Lecture M1 Slender (one dimensional) Structures

Reading: Crandall, Dahl and Lardner 3.1, 7.2

This semester we are going to utilize the principles we learnt last semester (i.e the 3 great principles and their embodiment in the 15 continuum equations of elasticity) in order to be able to analyze simple structural members. These members are: Rods, Beams, Shafts and Columns. The key feature of all these structures is that one dimension is longer than the others (i.e. they are one dimensional).

Understanding how these structural members carry loads and undergo deformations will also take us a step nearer being able to design and analyze structures typically found in aerospace applications. Slender wings behave much like beams, rockets for launch vehicles carry axial compressive loads like columns, gas turbine engines and helicopter rotors have shafts to transmit the torque between the components andspace structures consist of trusses containing rods. You should also be aware that real aerospace structures are more complicated than these simple idealizations, but at the same time, a good understanding of these idealizations is an important starting point for further progress.

There is a basic logical set of steps that we will follow for each in turn.

1) We will make general modeling assumptions for the particular class of structural member

In general these will be on:
a) Geometry
b) Loading/Stress State
c) Deformation/Strain State

2) We will make problem-specific modeling assumptions on the boundary conditions that apply (idealized supports, such as pins, clamps, rollers that we encountered with truss structures last semester)

a.) On stresses
b.) On displacements
} Applied at specified locations in structure

3) We will apply an appropriate solution method:

a) Exact/analytical (Unified, 16.20)
b Approximate (often numerical) (16.21). Such as energy methods (finite elements, finite difference - use computers)

Let us see how this works:

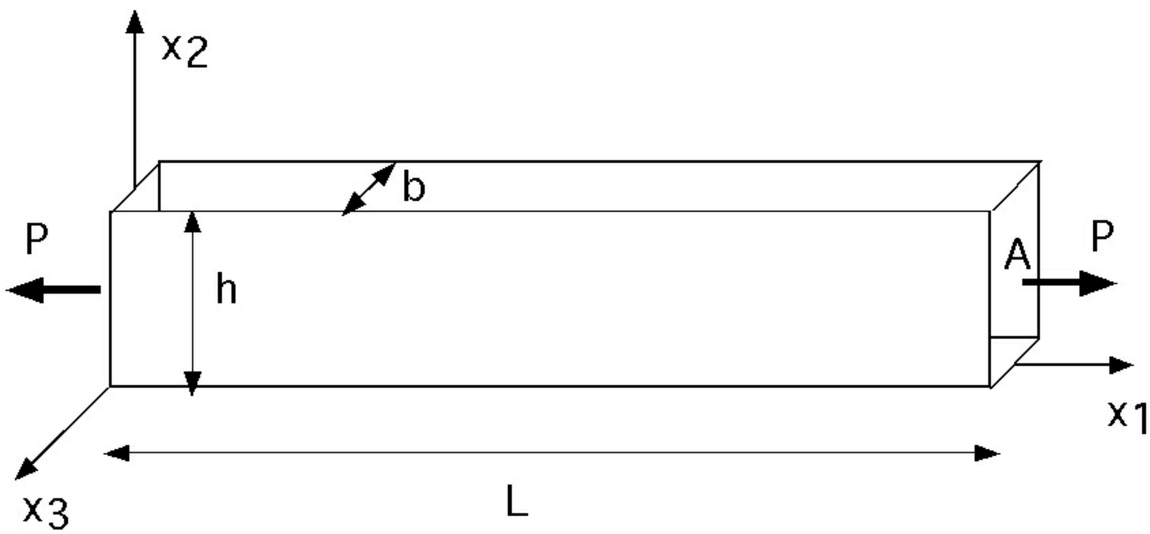
Rods (bars)

The first 1-D structure that we will analyzes is that of a rod (or bar), such as we encountered when we analyzed trusses. We are interested in analyzing for the stresses and deflections in a rod.

First start with a working definition - from which we will derive our modeling assumptions:

"A rod (or bar) is a structural member which is long and slender and is capable of carrying load along its axis via elongation"

Modeling assumptions
a.) Geometry



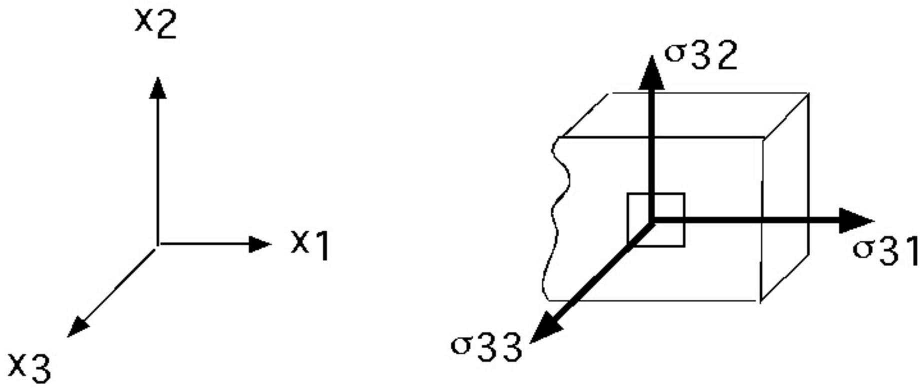
L = length (x$_1$ dimension)
b = width (x$_3$ dimension)
h = thickness (x$_2$ dimension)
Cross-section A (=bh)

assumption: L much greater than b, h (i.e it is a slender structural member)

(think about the implications of this - what does it imply about the magnitudes of stresses and strains?)

b.) Loading - loaded in x_1 direction only

Results in a number of assumptions on the boundary conditions

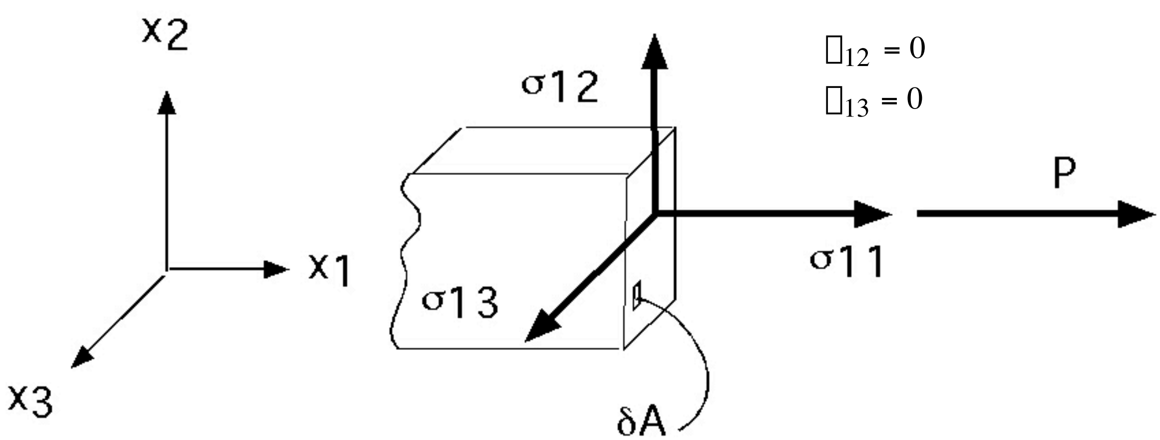


$$\sigma_{31} = 0$$
$$\sigma_{32} = 0$$
$$\sigma_{33} = 0$$

Similarly on the x_2 face - no force is applied

$$\sigma_{21} = \sigma_{12} = 0$$
$$\sigma_{32} = \sigma_{23} = 0$$
$$\sigma_{22} = 0$$

on x_1 face - take section perpendicular to x_1



$$\sigma_{12} = 0$$
$$\sigma_{13} = 0$$

and

$$\int_A \sigma_{11} dA = P$$

$$\iint \sigma_{11} dx_2 dx_3 = P$$

$$\Rightarrow \sigma_{11} = \frac{P}{bh} = \frac{P}{A}$$

c.) deformation

Rod cross-section deforms uniformly (is this assumption justified? - yes, there are no shear stresses, no changes in angle)

So much for modeling assumptions, Now let's apply governing equations and solve.

1. Equilibrium

$$\frac{\partial \sigma_{mn}}{\partial x_n} + f_n = 0$$

only σ_{11} is non-zero

$$\frac{\partial \sigma_{11}}{\partial x_1} + f_1 = 0$$

f_1 = body force =0 for this case

$$\frac{\partial \sigma_{11}}{\partial x_1} = 0 \Rightarrow \sigma_{11} = \text{constant} = \frac{\text{P}}{\text{A}}$$

Constitutive Laws

stress - strain equations:

$$\left.\begin{aligned} \varepsilon_{11} &= S_{1111}\sigma_{11} \\ \varepsilon_{22} &= S_{2211}\sigma_{11} \\ \varepsilon_{33} &= S_{3311}\sigma_{11} \end{aligned}\right\}$$

So long as not fully anisotropic - this is all that is required

Compliance Form

$$S_{1111} = \frac{1}{E_x}$$

$$S_{2211} = -\frac{\nu_{xy}}{E_x}$$

$$S_{3311} = -\frac{\nu_{xz}}{E_x}$$

For isotropic material gives:

$$\varepsilon_{11} = \frac{1}{E}\sigma_{11}$$

$$\varepsilon_{22} = \frac{-\nu}{E}\sigma_{11}$$

$$\varepsilon_{33} = \frac{-\nu}{E}\sigma_{11}$$

Now apply strain – displacement relations:

$$\varepsilon_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right)$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

Hence (for isotropic material):

$$\frac{\sigma_{11}}{E} = \frac{\partial u_1}{\partial x_1}$$

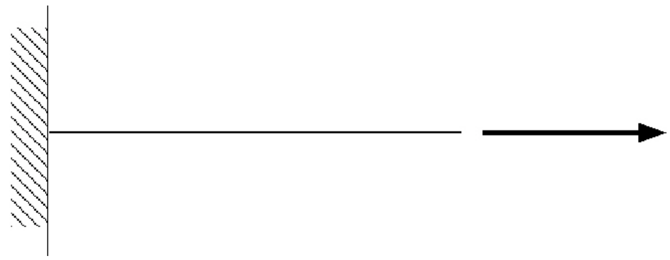
$$\Rightarrow \quad \frac{P}{AE} = \frac{\partial u_1}{\partial x_1}$$

integration gives:

$$u_1 = \frac{Px_1}{AE_1} + g(x_2, x_3)$$

Apply B. C. $u_1 = 0$ @ $x_1 = 0$ $\quad \Rightarrow \quad$ $g(x_2, x_3) = 0$

i.e. uniaxial extension only, fixed at root



$$\Rightarrow \quad u_1 = \frac{Px_1}{AE}.$$

similarly $u_2 = \dfrac{-\nu P}{AE} x_2$

$$u_3 = \frac{-\nu P}{AE} x_3$$

check: $\varepsilon_{12} = \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) = 0$

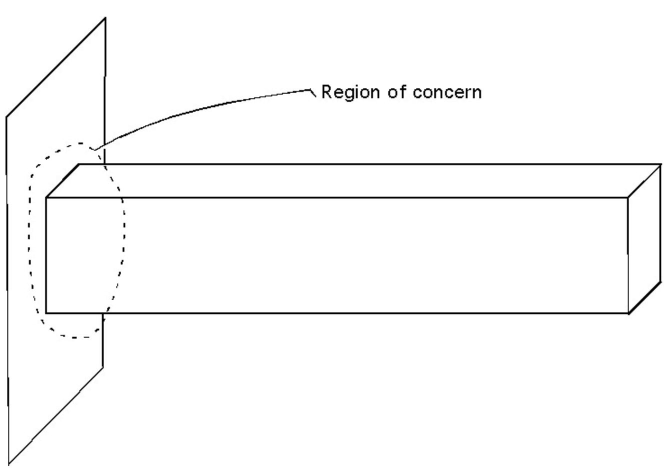
Assessment of assumptions

(Closer inspection reveals that our solutions are not exact.)

1) Cross section changes shape slightly. A is not a constant.

   If we solved the equations of elasticity simultaneously, we would account for this. Solving them sequentially is ok so long as deformations are small. (ΔA is second order.)

2) At attachment point boundary conditions are different from those elsewhere on the rod.



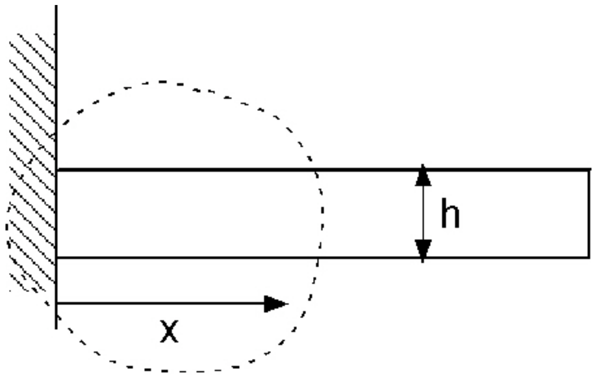
$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0$
(Remember recitation example last term – materials axially loaded in a rigid container. Also problem set question about thin adhesive joint.)

We deal with this by invoking St. Venant's principle:

"Remote from the boundary conditions internal stresses and deformations will be insensitive to the exact form of the boundary condition."

And the boundary condition can be replaced by a <u>statically equivalent</u> condition (equipollent) without loss of accuracy.

How far is remote?



This is the importance of the "long slender" wording of the rod definition.

This should have been all fairly obvious. Next time we will start an equivalent process for beams - which will require a little more thought.

M2 Statics of Beams

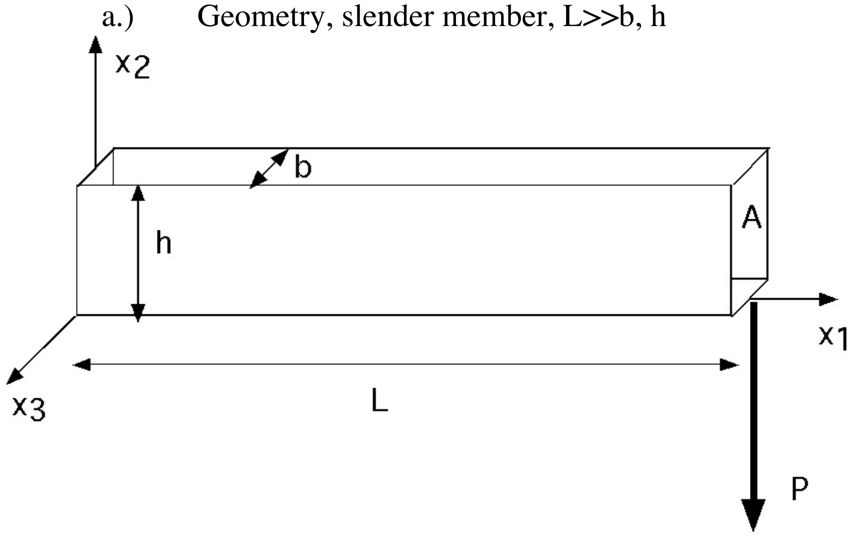
Reading: Crandall, Dahl and Lardner, 3.2-3.5, 3.6, 3.8

A beam is a structural member which is long and slender and is capable of carrying bending loads. I.e loads applied transverse to its long axis.

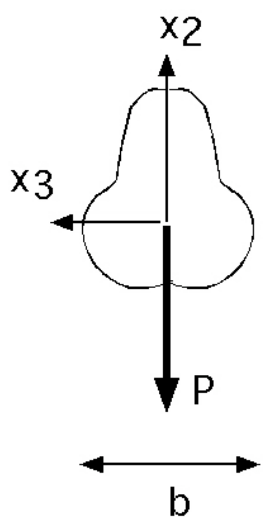
Obvious examples of aerospace interest are wings and other aerodynamic surfaces. Lift and weight act in a transverse direction to a long slender axis of the wing (think of glider wings as our prototype beam). Note, even a glider wing is not a pure beam – it will have to carry torsional loads (aerodynamic moments).

1.) Modeling assumptions

a.) Geometry, slender member, L>>b, h



At this stage, will assume arbitrary, symmetric cross-sections, i.e.:



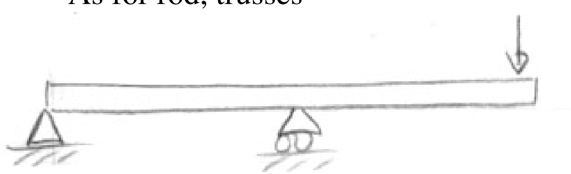
b.) Loading

- Similar to rod (traction free surfaces) but applied loads can be in the z direction

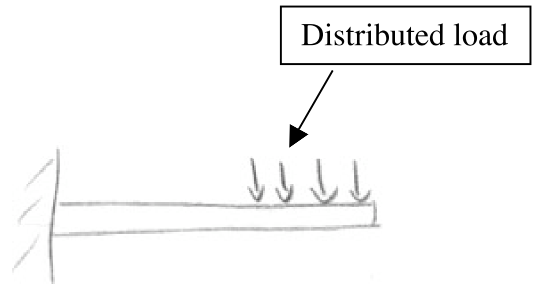
c.) Deformation

- We will talk about this later

2.) Boundary conditions
As for rod, trusses



Pinned, simply supported



Cantilever

Draw FBD, apply equilibrium to determine reactions.

3.) Governing equations

- Equations of elasticity

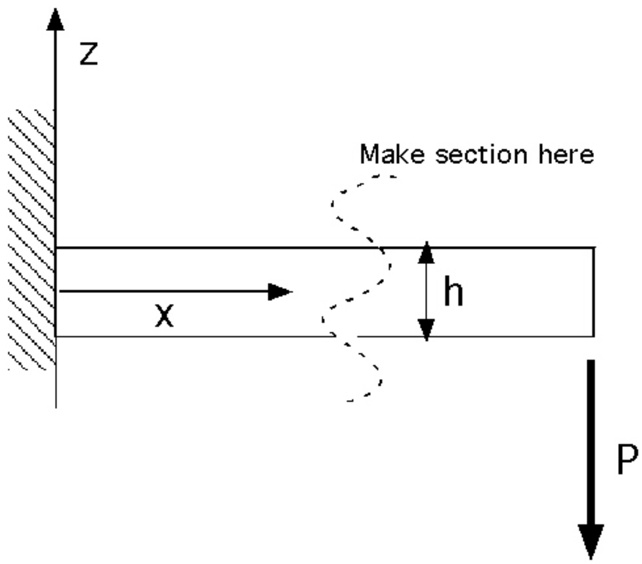
4.) Solution Method

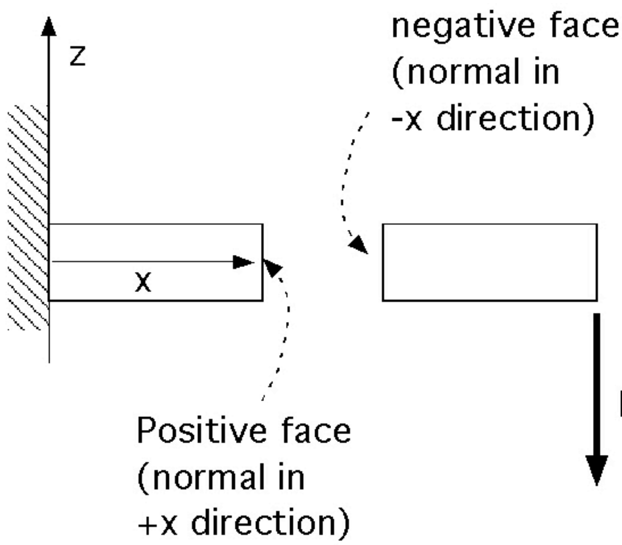
- Exact (exactly solve governing differential equations)
- Approximate (use numerical solution)

But first need to look at how beams transmit load.

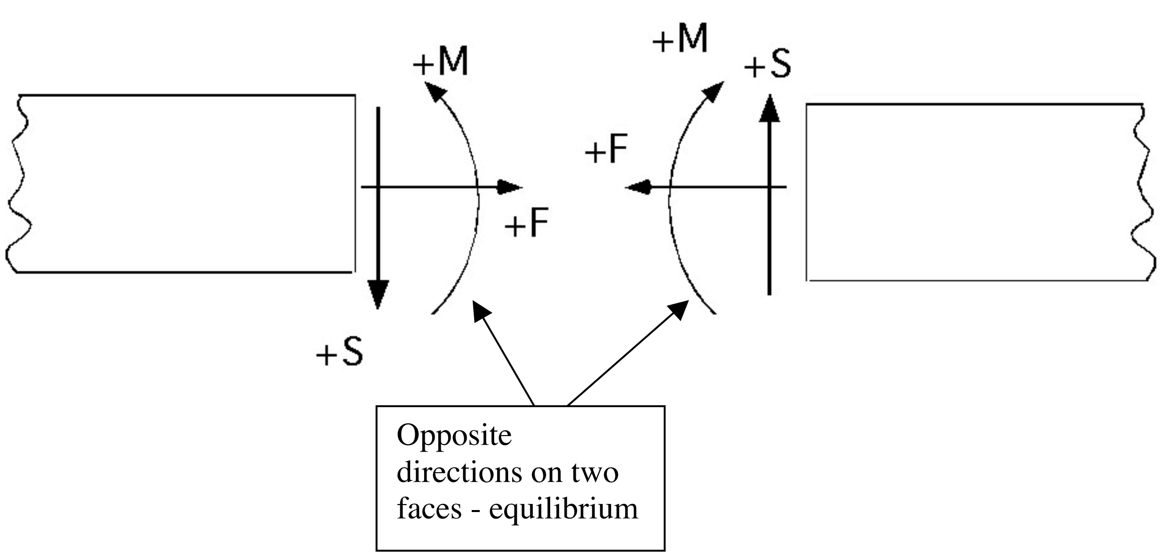
<u>Internal Forces</u>

Apply methods of sections to beam (also change coordinate system – to x, y, z – consistent with CDL). Method exactly as for trusses. Cut structure at location where we wish to find internal forces, apply equilibrium, obtain forces. In the case of a beam, the structure is continuous, rather than consisting of discrete bars, so we will find that the internal forces (and moments) are, in general, a continuous function of position.





Internal forces



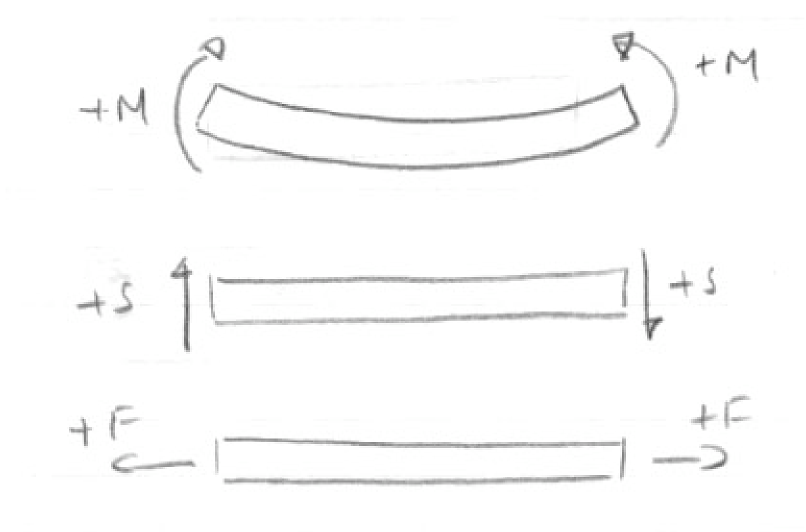
(Note Crandall Dahl and Lardner use V = -S)

where M = Bending movement
S = Shear force
F = Axial force

} Beam
} Bending
} bar, rod

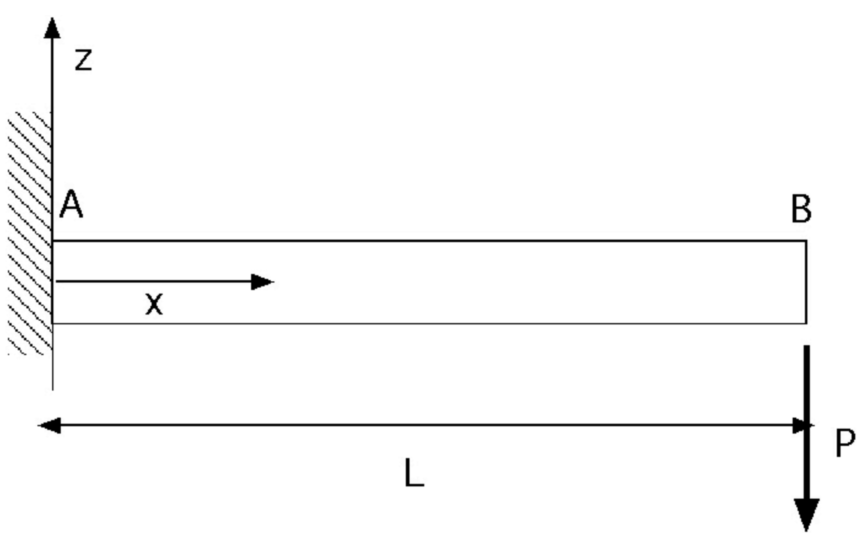
} "beam bar"

Also drawn as

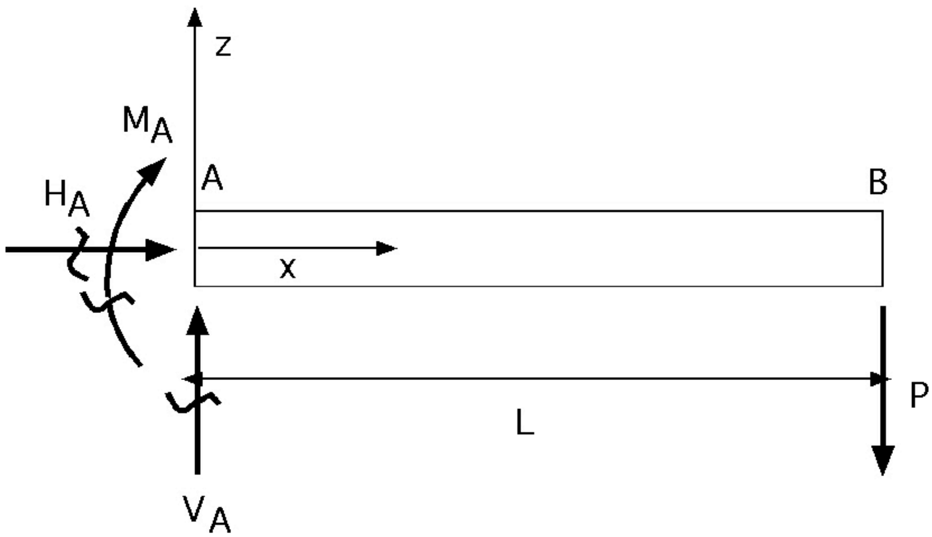


Example of calculating shear force and bending moment distribution along a beam.

Example 1. Cantilever beam.

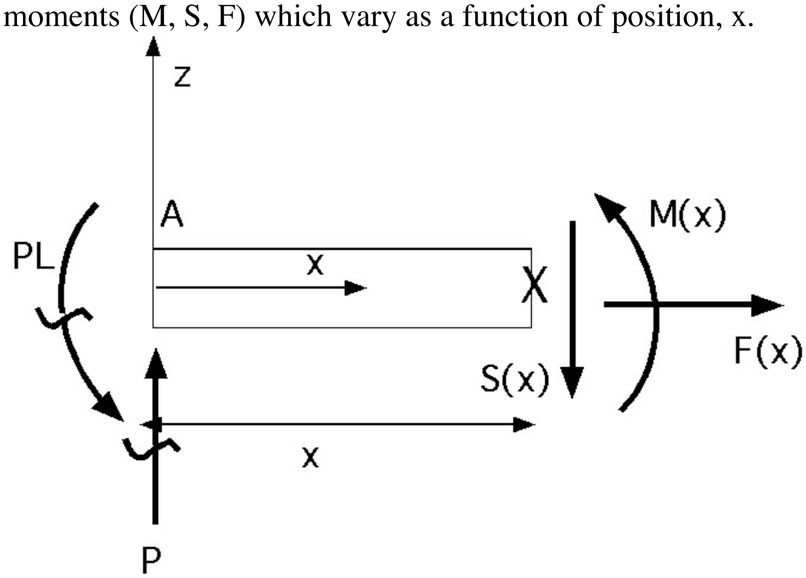


Free Body Diagram (note moment reaction at root)



Equilibrium: $H_A = 0, \quad V_A = P, \quad M_A = -PL$

Take cut at point X ,distance x from left hand end (root). 0< x < L. Replace the effect of the (discarded) right hand side of beam by an equivalent set of forces and moments (M, S, F) which vary as a function of position, x.



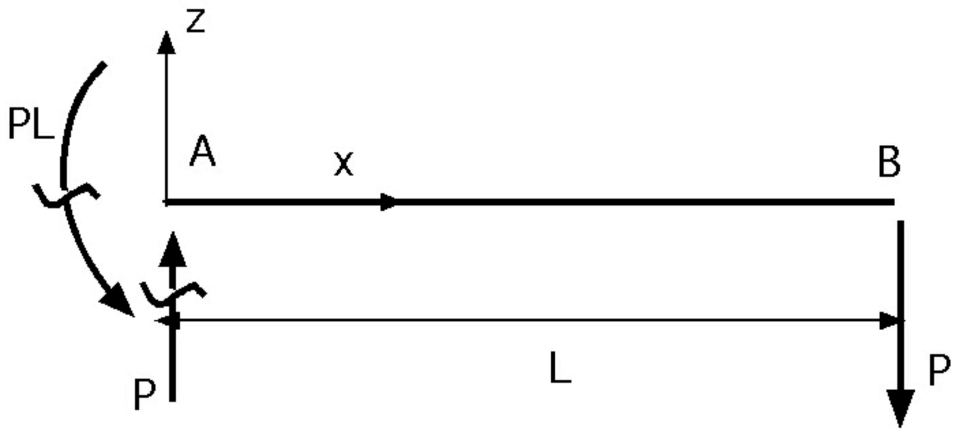
Apply equilibrium

$\sum F_x = 0 \; + \quad F(x) = 0$

$\sum F_z = 0 \uparrow + \quad P - S(x) = 0 \quad S(x) = P$

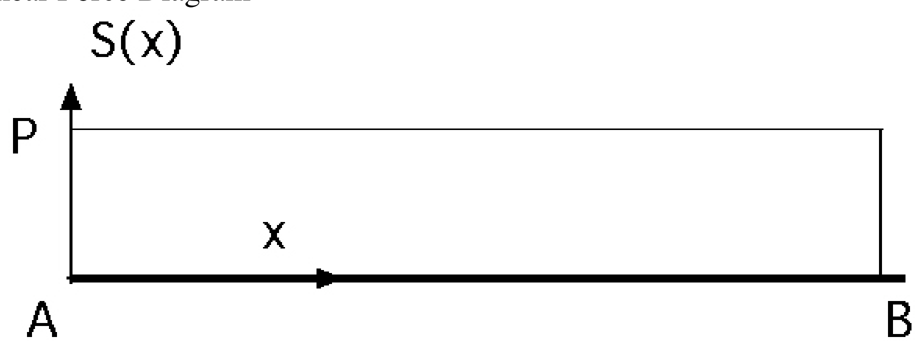
$\sum M_X = 0 \; \circlearrowright + \; P\ell + M(x) - Px = 0 \quad M(x) = -P(\ell - x)$

Draw "sketches" - bending moment, shear force, loading diagrams

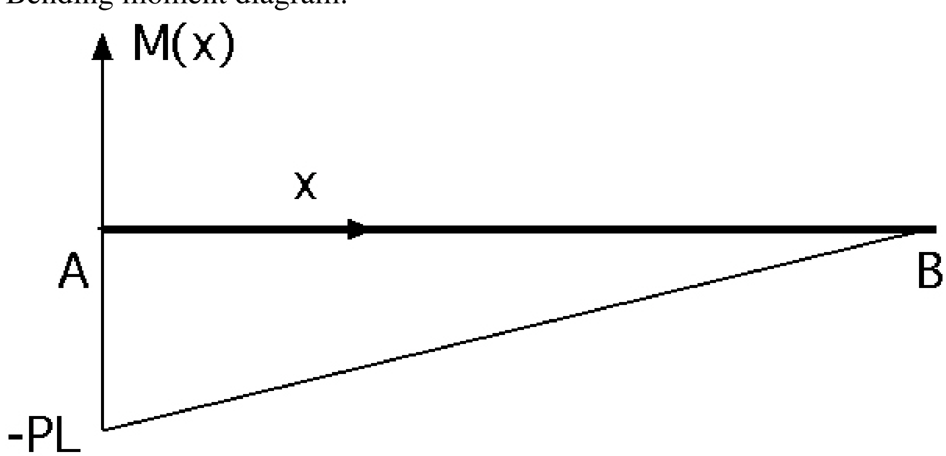


Axial Force diagram (zero everywhere in this case)

Shear Force Diagram



Bending moment diagram:

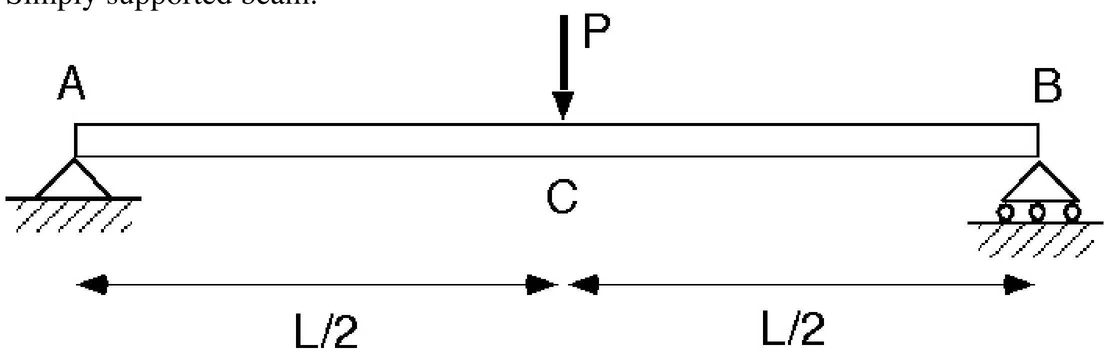


NOTE: At boundaries values go to reactions (moment at root, applied load at tip).

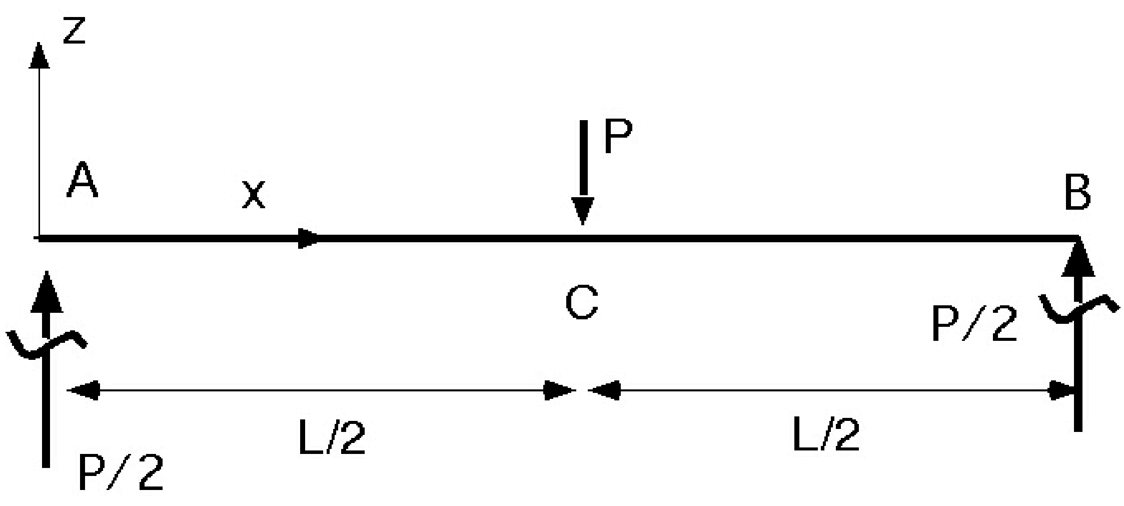
These representations are useful because they provide us with a visual indication of where the internal forces on the beam are highest, which will play a role in determining where failure might occur and how we should design the internal structure of the beam (put more material where the forces are higher).

Lecture M3 Shear Forces and Bending Moments in Beams continued: Example 2

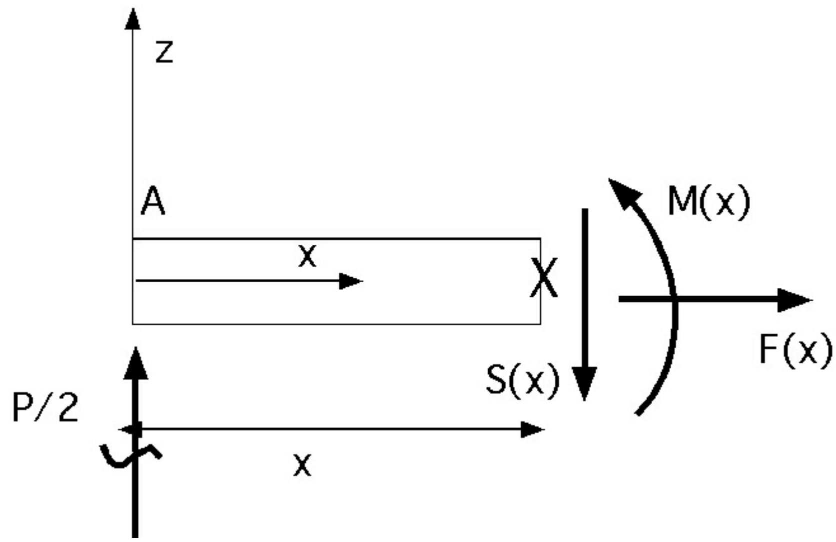
Simply supported beam:



Free body diagram:



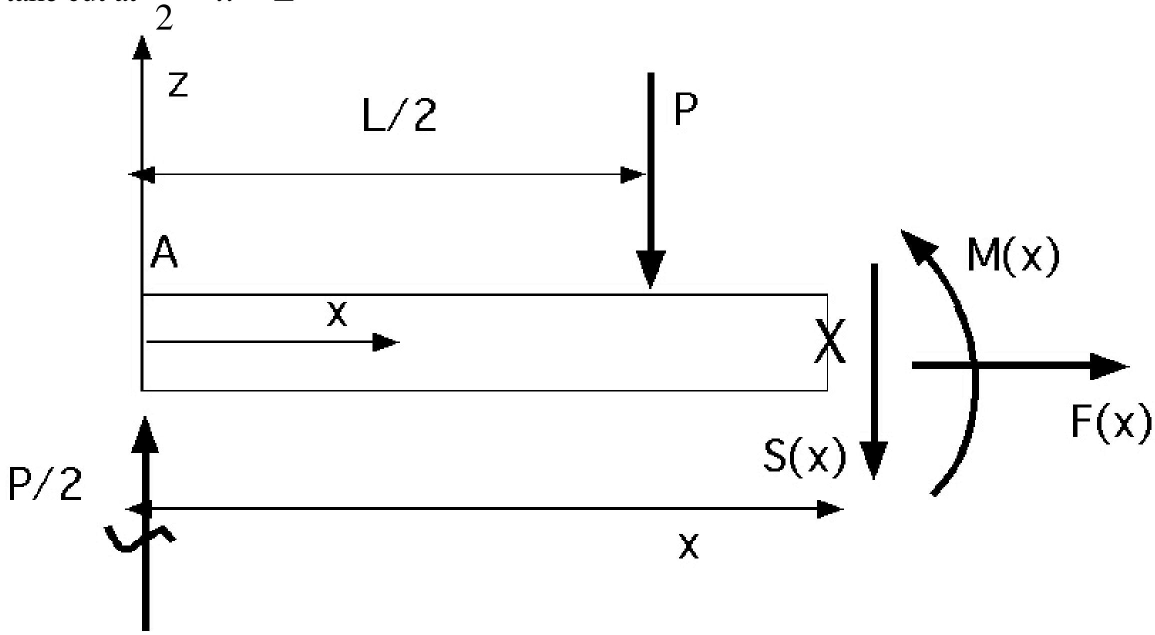
take cut at 0< x< L/2



Equilibrium $\square$ F(x) = 0

$$\left.\begin{array}{l} \uparrow \frac{P}{2} \square S(x) = 0 \quad \square \quad S(x) = \frac{P}{2} \\ \circlearrowleft \quad M_x = 0 \quad \circlearrowright + \quad \square \quad \square \frac{P}{2} x + M(x) = 0 \; \square \quad M(x) = \frac{Px}{2} \end{array}\right\} \quad 0 < x < L$$

take cut at $\frac{L}{2} < x < L$



Apply equilibrium (moments about X):

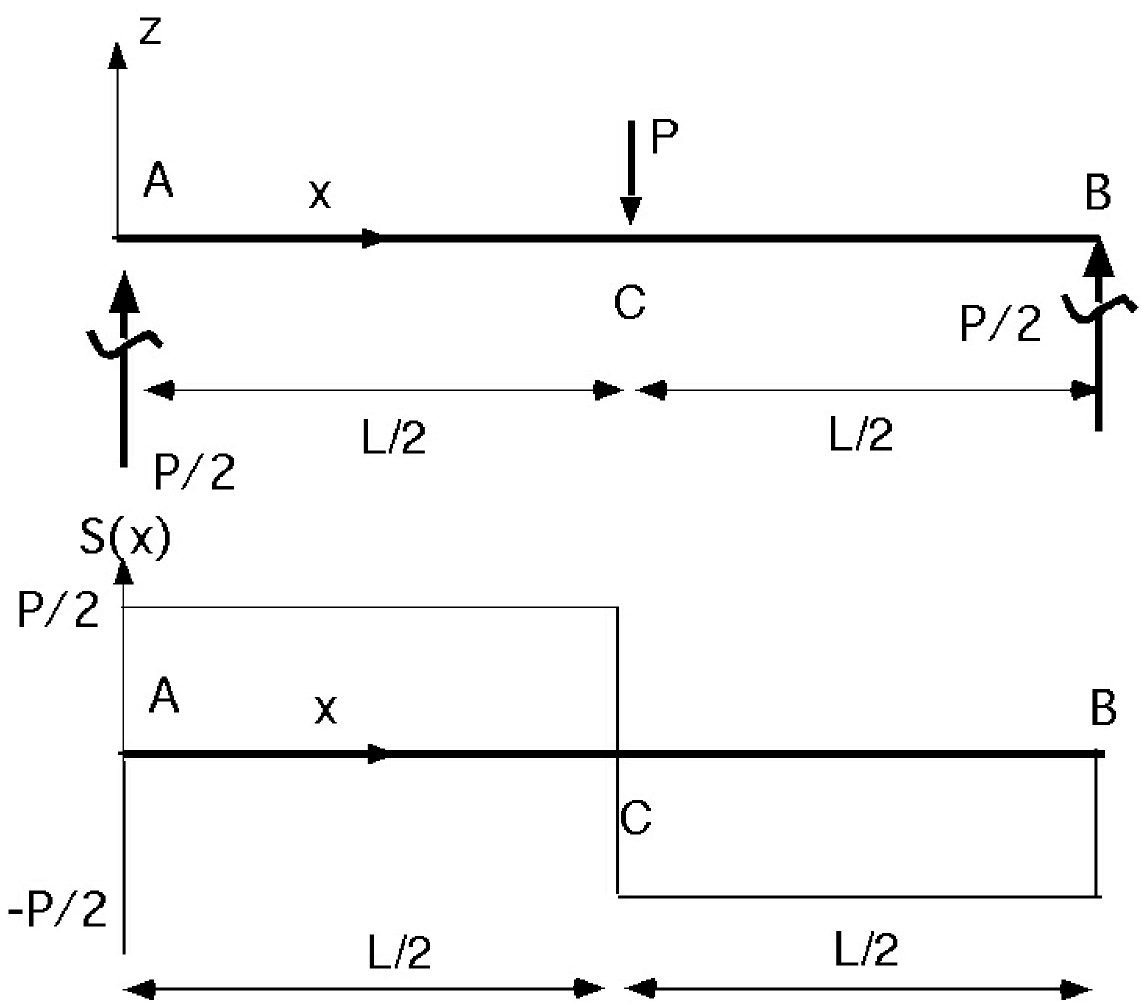
$$\sum \uparrow F+ = 0: \qquad \frac{P}{2} - P - S(x) = 0 \Rightarrow \quad S(x) = -\frac{P}{2}$$

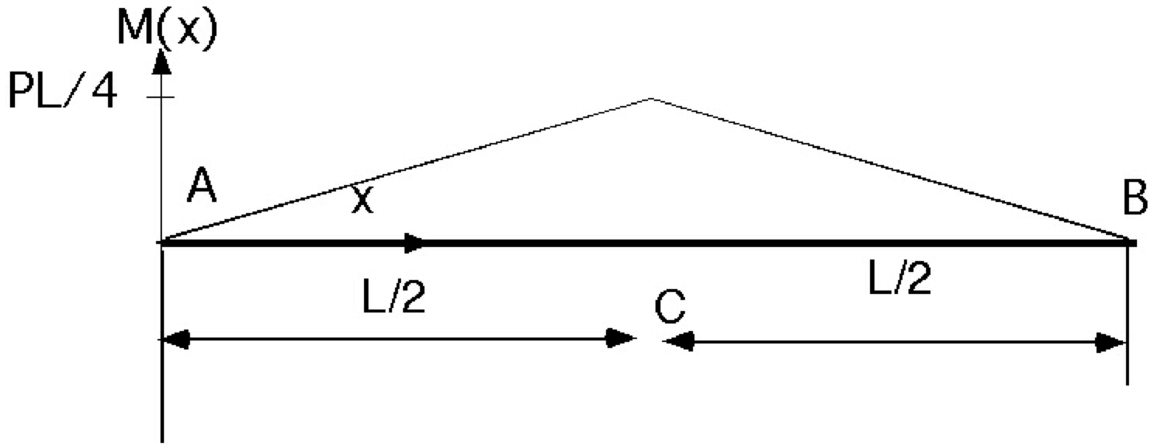
$$\sum M = 0: \qquad -\frac{P}{2}x + P\left(x - \frac{L}{2}\right) + M(x) = 0$$

$$M(x) = -\frac{P}{2}(x - L)$$

$$= \frac{P}{2}(L - x)$$

Draw Diagrams





Observations

- Shear is constant between point loads
- Bending moment varies linearly between discrete loads.
- Discontinuities occur in S and in slope of M at point of application of concentrated loads.
- Change in shear equals amount of concentrated loads.
- Values of S & M (and F) go to values of reactions at boundaries

Distributed loads

e.g. gravity, pressure, inertial loading. Can be uniform or varying with position.

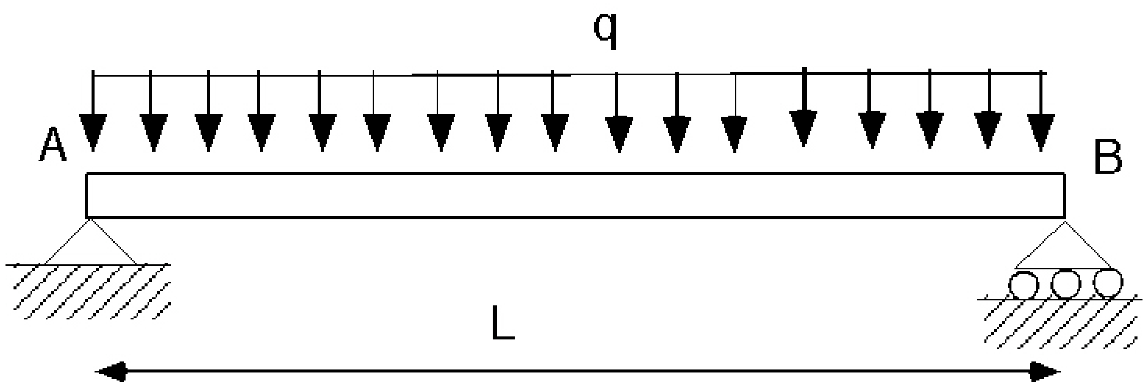
$q(x) = q$

$$q(x) = q_0\left(1 - \frac{x}{L}\right) = q_0 \ @\ x = 0, \quad = 0 \ @\ x = L$$

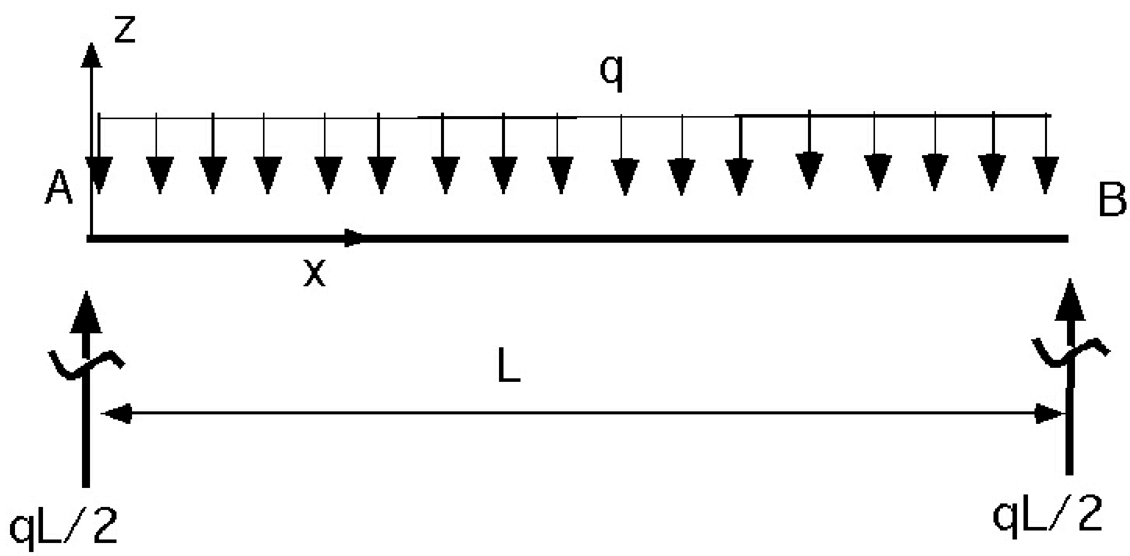
$[q_o]$ = [force/length]

Deal with distributed loads in essentially the same way as for point loads.

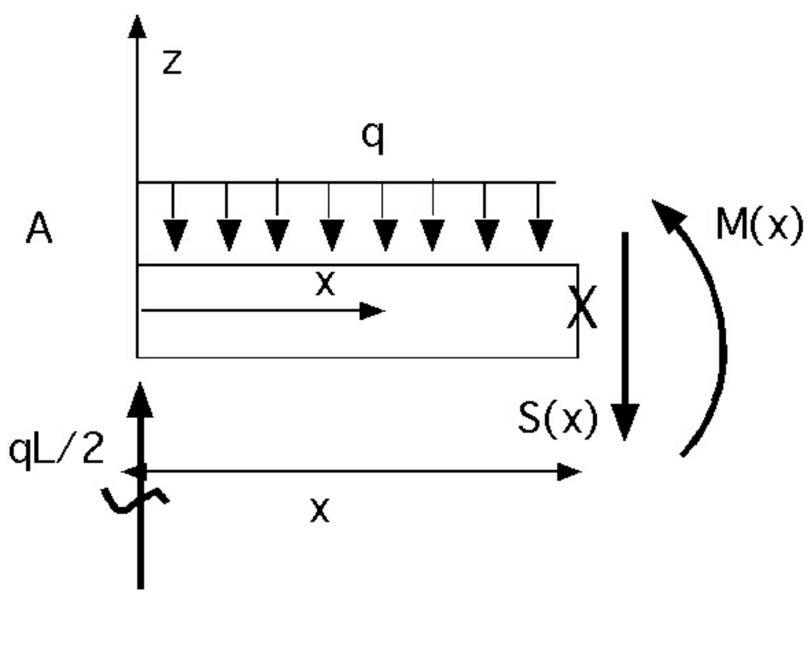
Example: Uniform distributed load, q (per unit length, applied to simply supported beam.



Free Body diagram:



Apply method of sections to obtain bending moments and shear forces:



Apply equilibrium:

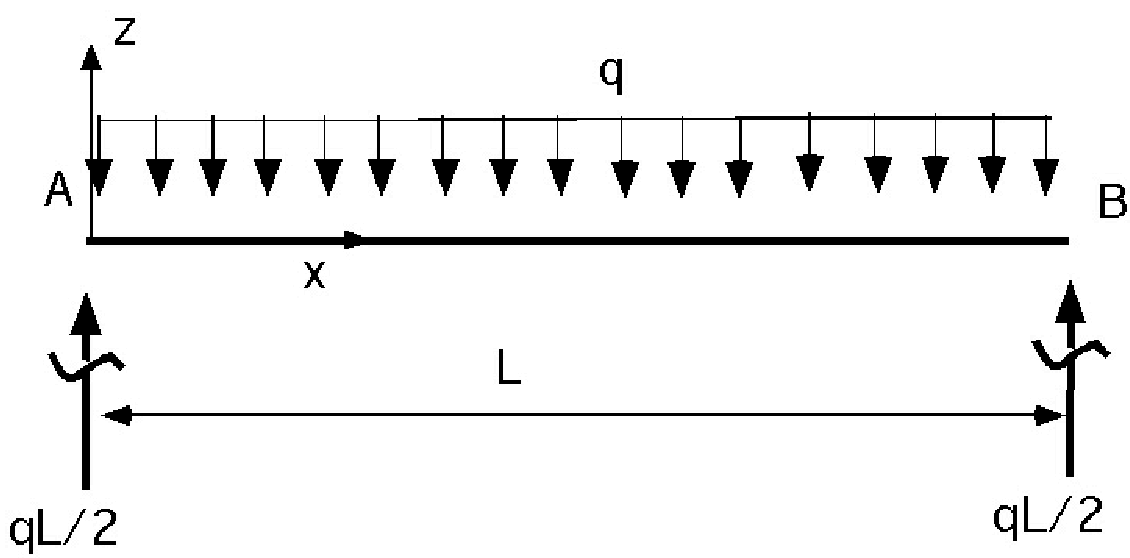
$$\Sigma \uparrow + \frac{qL}{2} - Sx - \int_0^x q\, dx = 0$$

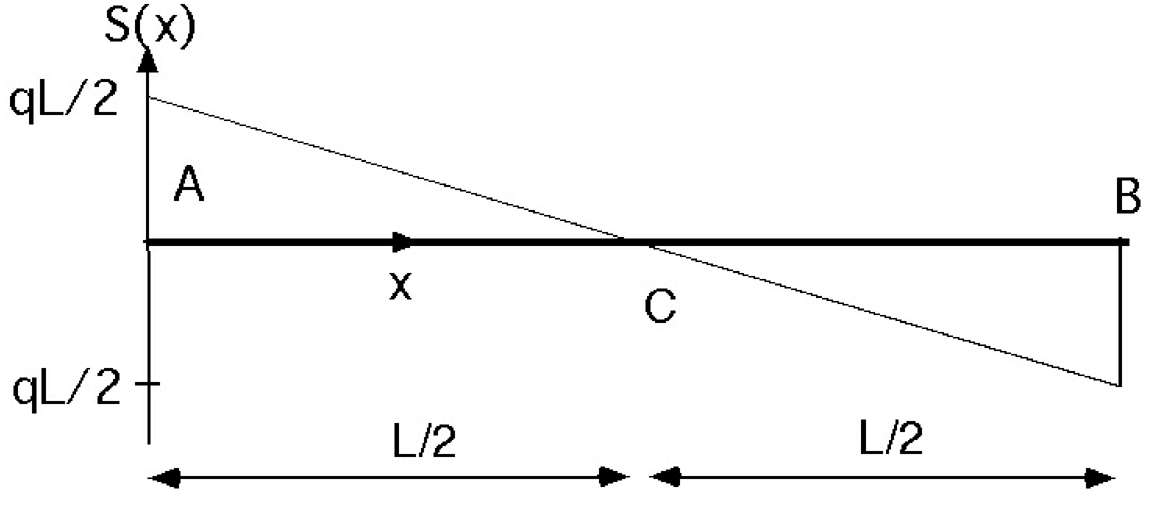
$$\Rightarrow \frac{qL}{2} - qx = S(x)$$

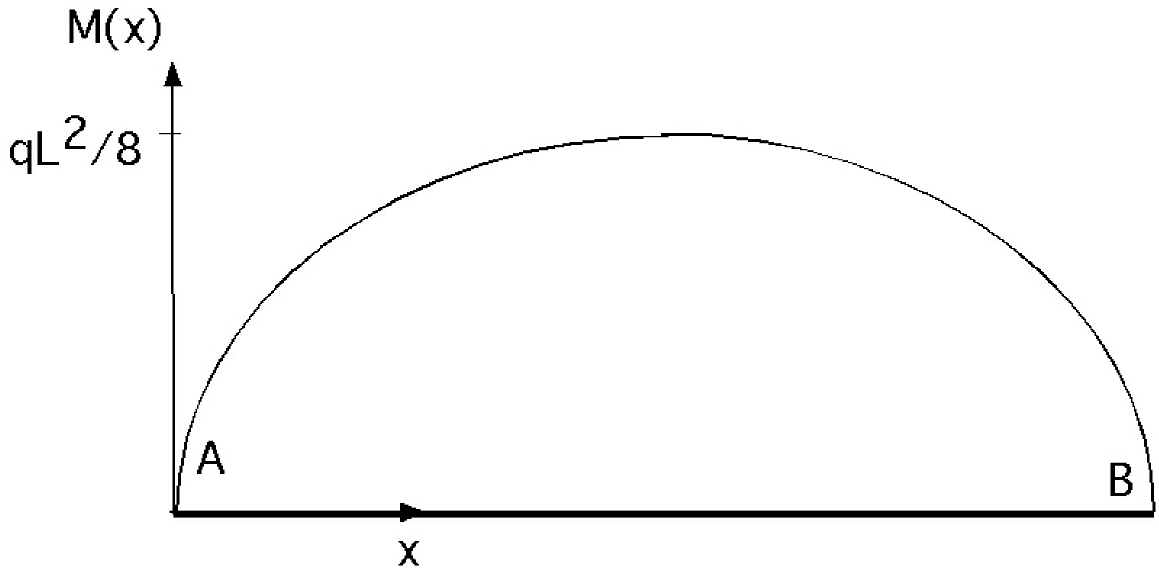
$$\Sigma M_x \circlearrowleft + \quad -\frac{qLx}{2} + M_x + \int_0^x qx dx = 0$$

$$-\frac{qL}{2} + M_x = \frac{qx^2}{2} = 0 \Rightarrow M_x = \frac{qLx}{2} - \frac{qx^2}{2}$$

Plot:





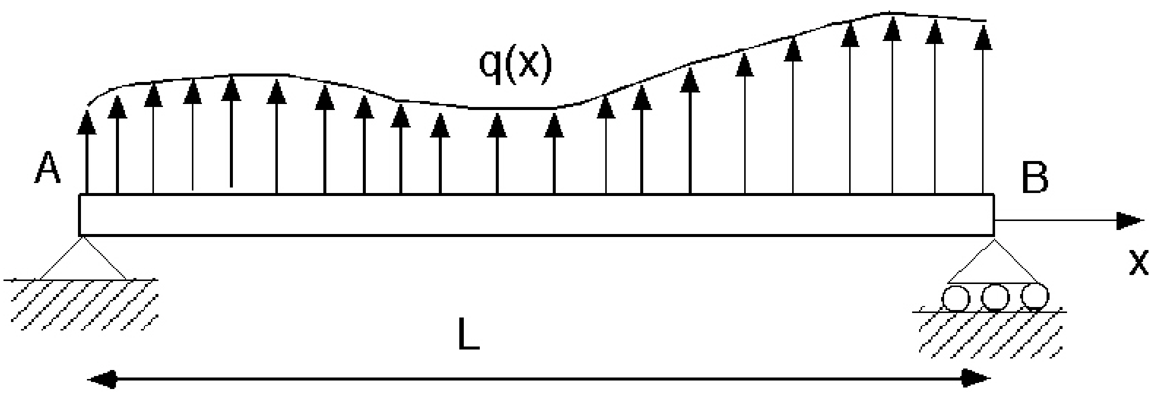


<u>Observations</u>

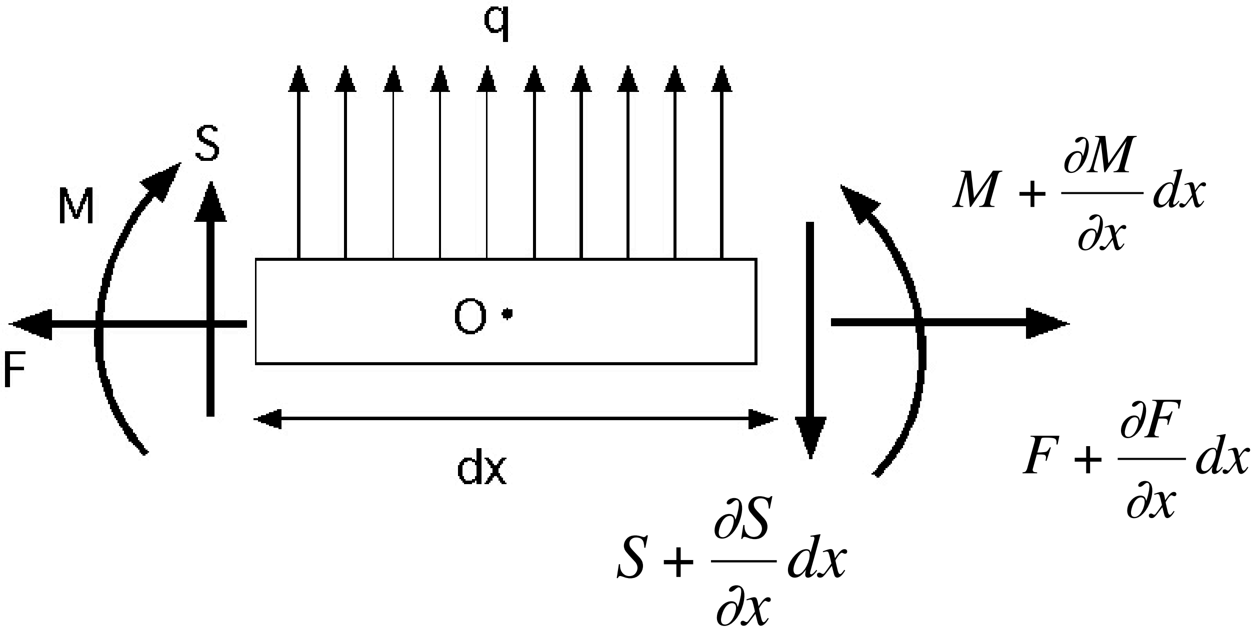
- Shear load varies linearity over constant distributed load.
- Moment varies quadratically (parabolically) over region where distributed load applied
- This suggests a relationship between M & S & P

General Relation Between q, S, M

Consider a beam under some arbitrary variable, distributed loading q(x):



Consider an infinitesimal element, length, dx, allow F, S, M to vary across element:



Now use equilibrium, replace partial derivatives by regular derivatives (F, S, M varying only in x).

$$\sum F_x = 0 \xrightarrow{+} -F + F + \frac{dF}{dx} dx = 0 \quad \frac{dF}{dx} = 0$$

$$\Sigma F_z = 0 \uparrow + \quad S - S - \frac{dS}{dx}dx + q(x)dx = 0$$

$$\frac{dS}{dx} = q(x)$$

$$\circlearrowright + \quad \Sigma M_0 \qquad -\not{M} + \not{M} + \frac{dM}{dx}dx - S\frac{dx}{2} - (S + \frac{dS}{dx}dx)\frac{dx}{2} = 0$$

note: q(x) has no net moment about O.

$$\frac{dM}{dx}dx - Sdx + \frac{1}{2}\frac{dS}{dx}(dx)^2 = 0$$

but $(dx)^2$ *is a higher order (small) term*

$$\Rightarrow \frac{dM}{dx} = S$$

Summarizing:

$$\frac{dF}{dx} = 0 \qquad (unless\ a\ bar)$$

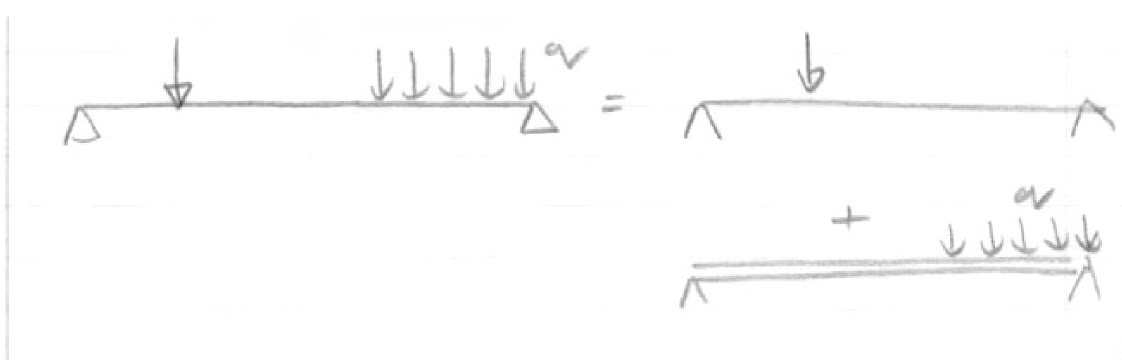
$$\frac{dS}{dx} = q$$

$$\frac{dM}{dx} = S \qquad (and \quad \frac{d^2M}{dx^2} = q)$$

Useful check, useful to automate process

<u>Superposition</u>

So long as the beam material is elastic and deformations are small all the structural problems are linear - can use superposition (as for trusses)



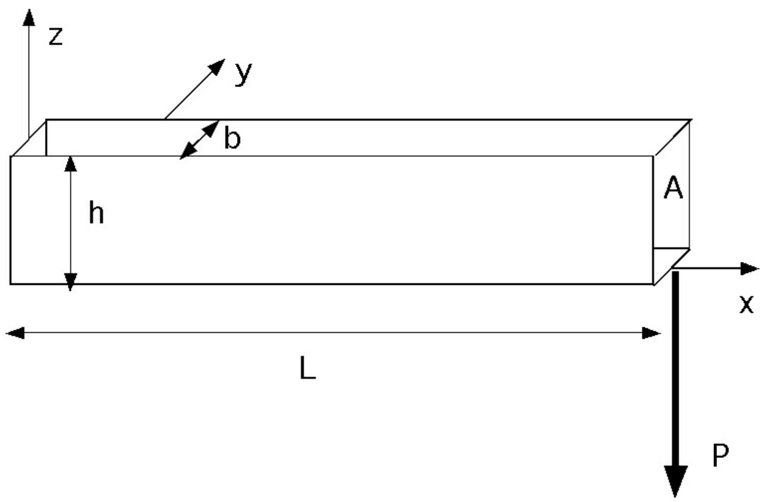
Lecture M4: Simple Beam Theory

Reading: Crandall, Dahl and Lardner 7.2-7.6

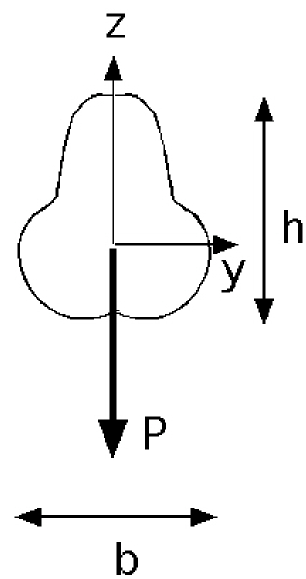
We have looked at the statics of a beam, seen that loads are transmitted by internal forces: axial forces, shear forces and bending moments.

Now look at how these forces imply stresses, strains, and deflections.

Recall model assumptions: slenderness



General, <u>symmetric</u>, cross section



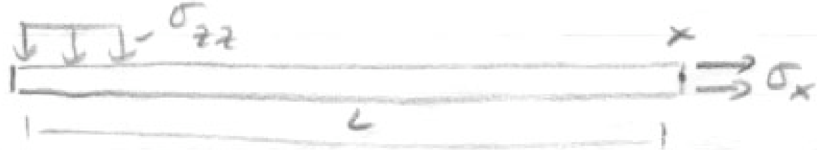
Geometry

$$L >> h, b$$

Loads

Leads to assumptions on stresses

Load in x - z plane $\Rightarrow$ $\begin{array}{l} \sigma_{yy} = \sigma_{xy} = \sigma_{yz} = 0 \\ \text{Also } L >> h, b \text{ implies } \sigma_{xx}, \sigma_{xz} >> \sigma_{zz} \end{array}$



Consider moment equilibrium of a cross section of a beam loaded by some distributed stress σ_{zz}, which is reacted by axial stresses on the cross section, σ_{xx}

$$\sum M_X = 0 = \sigma_{zz} L - \sigma_{xx} h \Rightarrow \frac{\sigma_{zz}}{\sigma_{xx}} \sim \frac{h}{L} \rightarrow 0$$

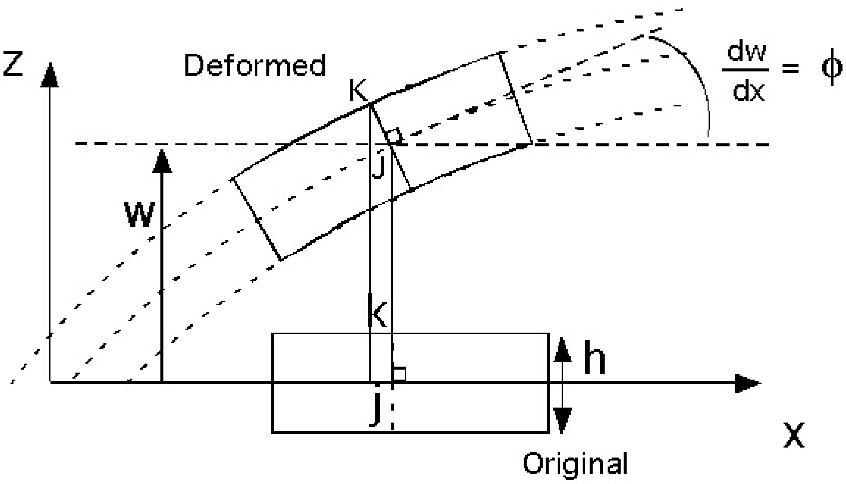
i.e. geometry of beam implies $\sigma_{xx} >> \sigma_{zz}$

Assumptions on Deformations:

The key to simple beam theory is the Bernoulli - Euler hypotheses (1750)
"Plane sections remain plane and perpendicular to the mid-plane after deformation."

It turns out that this is not really an assumption at all but a geometric necessity, at least for the case of pure bending.

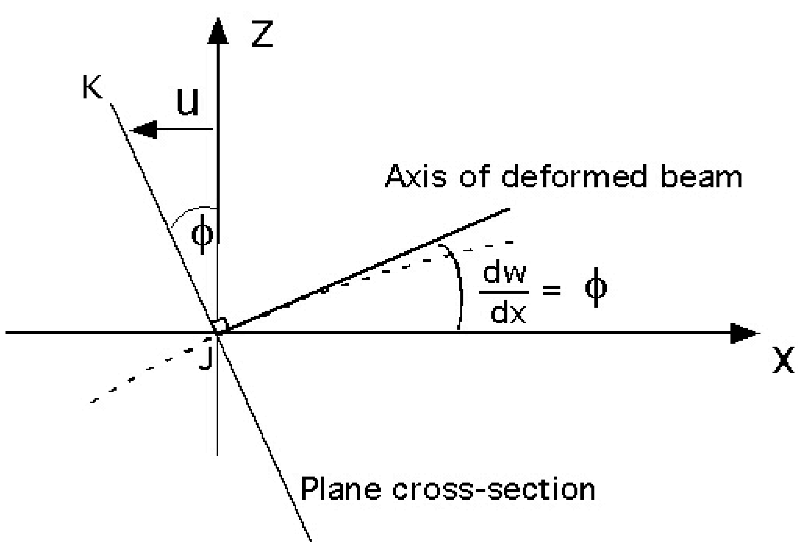
To see what the implications are of this, consider a beam element which undergoes transverse (bending)
deformation.



w= deflection of midplane/midline (function of x only)

Obtain deflection in x-direction, displacement u of point k to K, defined by rotation ϕ.

Note, key assumption, "plane sections remain plane"



axial displacement, u, of an arbitrary point on the cross-section arises from rotation of cross, sections

u= -z tanθ

Note, negative sign here due to use of consistent definitions of positive directions for w, x and dw/dx.

If deformations/angles are small $\tan\theta \approx \theta$

$$\theta \approx -\frac{dw}{dx} \Rightarrow \quad u = -z\frac{dw}{dx} \tag{1}$$

Hence obtain deformation field

$$u(x,y,z) = -z\frac{dw}{dx}$$

$$v(x,y,z) = 0 \quad \textit{nothing happening in y direction}$$

$$w(x,y,z) = w(x) \tag{2}$$

Deflection out of original plane
i.e., cross sections remain rigid

Hence we can obtain distributions of strain. (compatible with deformation). In absence of deformations in transverse directions partial derivatives can be replaced by regular derivatives, i.e. $\frac{\partial\_}{\partial\_} \equiv \frac{d\_}{d\_}$. Hence.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{d^2 w}{dx^2} \tag{3}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

$$\gamma_{xy} = 2\varepsilon_{12} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0, \quad \gamma_{yz} = \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

$$\gamma_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \left(-\frac{dw}{dx} + \frac{dw}{dx}\right) = 0$$

If no shear - consistent with B-E assumption of plane sections remaining plane.

for constant bending moment: $\frac{dM}{dx} = S = 0 \quad (\textit{will revisit for } S \neq 0)$

Next use stress - strain (assume orthotropic - for generality)

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} \qquad (4)$$

$$\varepsilon_{yy} = -\nu_{xy}\frac{\sigma_{xx}}{E_x}$$

$$\varepsilon_{zz} = -\nu_{xz}\frac{\sigma_{xx}}{E_x}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G_{yz}}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G_{xz}}$$

Note inconsistency - $\gamma_{xz} = 0, \quad \tau_{xz} \neq 0$ (shear forces are non zero)

The inconsistency on the shear stress/strain arises from the plane/sections remain plane assumption. Does not strictly apply when there is varying bending moment (and hence non-zero shear force). However, displacements due to γ_{xz} are very small compared to those due to ε_{xx}, and therefore negligible.

Finally apply equilibrium:

$$\frac{\partial \sigma_{mn}}{\partial x_m} + f_m = 0 \qquad (5)$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \Rightarrow \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0 \Rightarrow 0 = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial y} = 0 \Rightarrow \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$5 \ equations, 5 \ unknowns: \quad w, u, \varepsilon_{xx}, \sigma_{xx}, \tau_{xz}$$

To be continued…