Block 2   Stress and Strain

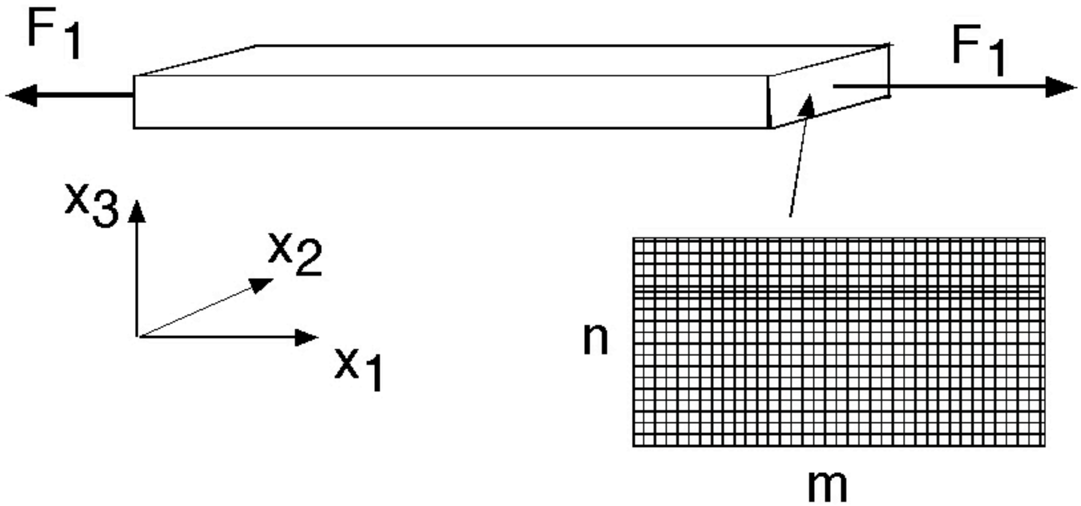
M10 Stress

We want to look more closely at how structures transmit loads. In general, structures are not made up of discrete, 1-D bars, but continuous, complex, 3-D shapes. If we want to understand how they transmit load we needs more tools. To this end we will introduce the concept of stress (and later strain). We will also apply three great principles in terms of stress and strain.

Definition   Stress is the measure of the intensity (per area) of force acting at a point.

Use new coordinate system $x, y, z \rightarrow x_1, x_2, x_3$

Example:



Divide x_1 face into m x n squares

On a 2nd square

$$\Delta F_1 = F_1 / nm$$

$$\Delta A = A / nm$$

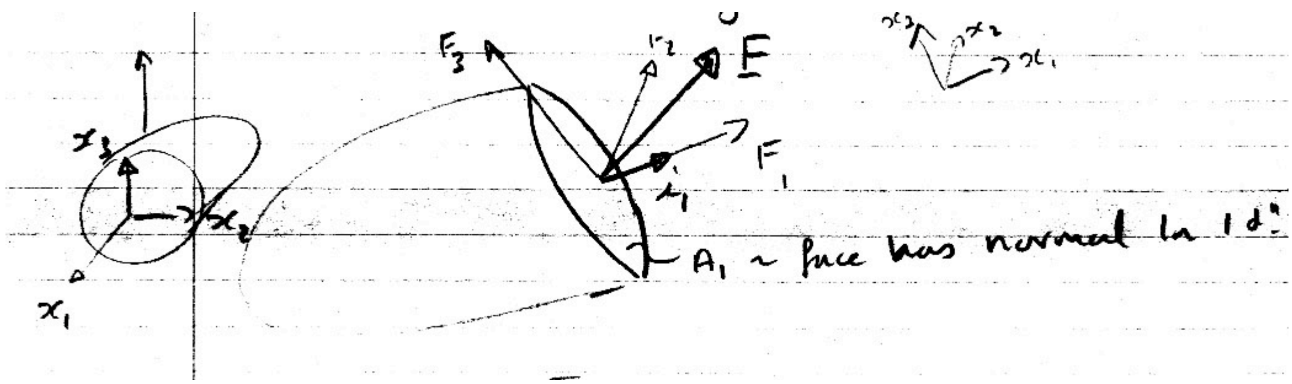
The stress is the intensity, force per unit aresa, so let *m* and *n* go to infinity.

$$\sigma_1 = \lim_{\substack{n \to \infty \\ m \to \infty}} \frac{\Delta F_1}{\Delta A} \qquad units \left[\frac{force}{length^2}\right]$$

where σ_1 is the stress at a point on the x_1 - face

- Magnitude σ_1
- Direction $\underline{i}_1$

Consider a more general case:



Force $\underline{F}$ acting on x_1 face (i_1 is normal to plane of face)

Resolve force into 3 components

$$\underline{F} = F_1\underline{i}_1 + F_2\underline{i}_2 + F_3\underline{i}_3$$

Then take the limit as the force on the face is carried by a smaller set of areas (bigger grid).

$$\lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = 0 \qquad \text{(but stress vector has other components)}$$

$$\underline{\tau} = \sigma_{11}\underline{i}_1 + \sigma_{12}\underline{i}_2 + \sigma_{13}\underline{i}_3$$

$$\boxed{\frac{\Delta F_1}{\Delta A_1}} \quad \boxed{\frac{\Delta F_2}{\Delta A_1}} \quad \boxed{\frac{\Delta F_3}{\Delta A_1}}$$

Where $\underline{\tau}_1$ is the stress vector and σ_{1n} are the components acting on the x_1 normal face, in the x_1, x_2 and x_3 directions.

Get similar results if we looked at the x_2 and x_3 faces.

$$\underline{\tau}_2 = \sigma_{21}\underline{i}_2 + \sigma_{22}\underline{i}_2 + \sigma_{23}\underline{i}_3 \quad \text{on } i_2 \text{ face}$$

$$\underline{\tau}_3 = \sigma_{31}\underline{i}_1 + \sigma_{32}\underline{i}_2 + \sigma_{32}\underline{i}_3 \quad \text{on } i_3 \text{ face}$$

can write more succinctly as

$$\underline{\tau}_m = \sigma_{mn}\underline{i}_n$$

σ_{mn} is the stress tensor, $\underline{\tau}_m$ is the stress vector.

This method of representing a set of vector equations is an example of indicial, or tensor, notation:

<u>Tensor (Indicial) Notation</u>

"easy" to write complicated formulae

"easy" to mathematically manipulate

"elegant" rigorous

Example:

$$x_i = F_{ij}Y_j$$

Subscripts or indices

<u>By Convention</u>

<u>Latin</u> subscripts $m, n, i, j, p, q, etc.$

Take values 1, 2, 3, (3-D)

Greek Subscripts α, β, γ

Take values 1 or 2 (2-D)

<u>3 Rules for Tensor Manipulation</u>

1. A subscript occurring twice is a repeated (or dummy) index and is summed over 1, 2, (and) 3.

- Implicit summation

For $x_i = F_{ij}Y_j = \sum_{j=1}^{3} F_{ij}Y_j$

i.e: $x_i = F_{i1}Y_1 + F_{i2}Y_2 + F_{i3}Y_3$

2. A subscript occurring once in a term is called a <u>free index</u>, can take on the range 1, 2, (3) but is not summed. It represents separate equations.

$$x_1 = F_{11}Y_1 + F_{12}Y_2 + F_{13}Y_3 \qquad \text{(cf matrix multiplication)}$$
$$x_2 = F_{21}Y_1 + F_{22}Y_2 + F_{23}Y_3$$
$$x_3 = F_{31}Y_1 + F_{32}Y_2 + F_{33}Y_3$$

3. No index can appear in a term more than twice.

<u>Two Useful Tensor Parameters</u>

1. <u>Kronecker delta</u>

$$\delta_{mn} = \begin{cases} 1 & when\ m=n \\ 0 & when\ m \neq n \end{cases}$$

Why is this useful?

Consider scalar product of unit vectors

$i_m \,.\, i_m = 1$ parallel

$i_m \,.\, i_n = 0$ perpendicular

$\delta_{mn} = \underline{i}_m \cdot \underline{i}_n$

So Scalar product of two vectors becomes:

$$\underline{F}\,.\,\underline{G} = \left(F_m \underline{i}_m\right).\,G_n \underline{i}_n$$
$$= F_m G_n \left(\underline{i}_m\,.\,\underline{i}_n\right)$$
$$= F_m G_n \delta_{mn}$$

2. Permutation tensor e_{mnp}. ("permute" - to change the order of)

$$e_{mnp} = \begin{cases} 0 & \text{When any two indices are equal} \\ 1 & \text{When } mnp \text{ is } \underline{\text{even}} \text{ permutation of 1, 2, 3} \\ -1 & \text{When } mnp \text{ is an } \underline{\text{odd}} \text{ permutation of 1, 2, 3} \end{cases}$$

Swapped 2 pairs
1, 2, 3
3, 1, 2
2, 3, 1

Swapped 1 pair
1, 3, 2
2, 1, 3
3, 2, 1

(So where does this come from?)

Consider cross product of unit vectors.

$$\underline{i}_1 \times \underline{i}_1 = 0 \qquad \underline{i}_1 \times \underline{i}_1 = \underline{i}_3 \qquad \underline{i}_2 \times \underline{i}_1 = -\underline{i}_3$$
$$\underline{i}_2 \times \underline{i}_2 = 0 \qquad \underline{i}_2 \times \underline{i}_3 = \underline{i}_1 \qquad \underline{i}_3 \times \underline{i}_2 = -\underline{i}_1$$
$$\underline{i}_3 \times \underline{i}_3 = 0 \qquad \underline{i}_3 \times \underline{i}_1 = \underline{i}_2 \qquad \underline{i}_1 \times \underline{i}_3 = -\underline{i}_2$$

So $\underline{i}_1 \times \underline{i}_2 = e_{121}\underline{i}_1 + e_{122}\underline{i}_2 + i_{123}\underline{i}_3 = \underline{i}_3$

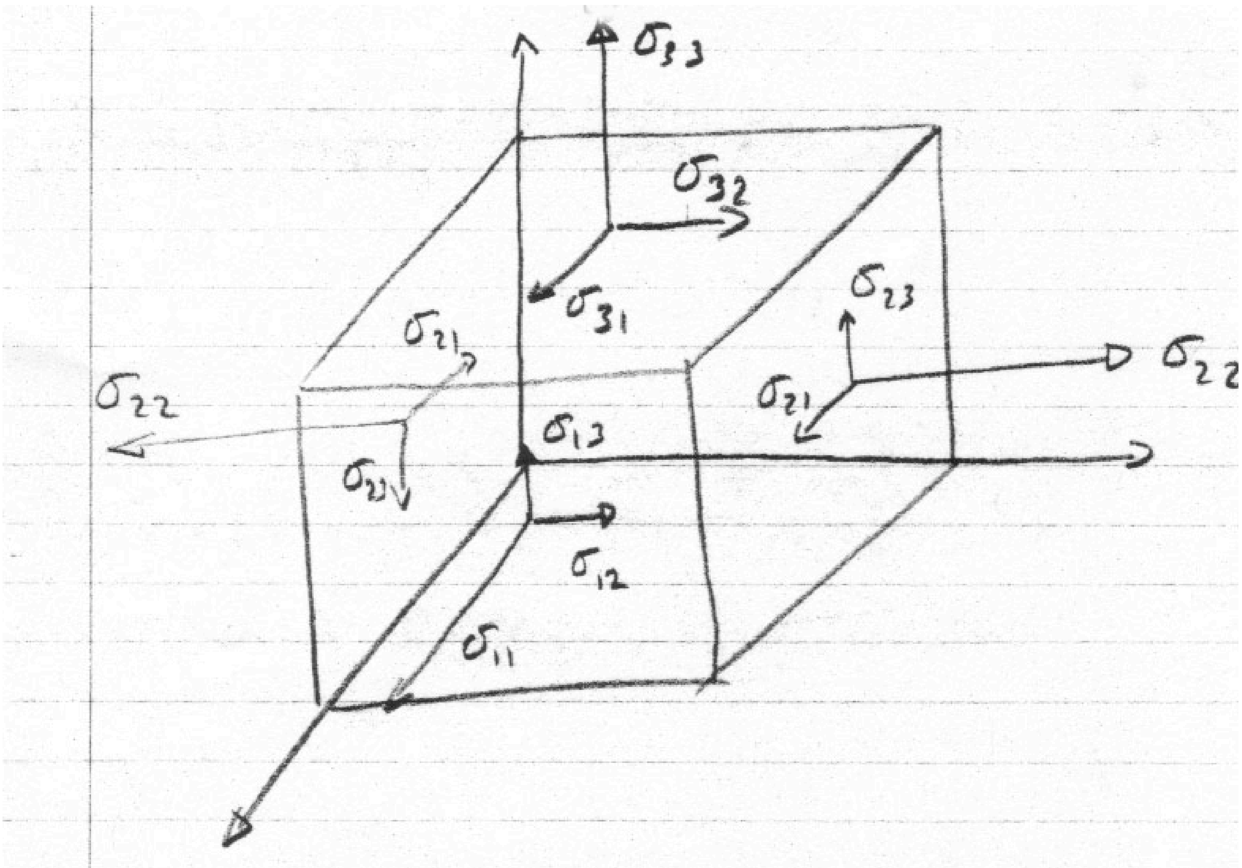
In general,

Now back to stress. We have learnt that stress is a tensor quantity, i.e. it can be represented by tensor (indicial) notation

Stress Tensor and Stress Types

σ_{mn} is the stress tensor

Consider a differential element:



σ_{mn}

Stress acts on face with normal in the x_m-direction

Acts in x_n direction

Types of Stress

Can identify two distinct types of stress:

Normal (Extensional) - acts normal to face

$\sigma_{11}, \sigma_{22}, \sigma_{33}$ - acts to extend element

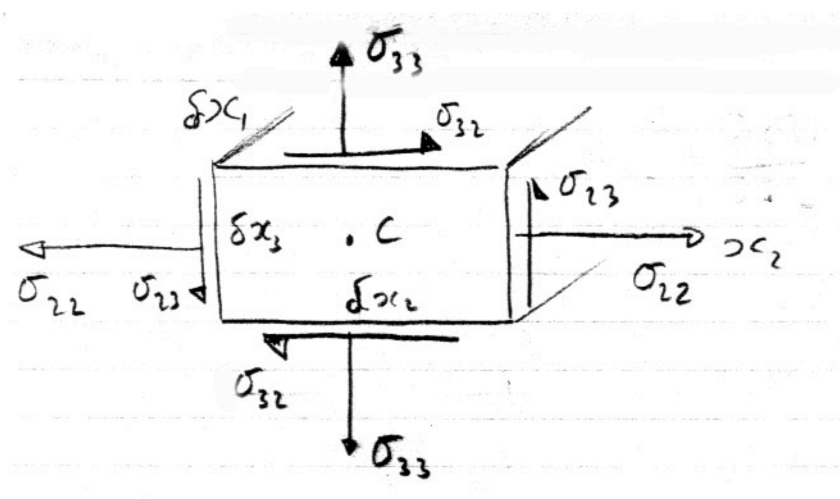
Shear Stress - acts parallel to face plane

$\sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{31}, \sigma_{32}$ acts to shear element

i.e. 9 components of stress, but it turns out that not all of these are independent:

Symmetry of Stress Tensor

Consider moment equilibrium of differential element:



Take moments about *C*

$$2\sigma_{23}\delta x_3\delta x_1\left(\frac{1}{2}\delta x_2\right) - 2\sigma_{32}\delta x_2\delta x_1\left(\frac{1}{2}\delta x_3\right) = 0$$

simplifies to: $\sigma_{23} = \sigma_{32}$

Therefore only 6 independent components of the stress tensor: three extensional stresses: σ_{11}, σ_{22}, σ_{33} and three shear stresses: σ_{12}, σ_{23}, σ_{31}