LECTURE S21

Last time, we found that the duration Δt and bandwidth $\Delta\omega$ of a signal $g(t)$ are given by

$$\left(\frac{\Delta t}{2}\right)^2 = \frac{\|t\,g(t)\|^2}{\|g(t)\|^2} = \frac{\int_{-\infty}^{\infty} t^2 g^2(t)\,dt}{\int_{-\infty}^{\infty} g^2(t)\,dt}$$

$$\left(\frac{\Delta \omega}{2}\right)^2 = \frac{\|j\omega\, G(j\omega)\|^2}{\|G(j\omega)\|^2} = \frac{\int_{-\infty}^{\infty} |j\omega G(j\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega}$$

$$= \frac{\|\dot{g}(t)\|^2}{\|g(t)\|^2} = \frac{\int_{-\infty}^{\infty} \dot{g}^2(t)\,dt}{\int_{-\infty}^{\infty} g^2(t)\,dt}$$

To connect frequency and time domain, we used Parseval's theorem:

$$\|g(t)\|^2 = \frac{1}{2\pi}\|G(j\omega)\|^2 = \|G(f)\|^2$$

To add next step, think of <u>signals</u> as <u>vectors</u>:

$$t\,g(t) \sim \underline{x}$$

$$\dot{g}(t) \sim \underline{y}$$

$$g(t) \sim \underline{z}$$

Then

$$\left(\frac{\Delta t}{2}\right)\left(\frac{\Delta \omega}{2}\right) = \frac{\|t\, g(t)\|\,\|\dot{g}(t)\|}{\|g(t)\|^2}$$

$$= \frac{\|\underline{x}\|\,\|\underline{y}\|}{\|\underline{z}\|^2}$$

The <u>Schwarz inequality</u> states that

$$\|\underline{x}\|\,\|\underline{y}\| \geq |\underline{x}\cdot\underline{y}|$$

In this case,

$$\underline{x}\cdot\underline{y} = \int_{-\infty}^{\infty} \underbrace{t}_{v}\,\underbrace{g(t)\,\dot{g}(t)\,dt}_{dv}$$

$$= -\int_{-\infty}^{\infty} \underbrace{\tfrac{1}{2}g^2(t)}_{v}\quad \underbrace{dt}_{dv}$$

$$= -\tfrac{1}{2}\|g(t)\|^2$$

Putting this all together,

$$\left(\frac{\Delta t}{2}\right)\left(\frac{\Delta\omega}{2}\right) \geq \frac{\tfrac{1}{2}\|g(t)\|^2}{\|g(t)\|^2}$$

$$\Longrightarrow \quad \boxed{\Delta t\,\Delta\omega \geq 2}$$

$$\left(\Delta t\,\Delta f \geq \frac{1}{\pi}\right)$$

To make equal to z, $\underline{x}$ and y must be <u>colinear</u>. That is

$$\dot{g}(t) = \underbrace{k}_{\text{unknown constant}} t\, g(t)$$

$$\Rightarrow \frac{dg}{dt} = k t g$$

$$\Rightarrow \frac{dg}{g} = kt\,dt$$

$$\Rightarrow \ln g = \frac{1}{2} k t^2 + c$$

$$\Rightarrow \quad g(t) = A\, e^{\frac{1}{2}kt^2}$$

For stable $g(t)$, must have $k<0$. Let $k = -1/T^2$. Then

$$g(t) = A\, e^{-t^2/2T^2} = \text{"Gaussian"}$$

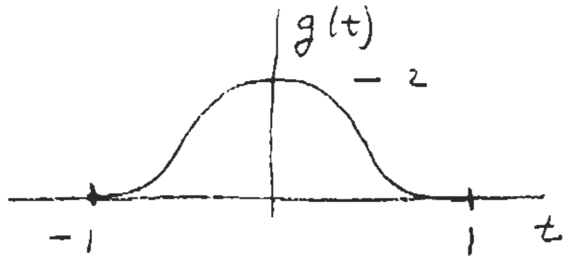
So Gaussian pulses (or close approximations) have the lowest bandwidth, for a given duration.

Pulses that "look" Gausian are close to lower bound.

Example "Raised Cosine"

$$g(t) = [1 + \cos \pi t]\, \sigma(t-1)\, \sigma(-t-1)$$

$$= \begin{cases} 1 + \cos \pi t, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$$



$$\int g^2(t)\, dt = 3$$

$$\int t^2 g^2(t)\, dt = 1 - 15/2\pi^2 = 0.2401$$

$$\int \dot{g}^2(t)\, dt = \pi^2 = 9.870$$

$$\Rightarrow \quad \Delta t = 2\sqrt{\frac{0.2401}{3}} = 0.5658 \text{ sec}$$

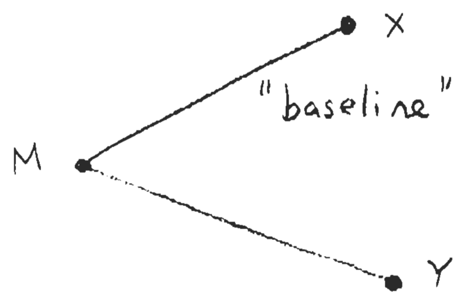
$$\Delta \omega = 2\sqrt{\frac{9.870}{3}} = 3.628 \text{ rad/sec}$$

$$\Delta t \Delta \omega = 2.052 \approx 2$$

So can get very close to 2.

Example Loran-C Navigation System

- Three transmitters on surface of earth:



- Each transmitter sends out pulses simultaneously (actually, there is a delay to prevent overlap)

- Receiver on ship/aircraft measures time of arrival of each pulse.

- Difference in arrival times is proportional to difference in distance to stations

- By measuring two time differences ($t_M - t_X$, $t_M - t_Y$), can determine position.

- Third TD ($t_X - t_Y$) is redundant.

- Navigation system is "hyperbolic."

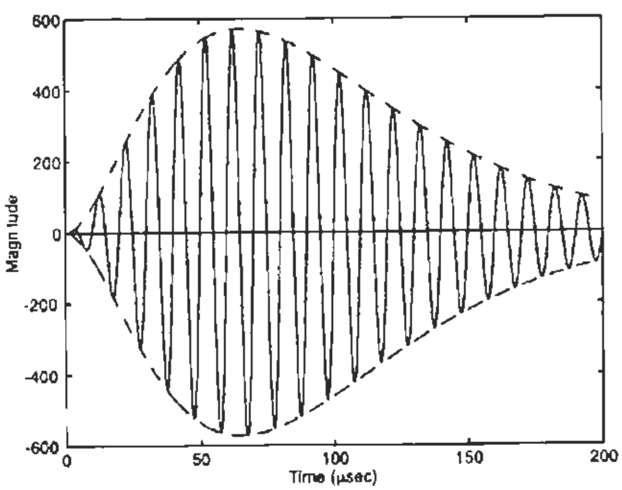
Choice of transmitted Pulse:

- Because Loran-C is a marine system, want signal propagation to follow curvature of earth
- 100 kHz band works well, is outside reserved bands.
- Want bandwidth as narrow as possible, to avoid disruption of other stations
- Want short pulse to avoid ambiguity.

Pulse shape is

$$g(t) = \underbrace{t^2 e^{-t/\tau}}_{u(t)} \sin 2\pi f t$$

$f = 100$ kHz $\qquad \tau = 32.5\,\mu sec$



Find duration, bandwidth of <u>envelope</u> $u(t)$.

$$\bar{t} = \frac{\int_0^\infty t\, u^2(t)\, dt}{\int_0^\infty u^2(t)\, dt} = \frac{5}{2}\tau = 81.25\ \mu\text{sec}$$

$$\Delta t = 2\left(\frac{\int_0^\infty (t-\bar{t})^2 g^2(t)\, dt}{\int_0^\infty g^2(t)\, dt}\right)^{1/2} = 72.67\ \mu\text{sec}$$

$$= \sqrt{5}\,\tau =$$

$$\Delta\omega = 2\left(\frac{\int_0^\infty \dot{u}^2(t)\, dt}{\int_0^\infty u^2(t)\, dt}\right)^{1/2}$$

$$= \frac{2}{\sqrt{3}\,\tau} = 35{,}520\ \text{rad/se}$$

$$\Delta f = \Delta\omega / 2\pi = 5.65\ \text{kHz}$$

In fact, most of the power in the signal (99%) lies between 90 kHz and 110 kHz.

$\Delta t\, \Delta\omega = 2.58 \sim 2$ Not bad!