Trajectory Calculation
Lab 2 Lecture Notes

Nomenclature

t	time	ρ	air density
h	altitude	g	gravitational acceleration
V	velocity, positive upwards	m	mass
F	total force, positive upwards	C_D	drag coefficient
D	aerodynamic drag	A	drag reference area
$\dot{()}$	time derivative $(= d()/dt)$	i	time index

Trajectory equations

The vertical trajectory of a rocket is described by the altitude and velocity, $h(t)$, $V(t)$, which are functions of time. These are called *state variables* of the rocket. Figure 1 shows plots of these functions for a typical ballistic trajectory. In this case, the *initial values* for the two state variables h_0 and V_0 are prescribed.

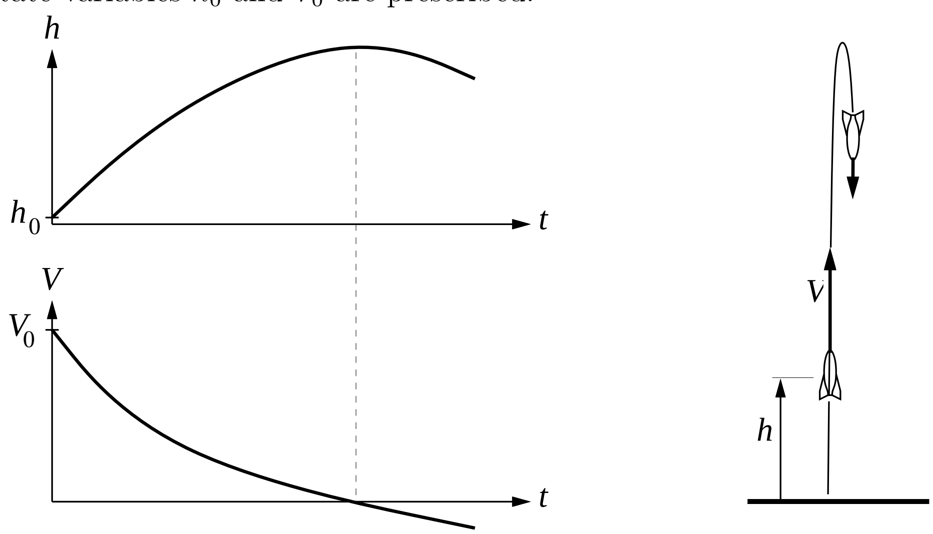

Figure 1: Time traces of altitude and velocity for a ballistic rocket trajectory.

The trajectories are governed by *Ordinary Differential Equations* (ODEs) which give the time rate of change of each state variable. These are obtained from the definition of velocity, and from Newton's 2nd Law.

$$\dot{h} = V \tag{1}$$

$$\dot{V} = F/m \tag{2}$$

Here, F is the total force on the rocket. For the ballistic case with no thrust, F consists only of the gravity force and the aerodynamic drag force.

$$F = \begin{cases} -mg - D & , \text{ if } V > 0 \\ -mg + D & , \text{ if } V < 0 \end{cases} \tag{3}$$

The two cases in (3) are required because F is defined positive up, so the drag D can subtract or add to F depending in the sign of V. In contrast, the gravity force contribution $-mg$ is always negative.

A convenient way to express the drag is

$$D = \frac{1}{2}\rho V^2 C_D A \tag{4}$$

The reference area A used to define the drag coefficient C_D is arbitrary, but a good choice is the rocket's frontal area. Although C_D in general depends on the Reynolds number, it can be often assumed to be constant throughout the ballistic flight. Typical values of C_D vary from 0.1 for a well streamlined body, to 1.0 or more for an unstreamlined or bluff body.

With the above total force and drag expressions, the governing ODEs are written as follows.

$$\dot{h} = V \tag{5}$$

$$\dot{V} = -g - \frac{1}{2}\rho V|V| \frac{C_D A}{m} \tag{6}$$

By replacing V^2 with $V|V|$, the drag contribution now has the correct sign for both the $V > 0$ and $V < 0$ cases.

Numerical Integration

In the presence of drag, or $C_D > 0$, the equation system (5),(6) cannot be integrated analytically. We must therefore resort to numerical integration.

Discretization

Before numerically integrating equations (5) and (6), we must first *discretize* them. We replace the continuous time variable t with a *time index* indicated by the subscript i, so that the state variables h,V, are defined only at discrete times $t_0, t_1, t_2 \ldots t_i \ldots$

$$\begin{aligned} t &\rightarrow t_i \\ h(t) &\rightarrow h_i \\ V(t) &\rightarrow V_i \end{aligned}$$

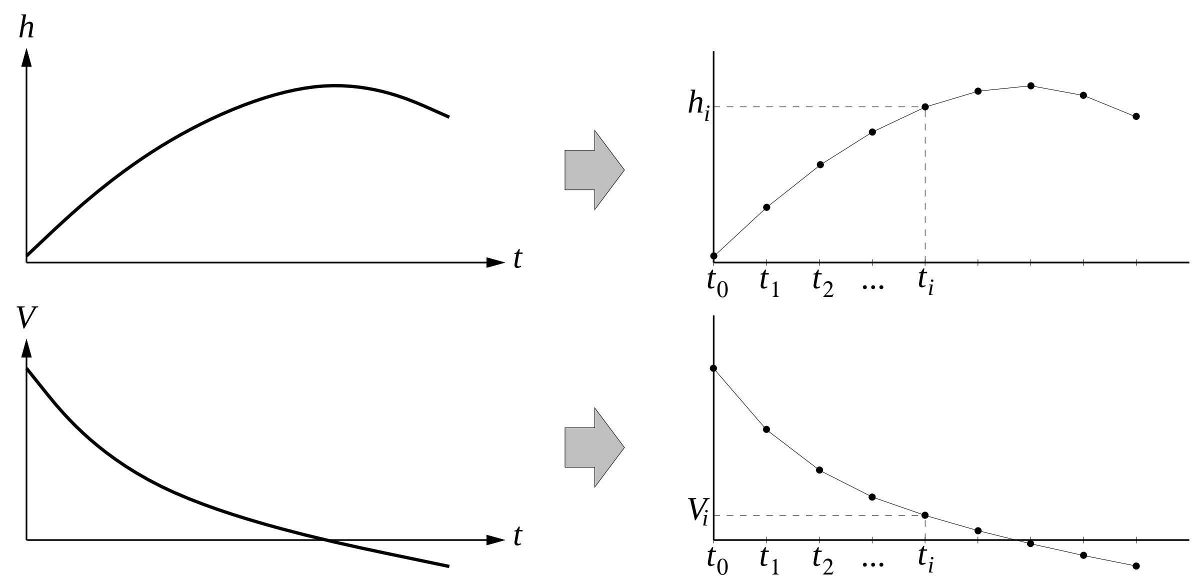

Figure 2: Continuous time traces approximated by discrete time traces.

The governing ODEs (5) and (6) can then be used to determine the discrete rates at each time level.

$$\dot{h}_i = V_i \tag{7}$$

$$\dot{V}_i = -g - \frac{1}{2}\rho V_i |V_i| \frac{C_D\, A}{m} \tag{8}$$

As shown in Figure 3, the rates can also be approximately related to the changes between two successive times.

$$\dot{h}_i = \frac{dh}{dt} \simeq \frac{\Delta h}{\Delta t} = \frac{h_{i+1} - h_i}{t_{i+1} - t_i} \tag{9}$$

$$\dot{V}_i = \frac{dV}{dt} \simeq \frac{\Delta V}{\Delta t} = \frac{V_{i+1} - V_i}{t_{i+1} - t_i} \tag{10}$$

Equating (7) with (9), and (8) with (10), gives the following *difference equations* governing the discrete state variables.

$$\frac{h_{i+1} - h_i}{t_{i+1} - t_i} = V_i \tag{11}$$

$$\frac{V_{i+1} - V_i}{t_{i+1} - t_i} = -g - \frac{1}{2}\rho V_i |V_i| \frac{C_D\, A}{m} \tag{12}$$

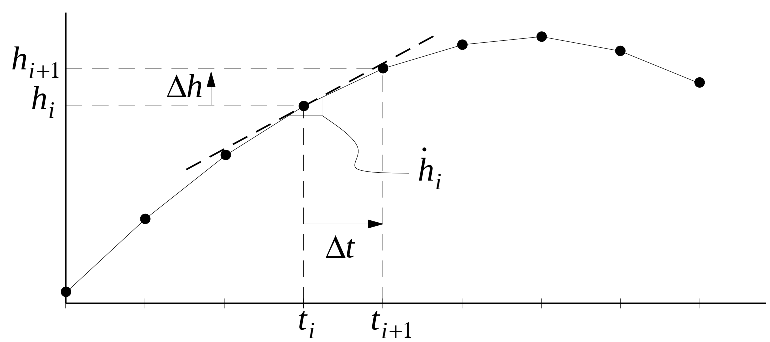

Figure 3: Time rate $\dot{h}$ approximated with finite difference $\Delta h/\Delta t$.

Time stepping (time integration)

Time stepping is the successive application of the difference equations (11),(12) to generate the sequence of state variables h_i,V_i. To start the process, it is necessary to first specify *initial conditions*, just like in the continuous case. These initial conditions are simply the state variable values h_0,V_0 corresponding to the first time index $i=0$. Then, given the values at any i, we can compute values at $i+1$ by rearranging equations (11) and (12).

$$h_{i+1} = h_i + V_i\,(t_{i+1} - t_i) \tag{13}$$

$$V_{i+1} = V_i + \left(-g - \frac{1}{2}\rho V_i |V_i| \frac{C_D\, A}{m}\right)(t_{i+1} - t_i) \tag{14}$$

Equations (13) and (14) are an example of *Forward Euler Integration*.

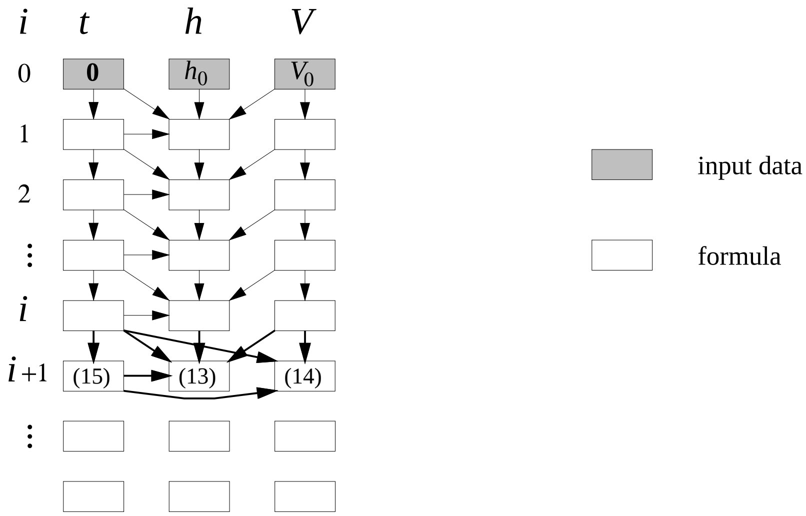

Figure 4: Spreadsheet for time stepping. Arrows show functional dependencies.

Numerical implementation

A spreadsheet provides a fairly simple means to implement the time stepping equations (13) and (14). Such a spreadsheet program is illustrated in Figure 4. The time t_{i+1} in equations (13) and (14) is most conveniently defined from t_i and a specified *time step*, denoted by Δt.

$$t_{i+1} \;=\; t_i + \Delta t \qquad (15)$$

It is most convenient to make this Δt to have the same value for all time indices i, so that equation (15) can be coded into the spreadsheet to compute each time value t_{i+1}, as indicated in Figure 4. This is much easier than typing in each t_i value by hand.

More spreadsheet rows can be added to advance the calculation in time for as long as needed. Typically there will be some *termination criteria*, which will depend on the case at hand. For the rocket, suitable termination criteria might be any of the following.

$h_{i+1} < h_0$ rocket fell back to earth
$h_{i+1} < h_i$ rocket has started to descend
$V_{i+1} < 0$ rocket has started to descend

Accuracy

The discrete sequences h_i, V_i are only approximations to the true analytic solutions $h(t)$, $V(t)$ of the governing ODEs. We can define *discretization errors* as

$$\mathcal{E}_{h_i} \;=\; h_i - h(t_i) \qquad (16)$$

$$\mathcal{E}_{V_i} \;=\; V_i - V(t_i) \qquad (17)$$

although $h(t)$ and $V(t)$ may or may not be available. A discretization method which is *consistent* with the continuous ODEs has the property that

$$|\mathcal{E}| \rightarrow 0 \qquad \text{as} \qquad \Delta t \rightarrow 0$$

The method described above is in fact consistent, so that we can make the errors arbitrarily small just by making Δt sufficiently small.