\infty \int_0^c (p_\ell - p_u)dx}$$

Similarly, the lift is:

$$L' = \vec{F} \bullet \left(\underbrace{-\sin\alpha_\infty}_{-\propto_\infty}\vec{i} + \underbrace{\cos\alpha_\infty}_{1}\vec{j}\right)$$

$$= -\vec{F} \bullet \vec{i}\,\alpha_\infty + \vec{F} \bullet \vec{j}$$

$$\boxed{L' = -\alpha_\infty \int_0^c \left(p_u \frac{dy_u}{dx} - p_\ell \frac{dy_\ell}{dx}\right)dx + \int_0^c (p_\ell - p_u)dx}$$

Or, manipulating the L' & D' slightly:

$$D' = \int_0^c \left[p_u \left(\frac{dy_u}{dx} - \alpha_\infty \right) - p_\ell \left(\frac{dy_\ell}{dx} - \alpha_\infty \right) \right] dx$$

$$L' = \int_0^c \left[p_\ell \left(1 + \alpha_\infty \frac{dy_\ell}{dx} \right) - p_u \left(1 + \alpha_\infty \frac{dy_u}{dx} \right) \right] dx$$

Then substituting in for the $c_p \equiv \dfrac{p - p_\infty}{\frac{1}{2}\rho u_\infty^2}$, we can write this as:

$$c_d = \frac{1}{c}\int_0^c c_{p_u} \left(\frac{dy_u}{dx} - \alpha_\infty \right) - c_{p_\ell} \left(\frac{dy_\ell}{dx} - \alpha_\infty \right) \Bigg] dx$$

$$c_\ell = \frac{1}{c}\int_0^c \left[c_{p_\ell} \left(1 + \alpha_\infty \frac{dy_\ell}{dx} \right) - c_{p_u} \left(1 + \alpha_\infty \frac{dy_u}{dx} \right) \right] dx$$

As shown in Anderson

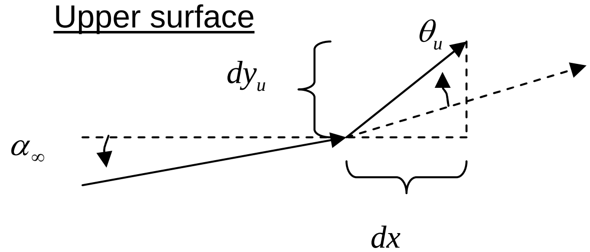
$$c_p = 2\theta \Big/ \sqrt{M_\infty^2 - 1}$$

$$\Rightarrow c_{p_u} = 2\left(\frac{dy_u}{dx} - \alpha_\infty \right) \Big/ \sqrt{M_\infty^2 - 1}$$

$$c_{p_\ell} = 2\left(-\frac{dy_\ell}{dx} + \alpha_\infty \right) \Big/ \sqrt{M_\infty^2 - 1}$$

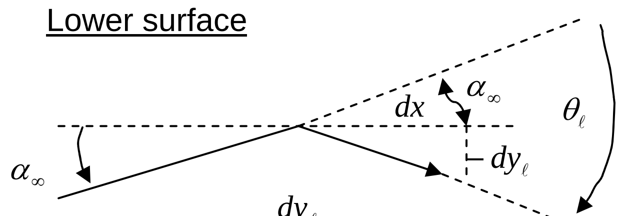
In supersonic linearized flow where θ is the flow direction relative to the freestream (and assuming an isolated airfoil).

Upper surface



$$\Rightarrow \theta_u = \frac{dy_u}{dx} - \alpha_\infty$$

Lower surface



$$\Rightarrow \theta_\ell = -\frac{dy_\ell}{dx} + \alpha_\infty$$

Next, let's substitute c_{p_u} *and* c_{p_ℓ} into c_ℓ:

$$c_\ell = \frac{1}{c}\int_0^c \frac{2}{\sqrt{M_\infty^2 - 1}}\left(-\frac{dy_\ell}{dx} + \alpha_\infty\right)\Bigg(1 + \underbrace{\alpha_\infty \frac{dy_\ell}{dx}}_{\text{small}}\Bigg) - \frac{2}{\sqrt{M_\infty^2 - 1}}\left(\frac{dy_u}{dx} - \alpha_\infty\right)\Bigg(1 + \underbrace{\alpha_\infty \frac{dy_u}{dx}}_{\text{small}}\Bigg)\Bigg]dx$$

$$c_\ell \approx \frac{2}{c\sqrt{M_\infty^2 - 1}}\int_0^c \left(-\frac{dy_\ell}{dx} - \frac{dy_u}{dx} + 2\alpha_\infty\right)dx$$

$$= \frac{2}{c\sqrt{M_\infty^2 - 1}}\left[-\int_0^c \frac{dy_\ell}{dx}dx - \int_0^c \frac{dy_u}{dx}dx + 2\alpha_\infty c\right]$$

But $\int_0^c \frac{dy_\ell}{dx}dx = y_\ell(c) - y_\ell(0) = 0 - 0 = 0$

And, similarly, $\int_0^c \frac{dy_u}{dx}dx = 0$

$$\Rightarrow \boxed{c_\ell = \frac{4\alpha_\infty}{\sqrt{M_\infty^2 - 1}}} \longleftarrow \text{Important result!!}$$

$*c_\ell$ is linear with α_∞ but note the slope is different than subsonic case $\left(\frac{dc_\ell}{d\alpha} = \frac{2\pi}{\sqrt{1 - M_\infty^2}}\right)$.

$*c_\ell$ <u>does</u> not depend on camber! All the y-dependence has disappeared in this result. Thus, c_ℓ also does not depend on thickness.

Now, let's look at c_d:

$$c_d = \frac{2}{c\sqrt{M_\infty^2 - 1}}\int_0^c \left[\left(\frac{dy_u}{dx} - \alpha_\infty\right)^2 + \left(\frac{dy_\ell}{dx} - \alpha_\infty\right)^2\right]dx$$

$$= \frac{2}{c\sqrt{M_\infty^2 - 1}}\int_0^c \left[\left(\frac{dy_u}{dx}\right)^2 \underbrace{- 2\alpha_\infty\left(\frac{dy_u}{dx} + \frac{dy_\ell}{dx}\right)}_{\text{This term will integrate to 0}} + \left(\frac{dy_\ell}{dx}\right)^2 + 2\alpha_\infty^2\right]dx$$

$$\Rightarrow c_d = \frac{2}{c\sqrt{M_\infty^2 - 1}}\int_0^c \left[\left(\frac{dy_u}{dx}\right)^2 + \left(\frac{dy_\ell}{dx}\right)^2 + 2\alpha_\infty^2\right]dx$$

$$\Rightarrow \quad \boxed{c_d = \frac{4\alpha_\infty^2}{\sqrt{M_\infty^2 - 1}} + \frac{2}{\sqrt{M_\infty^2 - 1}} \frac{1}{c} \int_0^c \left[\left(\frac{dy_u}{dx} \right)^2 + \left(\frac{dy_\ell}{dx} \right)^2 \right] dx \geq 0}$$

Note: c_d is >0 unless $\alpha_\infty = 0$ and airfoil is a plate $(y_u = y_\ell = const \Rightarrow y_u = y_\ell = 0)$.
A little manipulation gives another form dependent on the camber and thickness:

$$\boxed{c_d = \frac{4\alpha_\infty^2}{\sqrt{M_\infty^2 - 1}} + \frac{4}{\sqrt{M_\infty^2 - 1}} \frac{1}{c} \int_0^c \left(dy_c \right)^2 dx + \frac{1}{\sqrt{M_\infty^2 - 1}} \frac{1}{c} \int_0^c \left(\frac{d\tau}{dx} \right)^2 dx}$$

Thus, for a given c_ℓ, the lowest c_d occurs when the airfoil is a flat plate $(y_c = \tau = 0)$!!