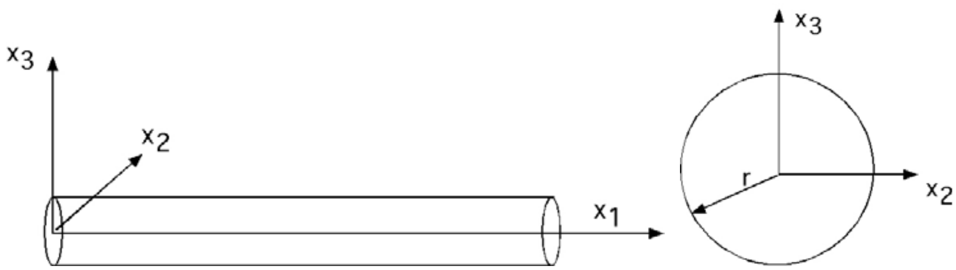
M9 Shafts: Torsion of Circular Shafts

Reading: Crandall, Dahl and Lardner 6.2, 6.3

A shaft is a structural member which is long and slender and subject to a torque (moment) acting about its long axis. We will only consider circular cross-section shafts in Unified. These have direct relevance to circular cross-section shafts such as drive shafts for gas turbine engines, propeller driven aircraft and helicopters (rotorcraft). However, the basic principles are more general and will provide you with a basis for understanding how structures with arbitrary cross-sections carry torsional moments. Torsional stiffness, and the shear stresses that arise from torsional loading are important for the design of aerodynamic surfaces such as wings, helicopter rotor blades and turbine fan blades.

Modelling assumptions

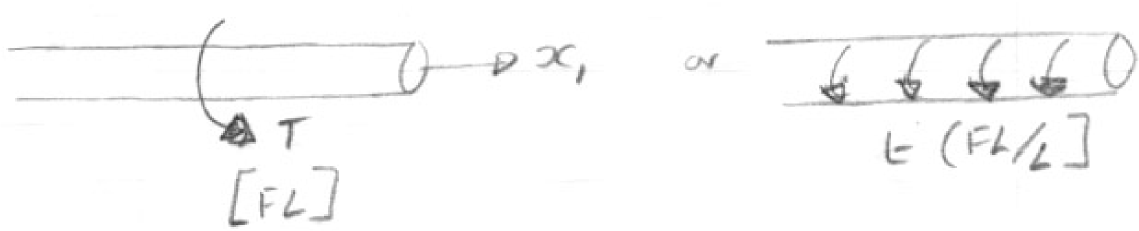
(a) Geometry (as for beam). Long slender, L >> r (b,h)



Note: For the time being we will work in tensor notation since this is all about shear stresses and tensor notation will make the analysis more straightforward. Remember we can choose the system of notation, coordinates to make life easy for ourselves!

(b) Loading

Torque about x_1 axis, T (units of Force x length). We may also want to consider the possibility of distributed torques (Force x length/unit length) (distributed aerodynamic moment along a wing, torques due to individual stages of a gas turbine)



No axial loads (forces) applied to boundaries (on curved surfaces with radial normal, or on x_1 face)

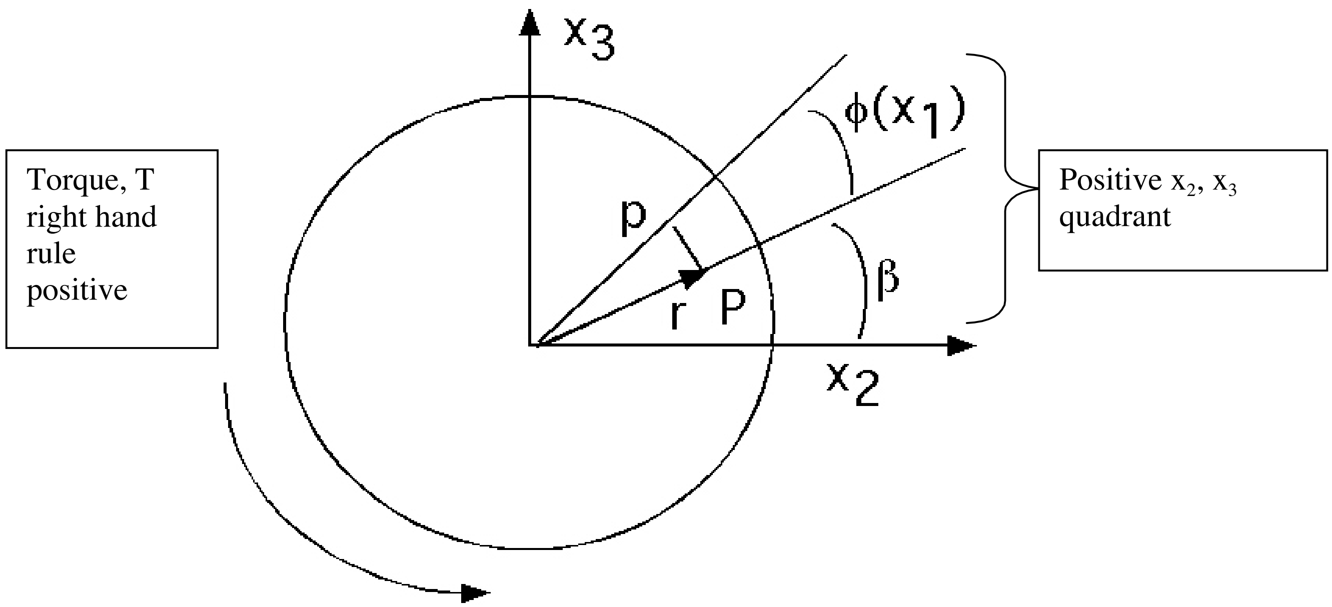
$$_{11} = {}_{22} = {}_{33} = 0$$

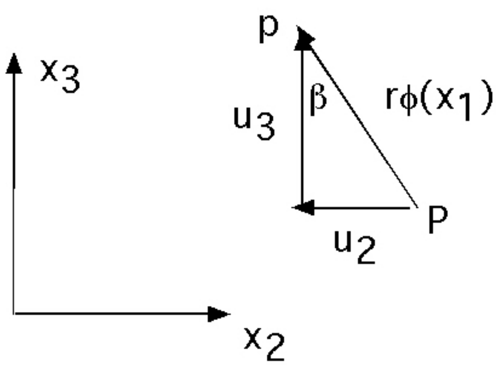
(c) Deformation

-Cross sections rotate as rigid bodies through twist angle , varies with x_1 (cf. beams – plane sections remain plane and perpendicular)

- No bending or extensional deformations in x_1 direction

Cross-section:





$$u_2 = \quad r \quad (x_1)\sin$$
$$u_3 = +r \quad (x_1)\cos$$

$$where\ r = \sqrt{x_2^2 + x_3^2}$$

$$\sin \quad = \frac{x_3}{r} \qquad \cos \quad = \frac{x_2}{r}$$

$$so\ u_2 = \sqrt{x_2^2 + x_3^2} \quad (x_1) \frac{x_3}{\sqrt{x_2^2 + x_3^2}} = \quad (x_1) x_3$$

$$u_3 = \sqrt{x_2^2 + x_3^2} \quad (x_1) \frac{x_2}{\sqrt{x_2^2 + x_3^2}} = \quad (x_1) x_2$$

So

$$u_1 = 0 \qquad (1)$$

$$u_2 = \quad (x_1) x_3 \qquad (2)$$

$$u_3 = \quad (x_1) x_2 \qquad (3)$$

Governing Equations

$$\left. \begin{aligned} {}_{11} &= \frac{du_1}{dx_1} = 0 \\ {}_{22} &= \frac{du_2}{dx_2} = 0 \\ {}_{33} &= \frac{du_3}{dx_3} = 0 \end{aligned} \right\}$$

Consistent with assumption of no axial,
or radial stresses, strains

$${}_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \quad \frac{1}{2} x_3 \frac{d}{dx_1} \qquad (4)$$

$${}_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2} x_2 \frac{d}{dx_1} \qquad (5)$$

$${}_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left(\quad (x_1) + \quad (x_1)\right) = 0$$

Next, apply constitutive laws, i.e. stress-strain relations (assume isotropic), remember we are working in tensor notation: $_{mn} = \frac{_{mn}}{2G}$, therefore need "2G"

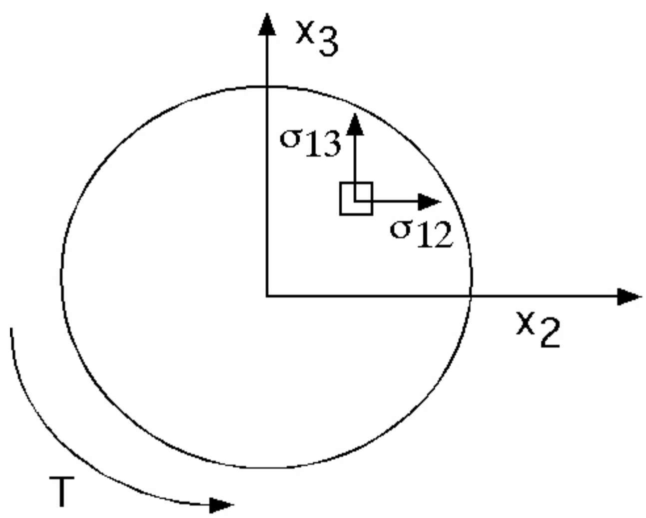
$$_{23} = \frac{_{23}}{2G} = 0$$

$$_{12} = \frac{_{12}}{2G} \qquad (6)$$

$$_{13} = \frac{_{13}}{2G} \qquad (7)$$

Net moment due to shear stresses must equal resultant torque on section:

$$equipollent\ torque, T = \quad \left(x_2 \ _{13} \quad x_3 \ _{12}\right) dx_2 dx_3 \qquad (8)$$



Apply equilibrium: $\frac{\partial \ _{mn}}{\partial x_m} + f_n = 0$

i.e.:

$$\frac{\partial \ _{11}}{\partial x_1} + \frac{\partial \ _{21}}{\partial x_2} + \frac{\partial \ _{31}}{\partial x_3} = 0$$

$$\frac{\partial \ _{12}}{\partial x_1} + \frac{\partial \ _{22}}{\partial x_2} + \frac{\partial \ _{32}}{\partial x_3} = 0$$

$$\frac{\partial \ _{13}}{\partial x_1} + \frac{\partial \ _{23}}{\partial x_2} + \frac{\partial \ _{33}}{\partial x_3} = 0$$

Retaining non-zero terms we obtain

$$\frac{\partial \quad_{21}}{\partial x_2} + \frac{\partial \quad_{31}}{\partial x_3} = 0 \qquad (9)$$

$$\frac{\partial \quad_{12}}{\partial x_1} = 0 \qquad (10)$$

$$\frac{\partial \quad_{13}}{\partial x_1} = 0 \qquad (11)$$

Solution

Go back to stress-displacement relationships (4,5, 6 and 7)

$$_{12} = 2G \;_{12} = 2G \quad \frac{1}{2} x_3 \frac{d}{dx_1}$$

$$_{12} = \quad Gx_3 \frac{d}{dx_1}$$

Similarly:

$$_{13} = 2G_{13} \;_{13} = 2G \; \frac{1}{2} x_2 \frac{d}{dx_1} \;= Gx_2 \frac{d}{dx_1}$$

Substitute into equation for resultant moment (8)

$$T = \quad \left(x_2 \;_{13} \quad x_3 \;_{12}\right) dx_2 dx_3$$

$$= \quad x_2^2 \frac{d}{dx_1} + x_3^2 \frac{d}{dx_1} \; dx_2 dx_3$$

$$T = G \frac{d}{dx_1} \quad \left(x_2^2 + x_3^2\right) dA$$

define J = $\underbrace{\quad x_2^2 + x_3^2 \; dA}_{\boxed{\text{Polar 2}^{\text{nd}}\text{ moment of area}}}$

$$= \frac{R^4}{2} \text{ for circular cross-section}$$

Hence $T = GJ\frac{d}{dx_1}$

Torque-twist relation

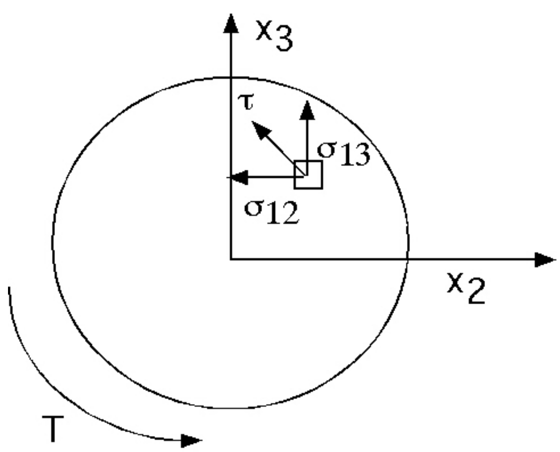
(note, we can compare $T = GJ\frac{d}{dx_1}$ for shafts with $M = EI\frac{\partial^2 w}{\partial x^2}$ for beams)

Hence, relate stress to torque

$$_{12} = \frac{Tx_3}{J}$$

$$_{13} = \frac{Tx_2}{J}$$

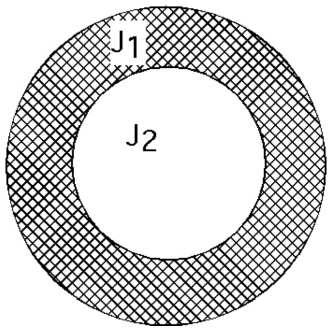
Express total shear as shear stress resultant,



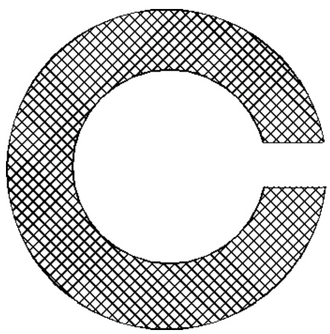
$$_{res} = \sqrt{\ {}^{2}_{12} + \ {}^{2}_{13}} = \frac{T}{J}\sqrt{x_3^2 + x_2^2} = \frac{Tr}{J}$$

compare $\ = \frac{Tr}{J}$ *with* $\ _{xx} = \frac{Mz}{I}$

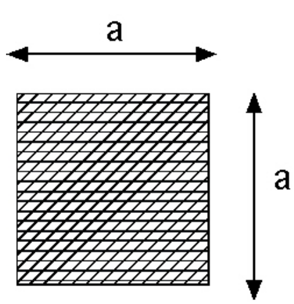
Model works well for circular cross-sections and cylindrical tubes $\quad J = J_1 \quad J_2$



Does not work for open sections:



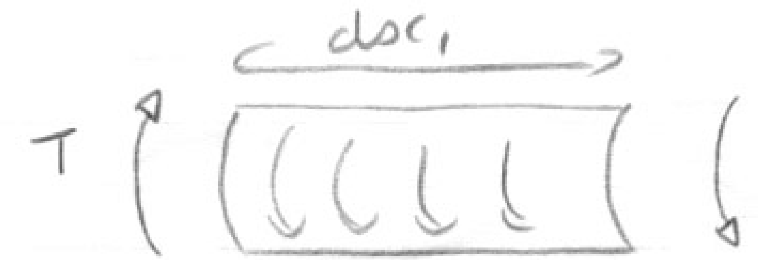
We can approximate for other sections, e.g. square cross-section, $J = 0.141a^4$.



For a full treatment of torsion of slender members see 16.20.

A note on distributed torques:

Distributed torques are similar to distributed loads on beams: Consider equilibrium of a differential element, length dx1, with a distributed torque t (Nm/m) applied. Leads to an increase in the resultant internal torque, T from T to T+dT.

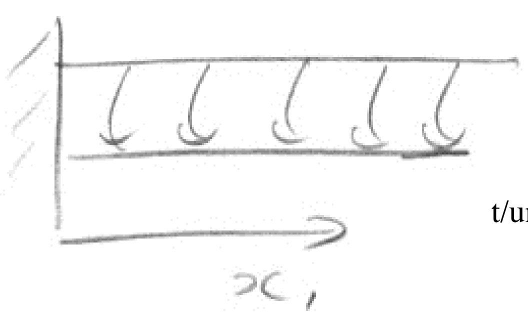


$$T + \frac{dT}{dx_1} dx_1$$

$$M_{x_1} = 0 \qquad\qquad T + t(x_1)dx_1 + T + \frac{dT}{dx_1} dx_1$$

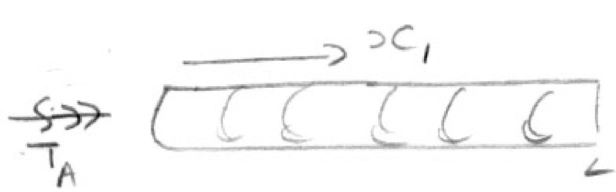
$$\frac{dT}{dx_1} = \quad t(x_1) \qquad \text{c.f.} \frac{dS}{dx} = q(x)$$

Can also consider macroscopic equilibrium of shaft fixed at one end with a uniform distributed torque/length, t, applied to it:



t/unit length

FBD



$$T_A + \int_0^L t dx_1 = 0$$

$$T_A = tL$$

M10 Introduction to Structural Instability

Reading Crandall, Dahl and Lardner: 9.2, 9.3

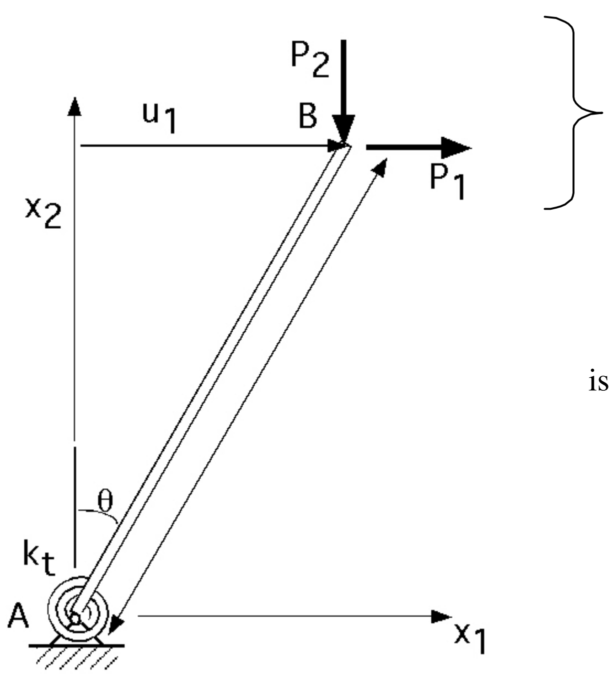
Elastic instabilities, of which buckling is the most important example, are a key limitation on structural integrity. The key feature of an elastic instability is the transition from a stable mode of deformation with increasing applied load to an unstable one, resulting in collapse (loss of load carrying capability) and possibly failure of the structure. Examples of elastic collapse are the buckling of bars in a truss under compressive load, the failure of columns under compressive load, the failure of the webs of "I" beams in shear, the failure of fuselage and wing skin panels in shear and many others. The only particular case we will consider here in Unified is the failure of a bar or column loaded in axial compression, however, as for the other slender members we have considered, the basic ideas will apply to more complex structures.

A structure is in stable equilibrium if, for all possible (small)displacements/deformations, a restoring force arises.

Before considering the case of a continuous structure we will consider a case in which we separate the stiffness of the structure from the geometry of the structure.

Introductory Example:

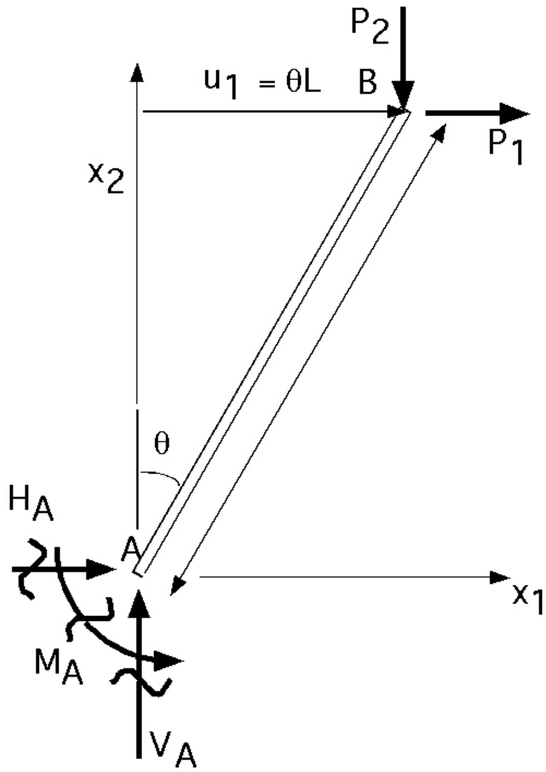
Rigid, massless bar with a torsional spring at one end, stiffness, k_t, which is also pinned. . The bar is loaded by a pair of horizontal and vertical forces at the free end, P_1, P_2. The bar undergoes a small angular displacement, .



Independent applied loads, P_1, P_2

is small – exaggerated in the figure

Draw free body diagram:



Horizontal equilibrium: $H_A = -P_1$

Vertical equilibrium: $V_A = P_2$

Taking moments, counterclockwise positive:

$$M \quad LP_1 \cos \quad\quad P_2 L \sin \quad = 0$$

$$(\cos \quad\quad 1 \quad \sin \quad\quad)$$

$$M \quad LP_1 \quad P_2 L \quad = 0$$

And $\quad k_t \quad\quad LP_1 \quad L \;\; P_2 = 0$

Hence: $\quad P_1 = \underbrace{\frac{K_t \quad P_2 L}{L}}$

Effective Stiffness - (note that
it includes the load, P_2)

i.e.

$$P_1 = k_{eff}$$

Rearranging gives: $= \dfrac{P_1}{\dfrac{k_t}{L} \quad P_2}$

Hence:

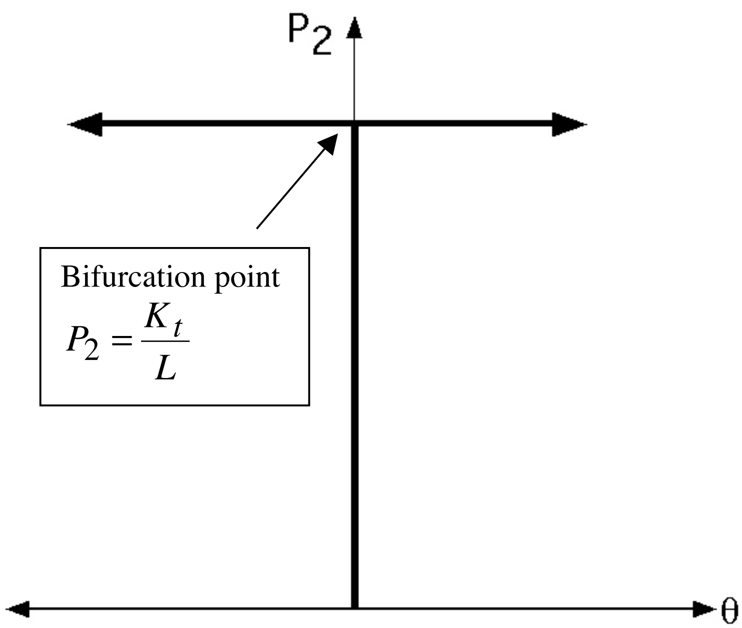
$$\begin{aligned} &for\ P_1 \quad 0 \\ &= 0\ for\ P_2 < k_t / L \\ &= \bullet\ \ for\ P_2 \quad K_t / L \end{aligned}$$

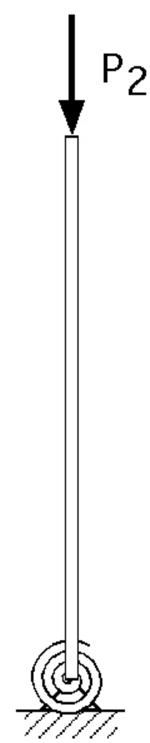
i.e. if $P_2 \geq \dfrac{K_t}{L}$ static instability, i.e., spring cannot provide in restoring moment

NOTE: If P_2 negative - i.e. upward -stiffness increases
But if $P_1$and P_2 removed or reduced, spring will allow bar to spring back to original configuration

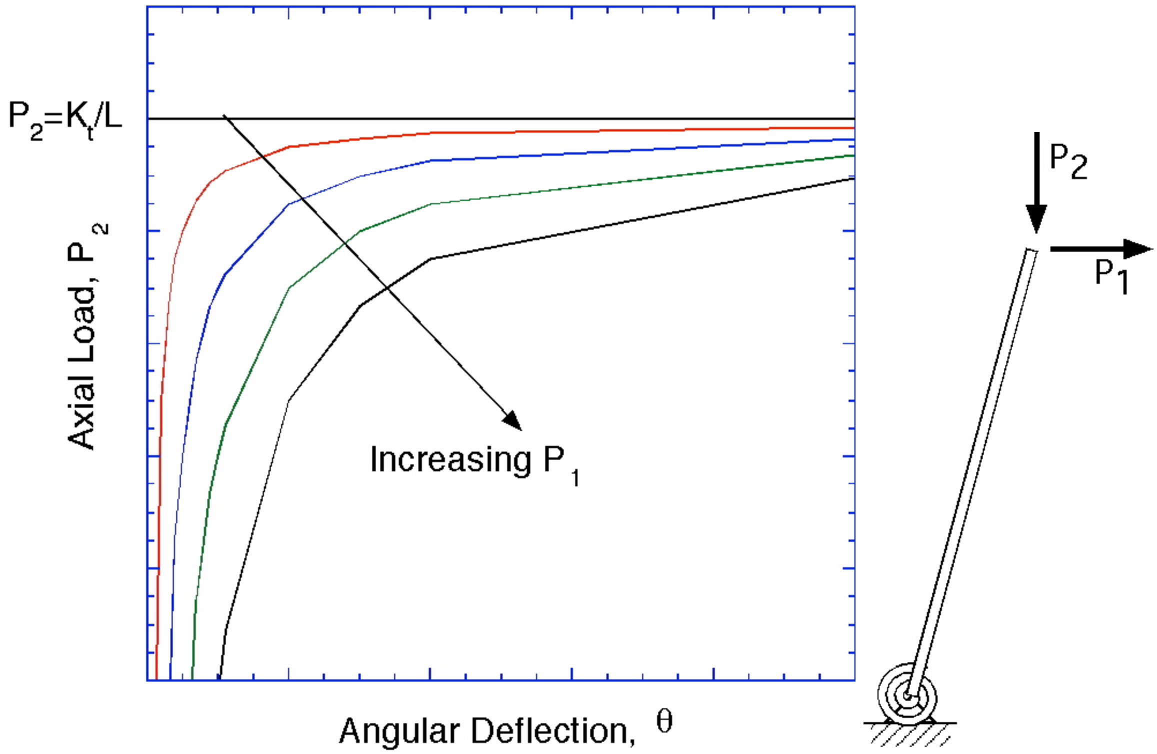
Plot Load vs. Displacement

a) for case when P_1=0. Obtain "bifurcation behavior"





b) case $P_1>0$



Next time we will apply these idea to a column, i.e. a continuous structure with a continuous distribution of stiffness. Need to think about what is the relevant structural stiffness, i.e. the equivalent of (K_t/L) in the spring/rigid rod system above.