Linearized Compressible Potential Flow Governing Equation

Recall the 2-D full potential eqn is:

$$\left[1-\frac{1}{a^2}(\phi_x)^2\right]\phi_{xx}+\left[1-\frac{1}{a^2}(\phi_y)^2\right]\phi_{yy}-\frac{2}{a^2}\phi_x\phi_y\phi_{xy}=0$$

Where $a^2 = a_o^2 - \frac{\gamma-1}{\gamma}\left[(\phi_x)^2+(\phi_y)^2\right]$

As you saw, for small perturbations to a uniform flow, the linearized form of the equation was:

$$\boxed{(1-M_\infty^2)\hat{\phi}_{xx}+\hat{\phi}_{yy}=0}$$

$\hat{\phi}=$ perturbation potential

Where

$\vec{u}=u_{\infty\vec{i}}+\vec{\hat{u}}$

$\vec{\hat{u}}=\hat{\phi}_{x\vec{j}}+\hat{\phi}_{y\vec{J}} \Rightarrow \hat{u}=\hat{\phi}_x$, $\hat{v}=\hat{\phi}_y$ } perturbation velocity

This equation is valid for both $M_\infty<1$ and $M_\infty>1$. Note, though, it is not correct for $M_\infty\to 1$ or for M_∞ large, say greater than about 2 or so.

So what happens to the linearized potential equation for $M_\infty>1$:

Subsonic Flow	Supersonic
$M_\infty<1$	$M_\infty>1$
$1-M_\infty^2>0$	$1-M_\infty^2<0$
$\Rightarrow$ Elliptic PDE (Laplace's eqn)	$\Rightarrow$ Hyperbolic PDE (wave eqn)

It also turns out that $(1-M_\infty^2)\hat{\phi}_{xx}+\hat{\phi}_{yy}=0$ is much easier to solve when $M_\infty>1$.

Define $\lambda=\sqrt{M_\infty^2-1}$

$\Rightarrow \lambda^2\hat{\phi}_{xx}-\hat{\phi}_{yy}=0$

Then,

$\hat{\phi}(x,y)=\hat{\phi}(\eta)$ where $\eta=x-\lambda y$

is a solution to the linearized potential. To see this:

$$\hat{\phi}_x = \hat{\phi}'\eta_x = \hat{\phi}' \leftarrow \hat{\phi}' \equiv \frac{d\hat{\phi}}{d\eta}$$

$$\hat{\phi}_y = \hat{\phi}'\eta_y = -\lambda\hat{\phi}'$$

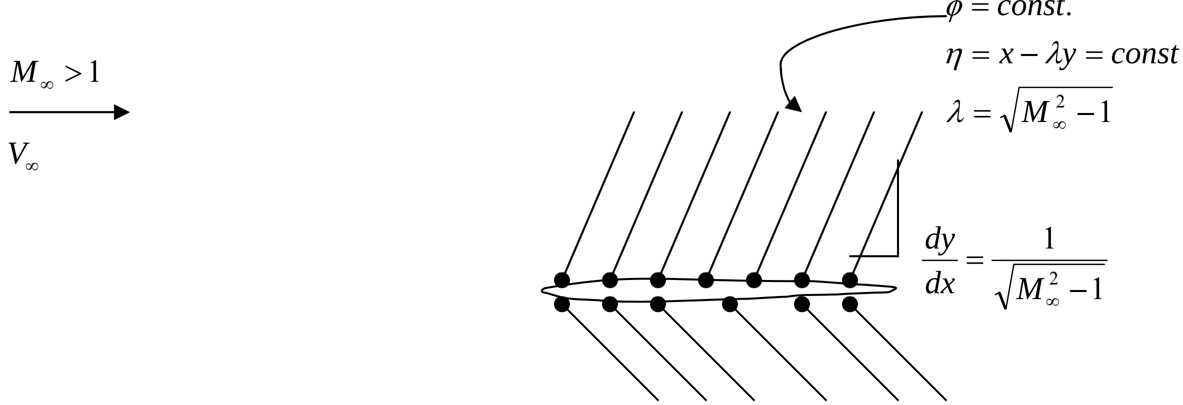
Similarly,

$$\hat{\phi}_{xx} = \hat{\phi}'' \qquad \Rightarrow \underbrace{\lambda^2\hat{\phi}_{xx} - \hat{\phi}_{yy} = \lambda^2\hat{\phi}'' - \lambda^2\hat{\phi}''}_{\hat{\phi}(\eta) \text{ is solution to linearized potential!}} \qquad = 0\,\surd$$

$$\hat{\phi}_{yy} = \lambda^2\hat{\phi}''$$

$\hat{\phi}(\eta)$ is solution to linearized potential!

This means that $\hat{\phi}$ is constant for lines described by $\eta = x - \lambda y = const.$ For example, consider an airfoil:

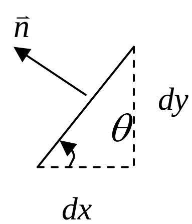


So, the values of $\hat{\phi}$ will be determined by the boundary conditions on the surface. Recall:

$\vec{u} \bullet \vec{n} = 0$ on boundary

For linearized flow, this b.c. reduces to:

$$\boxed{\hat{v} = V_\infty\theta}$$



Also, note that:

$$\hat{u} = \hat{\phi}_x = \hat{\phi}'$$

$$\hat{v} = \hat{\phi}_y = -\lambda\hat{\phi}' \quad \Rightarrow \hat{v} = -\lambda\hat{u}$$

$$\Rightarrow \qquad \boxed{\hat{u} = -\frac{V_\infty\theta}{\lambda}} \longleftarrow \text{ on boundary}$$

This is very useful because the linearized pressure coefficient is:

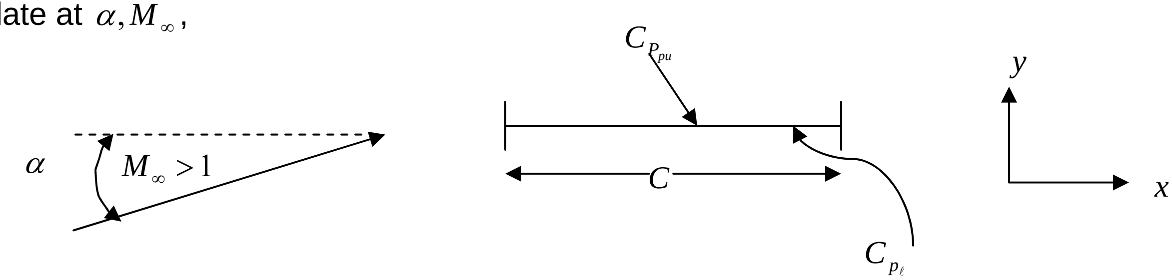
$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = -\frac{2\hat{u}}{V_\infty}$$

$$\Rightarrow \boxed{C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}} \longleftarrow \text{on boundary!}$$

$\Rightarrow C_p > 0$ for $\theta > 0$ (i.e. a compression)

$C_p < 0$ for $\theta < 0$ (i.e. an expansion)

Using linear potential theory, let's calculate the lift and drag coefficients for a flat plate at α, M_∞,



Find C_ℓ & C_d:

$$C_{p_u} = \frac{2\theta_u}{\sqrt{M_\infty^2 - 1}} =$$

$$C_{p_\ell} = \frac{2\theta_\ell}{\sqrt{M_\infty^2 - 1}} =$$

Then, the result is a force in the y - direction:

$$F_y = \int_o^c \left(\frac{1}{2}\rho_\infty V_\infty^2\right)\left(C_{p_\ell} - C_{p_u}\right)dx$$

$$F_y = \left(\frac{1}{2}\rho_\infty V_\infty^2 C\right)\left(C_{p_\ell} - C_{p_u}\right)$$

$$\Rightarrow C_{f_y} \equiv \frac{F_y}{\frac{1}{2}\rho_\infty V_\infty^2 c} = C_{p_\ell} - C_{p_u}$$

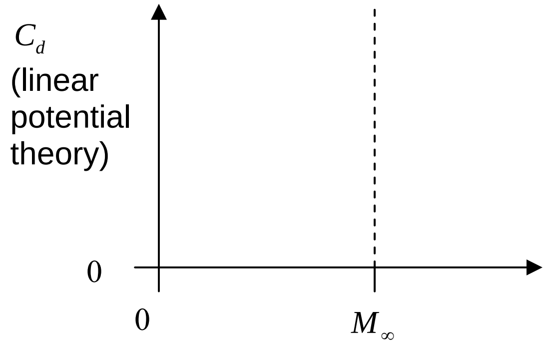
Finally, we need to rotate this into lift and drag directions:

$$C_\ell = -C_{f_x}\sin\alpha + C_{f_y}\cos\alpha$$

$$C_d = C_{f_x}\cos\alpha = C_{f_y}\sin\alpha$$

But, in our case, $C_{f_x} = 0 \;\&\; \alpha << 1$

$$\Rightarrow \boxed{\begin{aligned} C_\ell &\cong C_{f_y} \\ C_d &\cong C_{f_y}\alpha \end{aligned}}$$


C_d
(linear
potential
theory)
0
0
M_∞