Integral Boundary Layer Equations

Displacement Thickness

The displacement thickness δ^* is defined as:

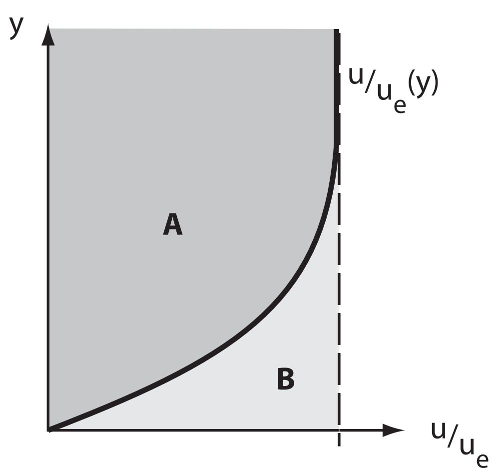
$$\delta^* = \underbrace{\int_0^\infty \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy}_{\substack{\textit{compressible} \\ \textit{flow}}} = \underbrace{\int_0^\infty \left(1 - \frac{u}{u_e}\right) dy}_{\substack{\textit{incompressible} \\ \textit{flow}}}$$

The displacement thickness has at least two useful interpretations:

Interpretation #1

$$\mathbf{A} = \int_0^\infty \frac{u}{u_e} dy$$

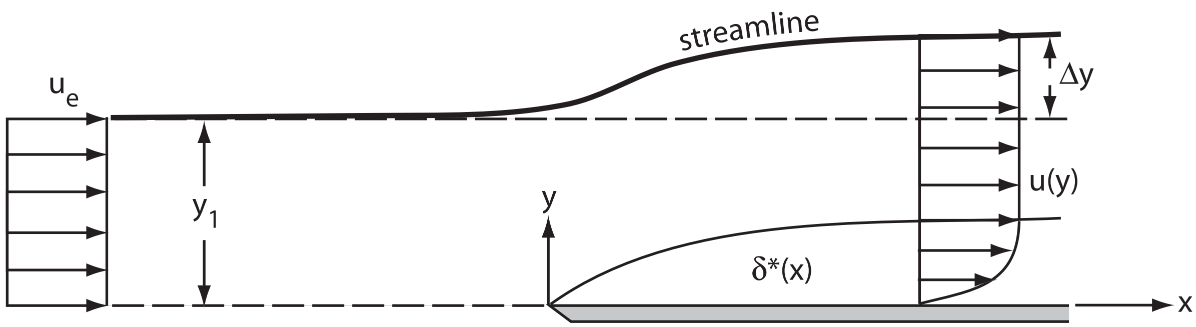
$$\mathbf{A} + \mathbf{B} = \int_0^\infty (1) dy$$



So, the difference is in area **B**.

$\Rightarrow \delta^*$ "represents" the decrease in mass flow due to viscous effects, i.e. lost $\dot{m}_{visc} = \rho_e u_e \delta^*$

Interpretation #2



Conservation of mass:

$$\int_0^{y_1} u_e dy = \int_0^{y_1+\Delta y} u dy$$

$$\int_0^{y_1} u_e dy = \int_0^{y_1} u dy + \Delta y u_e$$

$$\Rightarrow \Delta y u_e = \int_0^{y_1} (u_e - u) dy$$

$$\Delta y = \int_0^{y_1} \left(1 - \frac{u}{u_e}\right) dy$$

Taking the limit of $y_1 \to \infty$ gives

$$\Rightarrow \Delta y = \delta^* = \int_0^{\infty} \left(1 - \frac{u}{u_e}\right) dy$$

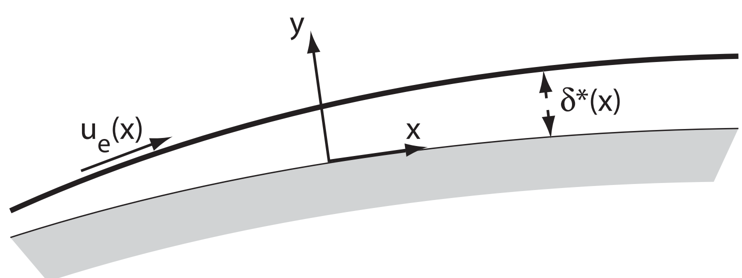
So, the external streamline is displaced by a distance δ^* away from the body due to viscous effects.

$\Rightarrow$ Outer flow sees an "effective body"

Karman's Integral Momentum Equation

This approach due to Karman leads to a useful approximate solution technique for boundary layer effects. It forms the basis of the boundary layer methods utilized in Prof. Drela's XFOIL code.

Basic idea: integrate b.l. equations in y to reduce to an ODE in x.



Derivation:

Add (ρu) x continuity + $x-$momentum

$$\Rightarrow \underbrace{\rho u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}_{(\rho u)\times\text{continuity}}+\underbrace{\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\rho u_e\frac{du_e}{dx}+\mu\frac{\partial^2 u}{\partial y^2}}_{x-\text{momentum}}$$

$$\Rightarrow \rho\left(\frac{\partial(u^2)}{\partial x}+\frac{\partial}{\partial y}(uv)\right)=\rho u_e\frac{du_e}{dx}+\frac{\partial}{\partial y}\underbrace{\left(\mu\frac{\partial u}{\partial y}\right)}_{\tau}$$

Now, we integrate from 0 to y_1:

$$\rho\int_0^{y_1}\frac{\partial(u^2)}{\partial x}dy+\rho uv\Big|_0^{y_1}=\rho u_e\frac{du_e}{dx}y_1+\tau\Big|_0^{y_1}$$

Note:

$$\rho uv\Big|_0^{y_1}=\rho u_e v(y_1)=\rho u_e\int_0^{y_1}\frac{\partial v}{\partial y}dy=-\rho u_e\int_0^{y_1}\frac{\partial u}{\partial x}dy$$

So, the equation becomes:

$$\rho \int_0^{y_1} \frac{\partial (u^2)}{\partial x} dy - \rho u_e \int_0^{y_1} \frac{\partial u}{\partial x} dy = \rho u_e \frac{du_e}{dx} y_1 + \tau \Big|_0^{y_1}$$

After a little more manipulation this can be turned into (note we let $y_1 \to \infty$ also):

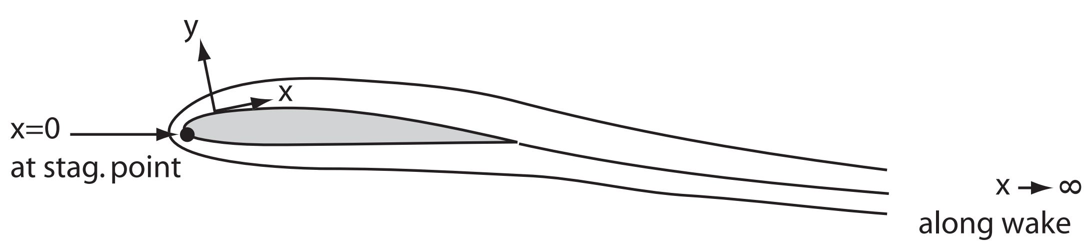
$$\tau_w = \frac{d}{dx}(\rho u_e^2 \theta) + \rho u_e \delta^* \frac{du_e}{dx} \qquad (1)$$

where θ ≡ momentum thickness = $\int_0^\infty \frac{\rho u}{\rho_e u_e}\left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$

incompressible form = $\int_o^\infty \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy$

<u>Insight</u>

Integrate (1) from stagnation point along airfoil & then down the wake



$$\int_0^\infty \tau_w dx = (\rho u_e{}^2 \theta)\Big|_0^\infty + \int_0^\infty \rho u_e \delta^* \frac{du_e}{dx} dx$$

But: $u_e = 0$ at stag. pt. $(x = 0)$ & $\underbrace{-\frac{dp}{dx} = \rho u_e \frac{du_e}{dx}}_{\text{Bernoulli}}$

$$\Rightarrow \underbrace{\rho u_e^2 \theta \Big|_{x \to \infty}}_{\text{drag (see Anderson Sec 2.6 for proof)}} = \int_0^\infty \tau_w dx + \int_0^\infty \delta^* \frac{dp}{dx} dx$$

$$D' = \underbrace{\int_0^\infty \tau_w dx}_{\text{friction drag}} + \underbrace{\int_0^\infty \delta^* \frac{dp}{dx} dx}_{\text{form drag}}$$

Another common form of the integral momentum equation is derived below:

$$\tau_w = \frac{d}{dx}(\rho_e u_e^2 \theta) + \rho_e u_e \delta^* \frac{du_e}{dx}$$

$$\boxed{\begin{array}{l} \dfrac{\tau_w}{\rho_e u_e^2} = \dfrac{d\theta}{dx} + \dfrac{\theta}{u_e}(2+H)\dfrac{du_e}{dx} \\ \text{where} \\ H = \dfrac{\delta^*}{\theta} \leftarrow \text{ known as "shape parameter"} \end{array}}$$