2.003 Problem Archive

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Contents

Problems **12**
1 1st Order Systems . . . 13
1.1 First Order Time Constant . . . 13
1.2 Rise and Settling Times . . . 14
1.3 First Order System Response . . . 15
2 2nd Order Systems . . . 16
2.1 Second-Order System Response . . . 16
2.2 Second Order Responses . . . 17
2.3 Second Order Derivation . . . 19
2.4 Second Order Derivation Continued . . . 20
3 Higher Order Systems . . . 21
3.1 Structure of Higher Order System Solutions . . . 21
3.2 Structure of Higher Order System Solutions . . . 22
4 Mechanical Systems . . . 23
4.1 Balloon . . . 23
4.2 Bungee Jumper . . . 25
4.3 Elevator Model . . . 26
4.4 Hoisting for Engineers . . . 27
4.5 Blocked Springs . . . 29
4.6 Engine Vibration . . . 31
4.7 Garage Door . . . 32
4.8 Rotating Damped . . . 34
4.9 Car Suspension 1 . . . 35
4.10 Disk Drive . . . 36
4.11 Crashworthiness . . . 38
4.12 Nonlinear Rotational System . . . 39
4.13 Developing Differential Equations . . . 40
4.14 Mass Spring System Frequency Response . . . 41
4.15 Modeling Practice . . . 42
4.16 Small Motion Transfer Function . . . 43
4.17 Mass Spring Damper System Frequency Response 1 . . . 44
4.18 Mass Spring Damper System Frequency Response 2 . . . 45
4.19 Propeller Shaft Vibration . . . 46
4.20 Safe Packaging . . . 47
4.21 Sliding Damped . . . 48
4.22 Toy Flywheel . . . 49
4.23 Truck Shocks . . . 50

4.24 Car Suspension 2 . . . 51
4.25 Kid-Skid . . . 52
4.26 Rolling Machine . . . 53
4.27 Sunday Bike Ride . . . 55
4.28 Compound Mass Spring Damper System 1 . . . 56
4.29 Compound Mass Spring Damper System 2 . . . 58
4.30 Wind Induced Building Vibrations . . . 59
5 Electrical Systems . . . 61
5.1 Camera Flash Circuit . . . 61
5.2 Batteries and Capacitors . . . 62
5.3 Loaded Motor . . . 66
6 Fluid Systems . . . 67
6.1 Rotational Fluid System . . . 67
6.2 Fluid Leak . . . 68
6.3 Where'd the water go? . . . 69
7 Thermal Systems . . . 71
7.1 Thermal Block Question . . . 71
7.2 The Hot Copper Block 1 . . . 72
7.3 The Hot Copper Block 2 . . . 74
7.4 Transistor on heat sink . . . 75
8 Circuits . . . 77
8.1 RC Transfer Function . . . 77
8.2 RLC Transfer Function . . . 78
8.3 Bandpass Filter . . . 79
8.4 LC Circuit Differential Equations . . . 81
8.5 Equivalent Impedance . . . 82
8.6 Circuit Response . . . 83
8.7 LRC Circuit 1 . . . 85
8.8 LRC Circuit 2 . . . 86
8.9 LR Circuit Step . . . 87
9 Op Amps . . . 89
9.1 Opamp Arithmetic . . . 89
9.2 Inverting Op-Amp . . . 90
9.3 Opamp Transfer Function . . . 91
9.4 Op-Amp Differentiator 1 . . . 92
9.5 Op-Amp Proportional plus Integral Controller . . . 93
9.6 Op-Amp Circuit Design 1 . . . 94
9.7 Op-Amp Block Diagram . . . 95
9.8 Op-Amp Circuit Design 2 . . . 96
9.9 Op-Amp Differentiator 2 . . . 97
9.10 RC Op-Amp Frequency Response . . . 98
10 Differential Equations . . . 99
10.1 Differential Equations 1 . . . 99
10.2 Differential Equations 2 . . . 100
10.3 Differential Equations 3 . . . 101
10.4 Euler's Theorem 1 . . . 102
10.5 Euler's Theorem 2 . . . 103
10.6 Force Steps . . . 104

10.7 Input for Zero Output . . . 105
10.8 Zero Input Response . . . 106
10.9 Zero Step Response 1 . . . 107
10.10 Zero Step Response 2 . . . 108
10.11 Nonlinear String System . . . 109
10.12 Homogenous and Particular Solutions . . . 110
11 Frequency Response . . . 112
11.1 Circuit Bode Plots . . . 112
11.2 Sketch Bode Plots . . . 113
11.3 LC Circuit Frequency Response . . . 114
11.4 LRC Circuit Frequency Response . . . 115
11.5 Pole-Zero Plots . . . 116
12 Signals, Transforms, and Transfer Functions . . . 117
12.1 First Order Zeros . . . 117
12.2 Laplace Practice . . . 118
12.3 Step-Zero . . . 119
12.4 Laplace to Time . . . 120
12.5 Time to Laplace . . . 121
12.6 Laplace Transform . . . 122
12.7 Signal in Time and Frequency Domain . . . 123
12.8 Transfer Function and Time Constants . . . 124
12.9 Fourier Series Input . . . 125
12.10 Spring Mass System . . . 127
12.11 Transfer Function . . . 128
13 Controls . . . 129
13.1 Rooftop Antenna . . . 129
13.2 Lead Controller . . . 131
13.3 Third Order . . . 132
13.4 Error Constants . . . 133
13.5 System Type . . . 134
13.6 Unity Feedback . . . 135
13.7 P and PI Controllers . . . 136
14 Motors and Transformers . . . 137
14.1 Locked Load . . . 137
14.2 Motoring with a Capacitor . . . 139
14.3 Leadscrew with Translating Stage . . . 140
14.4 Mass Pulley System . . . 141
14.5 DC Motors . . . 142
14.6 Gear Train . . . 143
14.7 Non-Ideal Transformer . . . 144
14.8 Equivalent Mass and Inertia . . . 145
14.9 Inertia in Geartrain . . . 146
14.10 Motor Driving Inertia Through Gear Train . . . 147
15 MATLAB and Simulink . . . 148
15.1 Matlab Spirograph . . . 148
15.2 MATLAB Plotting . . . 149
15.3 Simulink Introduction . . . 150
15.4 Wackygraph . . . 152

16 Case Studies . . . 153
16.1 CD Player 1 . . . 153
16.2 CD Player 2 . . . 154
16.3 CD Player 3 . . . 155
16.4 Pinewood Derby 1 . . . 156
16.5 Pinewood Derby 2 . . . 157
16.6 Pinewood Derby 3 . . . 158
16.7 Pinewood Derby 4 . . . 159
16.8 Pinewood Derby 5 . . . 160
16.9 Engine Block Vibration 1 . . . 161
16.10 Engine Block Vibration 2 . . . 162
16.11 Engine Block Vibration 3 . . . 163
16.12 Plate On Springs 1 . . . 164
16.13 Plate On Springs 2 . . . 165
16.14 Mousetrap Dynamics 1 . . . 166
16.15 Mousetrap Dynamics 2 . . . 168
16.16 Mousetrap Dynamics 3 . . . 169
16.17 Mousetrap Dynamics 4 . . . 170
16.18 Hydraulic Elevator Design . . . 171
16.19 Servo Position Control . . . 172
16.20 Servo Frequency Compensation . . . 173
16.21 Servo Torque Disturbance . . . 174
16.22 Plate On Springs Damped 1 . . . 175
16.23 Plate On Springs Damped 2 . . . 181
16.24 Plate On Springs Damped 3 . . . 182
16.25 Plate On Springs Damped 4 . . . 183
16.26 Plate On Springs Damped 5 . . . 184
16.27 Plate On Springs Damped 6 . . . 185
16.28 Shipping Crate 1 . . . 186
16.29 Shipping Crate 2 . . . 187
17 Quiz Problems . . . 188
17.1 Fun with Block Diagrams . . . 188
17.2 Complex Translation . . . 189
17.3 Cylinder Step Response . . . 190
17.4 JKC Frequency Response 1 . . . 191
17.5 Mass Spring Damper Dynamics . . . 192
17.6 Op-Amp Analysis . . . 193
17.7 RLC Circuit Analysis . . . 194
17.8 Sailing for Engineers . . . 195
17.9 Second Order Step Response . . . 196
17.10 Spring Damper Dynamics . . . 197
17.11 Derive Blocks for Op-amp Circuit . . . 198
17.12 Automobile Fender Spring/Damper System . . . 200
17.13 Cu Flywheel with Eddy Current Damper . . . 201
17.14 Current Driven RC Circuit . . . 202
17.15 Driven Mass Spring System . . . 203
17.16 Flywheel Driven by Hanging Mass . . . 204
17.17 Homogeneous Second Order DE . . . 205

17.18 Match Pole/Zero Plots with Step Response . . . 206
17.19 Opamp Block Transfer Functions . . . 208
17.20 Piston with 2nd Order Translation . . . 210
17.21 Pole Zero Bode Matching . . . 211
17.22 Power Semiconductor Thermal Problem . . . 213
17.23 Submersible Capsule Hoist System . . . 214
17.24 Transfer Function from Pole Zero Plot . . . 216
17.25 Vaccine Cooler . . . 217
17.26 Voltage Driven RRC Circuit . . . 219
17.27 Linear Mechanical System with Position Input . . . 220
17.28 Tank with Pump Inlet Lower than Outlet . . . 221
17.29 Thermal Power Chip Analysis . . . 223
17.30 Current Divider in Simple Network . . . 224
17.31 Voltage and Current Source Driving network . . . 225
17.32 Rotary Motor with Damper . . . 226
17.33 Patient Rehabilitation . . . 227
17.34 Delay Transfer Function . . . 228
17.35 Laplace and Complex Numbers . . . 229
17.36 Mechanical Stiffness . . . 230
18 Math Techniques . . . 231
18.1 Complex Expression Reduction . . . 231
18.2 Complex Expressions . . . 232
18.3 Matrix Operation Practice . . . 233
19 Recitation Problems . . . 234
19.1 Recitation 1 Problem . . . 234
19.2 Recitation 2 Problem . . . 235
19.3 Recitation 3 Problem . . . 236
19.4 Recitation 4 Problem . . . 237
19.5 Recitation 5 Problem . . . 238
19.6 Recitation 6 Problem . . . 240
19.7 Recitation 7 Problem . . . 241
19.8 Recitation 8 Problem . . . 242
19.9 Recitation 9 Problem . . . 243
19.10 Recitation 10 Problem . . . 245
19.11 Recitation 10A Problem . . . 246
19.12 Recitation 11 Problem . . . 247
19.13 Recitation 12 Problem . . . 248
19.14 Recitation 13 Problem . . . 249
20 Recitation Quizzes . . . 250
20.1 Recitation 1 Quiz . . . 250
20.2 Recitation 2 Quiz . . . 251
20.3 Recitation 3 Quiz . . . 252
20.4 Recitation 4 Quiz . . . 253
20.5 Recitation 5 Quiz . . . 254
20.6 Recitation 6 Quiz . . . 255
20.7 Recitation 7 Quiz . . . 256
20.8 Recitation 8 Quiz . . . 257
20.9 Recitation 9 Quiz . . . 258

20.10 Recitation 10 Quiz . . . 259
20.11 Recitation 10A Quiz . . . 260
20.12 Recitation 11 Quiz . . . 261
20.13 Recitation 12 . . . 262
20.14 Recitation 12 . . . 263

Solutions **264**
1 1st Order Systems . . . 265
1.1 First Order Time Constant . . . 265
1.2 Rise and Settling Times . . . 266
1.3 First Order System Response . . . 267
2 2nd Order Systems . . . 269
2.1 Second-Order System Response . . . 269
2.2 Second Order Responses . . . 275
2.3 Second Order Derivation . . . 276
2.4 Second Order Derivation Continued . . . 277
3 Higher Order Systems . . . 278
3.1 Structure of Higher Order System Solutions . . . 278
3.2 Structure of Higher Order System Solutions . . . 279
4 Mechanical Systems . . . 283
4.1 Balloon . . . 283
4.2 Bungee Jumper . . . 286
4.3 Elevator Model . . . 288
4.4 Hoisting for Engineers . . . 290
4.5 Blocked Springs . . . 291
4.6 Engine Vibration . . . 295
4.7 Garage Door . . . 296
4.8 Rotating Damped . . . 298
4.9 Car Suspension 1 . . . 301
4.10 Disk Drive . . . 303
4.11 Crashworthiness . . . 306
4.12 Nonlinear Rotational System . . . 312
4.13 Developing Differential Equations . . . 313
4.14 Mass Spring System Frequency Response . . . 316
4.15 Modeling Practice . . . 317
4.16 Small Motion Transfer Function . . . 318
4.17 Mass Spring Damper System Frequency Response 1 . . . 319
4.18 Mass Spring Damper System Frequency Response 2 . . . 321
4.19 Propeller Shaft Vibration . . . 324
4.20 Safe Packaging . . . 327
4.21 Sliding Damped . . . 329
4.22 Toy Flywheel . . . 332
4.23 Truck Shocks . . . 335
4.24 Car Suspension 2 . . . 336
4.25 Kid-Skid . . . 338
4.26 Rolling Machine . . . 341
4.27 Sunday Bike Ride . . . 343
4.28 Compound Mass Spring Damper System 1 . . . 346

4.29 Compound Mass Spring Damper System 2 . . . 351
4.30 Wind Induced Building Vibrations . . . 353
5 Electrical Systems . . . 356
5.1 Camera Flash Circuit . . . 356
5.2 Batteries and Capacitors . . . 357
5.3 Loaded Motor . . . 362
6 Fluid Systems . . . 363
6.1 Rotational Fluid System . . . 363
6.2 Fluid Leak . . . 365
6.3 Where'd the water go? . . . 366
7 Thermal Systems . . . 368
7.1 Thermal Block Question . . . 368
7.2 The Hot Copper Block 1 . . . 370
7.3 The Hot Copper Block 2 . . . 375
7.4 Transistor on heat sink . . . 378
8 Circuits . . . 382
8.1 RC Transfer Function . . . 382
8.2 RLC Transfer Function . . . 384
8.3 Bandpass Filter . . . 386
8.4 LC Circuit Differential Equations . . . 389
8.5 Equivalent Impedance . . . 390
8.6 Circuit Response . . . 391
8.7 LRC Circuit 1 . . . 395
8.8 LRC Circuit 2 . . . 396
8.9 LR Circuit Step . . . 399
9 Op Amps . . . 403
9.1 Opamp Arithmetic . . . 403
9.2 Inverting Op-Amp . . . 404
9.3 Opamp Transfer Function . . . 405
9.4 Op-Amp Differentiator 1 . . . 406
9.5 Op-Amp Proportional plus Integral Controller . . . 408
9.6 Op-Amp Circuit Design 1 . . . 410
9.7 Op-Amp Block Diagram . . . 412
9.8 Op-Amp Circuit Design 2 . . . 413
9.9 Op-Amp Differentiator 2 . . . 415
9.10 RC Op-Amp Frequency Response . . . 420
10 Differential Equations . . . 423
10.1 Differential Equations 1 . . . 423
10.2 Differential Equations 2 . . . 425
10.3 Differential Equations 3 . . . 428
10.4 Euler's Theorem 1 . . . 430
10.5 Euler's Theorem 2 . . . 432
10.6 Force Steps . . . 433
10.7 Input for Zero Output . . . 434
10.8 Zero Input Response . . . 435
10.9 Zero Step Response 1 . . . 437
10.10 Zero Step Response 2 . . . 438
10.11 Nonlinear String System . . . 441

10.12 Homogenous and Particular Solutions . . . 442
11 Frequency Response . . . 446
11.1 Circuit Bode Plots . . . 446
11.2 Sketch Bode Plots . . . 447
11.3 LC Circuit Frequency Response . . . 449
11.4 LRC Circuit Frequency Response . . . 452
11.5 Pole-Zero Plots . . . 456
12 Signals, Transforms, and Transfer Functions . . . 457
12.1 First Order Zeros . . . 457
12.2 Laplace Practice . . . 461
12.3 Step-Zero . . . 462
12.4 Laplace to Time . . . 464
12.5 Time to Laplace . . . 465
12.6 Laplace Transform . . . 466
12.7 Signal in Time and Frequency Domain . . . 467
12.8 Transfer Function and Time Constants . . . 468
12.9 Fourier Series Input . . . 472
12.10 Spring Mass System . . . 480
12.11 Transfer Function . . . 482
13 Controls . . . 484
13.1 Rooftop Antenna . . . 484
13.2 Lead Controller . . . 493
13.3 Third Order . . . 494
13.4 Error Constants . . . 497
13.5 System Type . . . 498
13.6 Unity Feedback . . . 503
13.7 P and PI Controllers . . . 505
14 Motors and Transformers . . . 506
14.1 Locked Load . . . 506
14.2 Motoring with a Capacitor . . . 509
14.3 Leadscrew with Translating Stage . . . 511
14.4 Mass Pulley System . . . 512
14.5 DC Motors . . . 514
14.6 Gear Train . . . 515
14.7 Non-Ideal Transformer . . . 517
14.8 Equivalent Mass and Inertia . . . 520
14.9 Inertia in Geartrain . . . 522
14.10 Motor Driving Inertia Through Gear Train . . . 523
15 MATLAB and Simulink . . . 524
15.1 Matlab Spirograph . . . 524
15.2 MATLAB Plotting . . . 527
15.3 Simulink Introduction . . . 529
15.4 Wackygraph . . . 530
16 Case Studies . . . 534
16.1 CD Player 1 . . . 534
16.2 CD Player 2 . . . 536
16.3 CD Player 3 . . . 538
16.4 Pinewood Derby 1 . . . 541

16.5 Pinewood Derby 2 . . . 543
16.6 Pinewood Derby 3 . . . 544
16.7 Pinewood Derby 4 . . . 546
16.8 Pinewood Derby 5 . . . 548
16.9 Engine Block Vibration 1 . . . 551
16.10 Engine Block Vibration 2 . . . 555
16.11 Engine Block Vibration 3 . . . 558
16.12 Plate On Springs 1 . . . 561
16.13 Plate On Springs 2 . . . 563
16.14 Mousetrap Dynamics 1 . . . 564
16.15 Mousetrap Dynamics 2 . . . 565
16.16 Mousetrap Dynamics 3 . . . 566
16.17 Mousetrap Dynamics 4 . . . 567
16.18 Hydraulic Elevator Design . . . 568
16.19 Servo Position Control . . . 574
16.20 Servo Frequency Compensation . . . 575
16.21 Servo Torque Disturbance . . . 576
16.22 Plate On Springs Damped 1 . . . 577
16.23 Plate On Springs Damped 2 . . . 584
16.24 Plate On Springs Damped 3 . . . 586
16.25 Plate On Springs Damped 4 . . . 588
16.26 Plate On Springs Damped 5 . . . 589
16.27 Plate On Springs Damped 6 . . . 590
16.28 Shipping Crate 1 . . . 593
16.29 Shipping Crate 2 . . . 594
17 Quiz Problems . . . 596
17.1 Fun with Block Diagrams . . . 596
17.2 Complex Translation . . . 597
17.3 Cylinder Step Response . . . 598
17.4 JKC Frequency Response 1 . . . 599
17.5 Mass Spring Damper Dynamics . . . 600
17.6 Op-Amp Analysis . . . 601
17.7 RLC Circuit Analysis . . . 603
17.8 Sailing for Engineers . . . 605
17.9 Second Order Step Response . . . 606
17.10 Spring Damper Dynamics . . . 607
17.11 Derive Blocks for Op-amp Circuit . . . 608
17.12 Automobile Fender Spring/Damper System . . . 609
17.13 Cu Flywheel with Eddy Current Damper . . . 610
17.14 Current Driven RC Circuit . . . 611
17.15 Driven Mass Spring System . . . 612
17.16 Flywheel Driven by Hanging Mass . . . 613
17.17 Homogeneous Second Order DE . . . 614
17.18 Match Pole/Zero Plots with Step Response . . . 615
17.19 Opamp Block Transfer Functions . . . 616
17.20 Piston with 2nd Order Translation . . . 617
17.21 Pole Zero Bode Matching . . . 618
17.22 Power Semiconductor Thermal Problem . . . 619

17.23 Submersible Capsule Hoist System . . . 620
17.24 Transfer Function from Pole Zero Plot . . . 621
17.25 Vaccine Cooler . . . 622
17.26 Voltage Driven RRC Circuit . . . 623
17.27 Linear Mechanical System with Position Input . . . 624
17.28 Tank with Pump Inlet Lower than Outlet . . . 625
17.29 Thermal Power Chip Analysis . . . 626
17.30 Current Divider in Simple Network . . . 627
17.31 Voltage and Current Source Driving network . . . 628
17.32 Rotary Motor with Damper . . . 629
17.33 Patient Rehabilitation . . . 630
17.34 Delay Transfer Function . . . 631
17.35 Laplace and Complex Numbers . . . 632
17.36 Mechanical Stiffness . . . 633
18 Math Techniques . . . 634
18.1 Complex Expression Reduction . . . 634
18.2 Complex Expressions . . . 635
18.3 Matrix Operation Practice . . . 636
19 Recitation Problems . . . 637
19.1 Recitation 1 Problem . . . 637
19.2 Recitation 2 Problem . . . 638
19.3 Recitation 3 Problem . . . 644
19.4 Recitation 4 Problem . . . 646
19.5 Recitation 5 Problem . . . 647
19.6 Recitation 6 Problem . . . 649
19.7 Recitation 7 Problem . . . 651
19.8 Recitation 8 Problem . . . 652
19.9 Recitation 9 Problem . . . 653
19.10 Recitation 10 Problem . . . 654
19.11 Recitation 10A Problem . . . 655
19.12 Recitation 11 Problem . . . 657
19.13 Recitation 12 Problem . . . 658
19.14 Recitation 13 Problem . . . 659
20 Recitation Quizzes . . . 660
20.1 Recitation 1 Quiz . . . 660
20.2 Recitation 2 Quiz . . . 661
20.3 Recitation 3 Quiz . . . 662
20.4 Recitation 4 Quiz . . . 663
20.5 Recitation 5 Quiz . . . 664
20.6 Recitation 6 Quiz . . . 665
20.7 Recitation 7 Quiz . . . 666
20.8 Recitation 8 Quiz . . . 667
20.9 Recitation 9 Quiz . . . 668
20.10 Recitation 10 Quiz . . . 669
20.11 Recitation 10A Quiz . . . 670
20.12 Recitation 11 Quiz . . . 671
20.13 Recitation 12 . . . 672
20.14 Recitation 12 . . . 673

Chapter

Problems

1 1st Order Systems

1.1 First Order Time Constant

This problem concerns the first-order response shown in Figure 1.1.1.

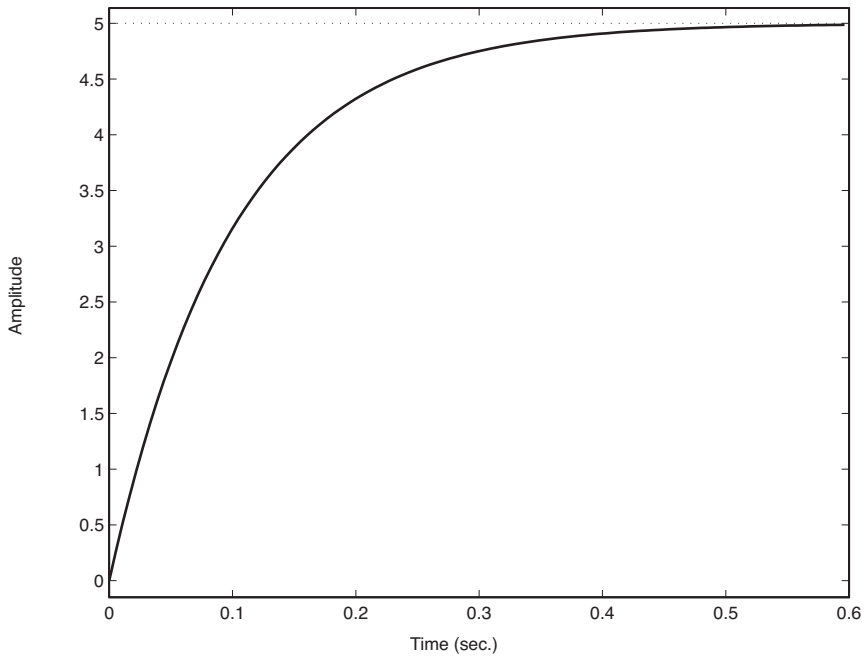


Figure 1.1.1: Plot of Response

(a) What is the time constant of this response? (You can work directly on the given time response plot and turn this in as part of your homework.)

(b) Show us a mechanical system which will give this response, under the assumption that the indicated response is angle in radians. What are possible numerical values of the system parameters and initial conditions that go with this response? Be sure to show your reasoning.

1.2 Rise and Settling Times

Consider the first-order system

$$\tau\dot{y} + y = u$$

driven with a unit step from zero initial conditions. The input to this system is u and the output is y. Derive expressions for the 10–90% rise time t_r and the settling time t_s, where the settling is to within an error $\pm\Delta$ from the final value of 1. How many time constants are required in order to settle to within an error of $\Delta = 10^{-6}$?

1.3 First Order System Response

You are given an equation of motion of the form:

$$\dot{y} + 5y = 10u$$

(a) What is the time constant for this system?

(b) If $u = 10$, what is the final or steady-state value for $y(t)$? Now assume that $u = 0$ (no input) and the system was started at some initial position y_o, which you do not know. But, you do know that 0.5 seconds later it was at a position $y(0.5) = 2$.

(c) What was the initial condition $y(0)$ that would lead to this result?

(d) Sketch as accurately as you can, the time response for case (c) starting from $t = 0$ until the response is nearly zero, and indicate where the one data point at $t = 0.5$ would lie.

(e) When does the response reach 2% of the initial value?

(f) When does the response reach a value of 0.02?

2 2nd Order Systems

2.1 Second-Order System Response

For a system described by the homogenous equation:

$$20\ddot{y} + 160\dot{y} + 8000y = 0$$

Determine the solution y (t) for three different initial conditions:

(a) $y(0) = 0$
$\dot{y}(0) = 10$

(b) $y(0) = 10$
$\dot{y}(0) = 0$

(c) $y(0) = 1$
$\dot{y}(0) = 10$

(d) For all three cases, create a separate plot of $y(t)$ using MATLAB, but be sure to use the same scale on all three plots.

(e) Compare the time for each of the three plots to reach steady-state. Comment on how you can make this comparison and why the three answers do or do not differ.

2.2 Second Order Responses

This problem concerns the second-order response shown in Figure 2.2.1.

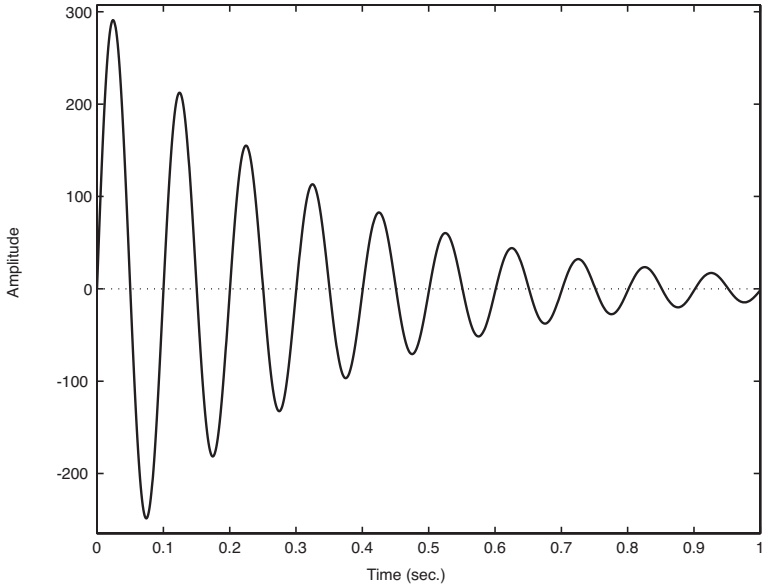


Figure 2.2.1: Second Order Response

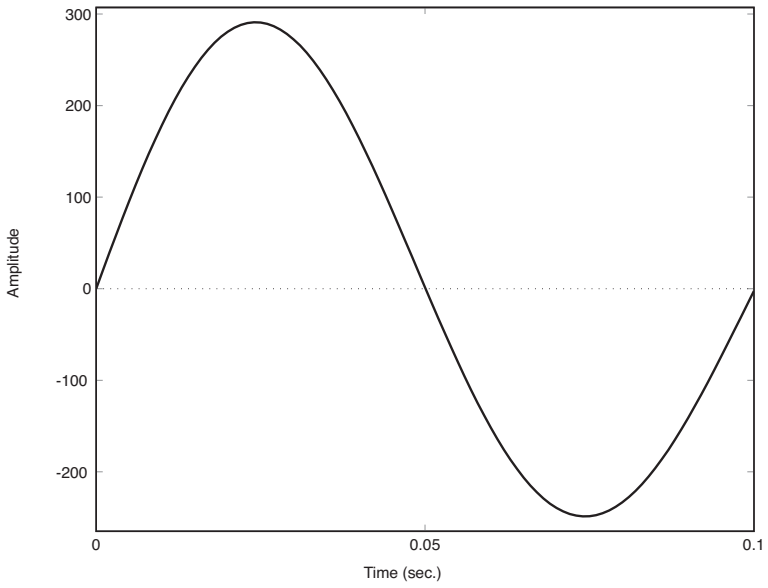


Figure 2.2.2: Zoomed Second Order Response

(a) What are the natural frequency and damping ratio of this response? (You can work directly

on the given time response plot and turn this in as part of your homework paper.)

(b) Show us a mechanical system which will give this response, under the assumption that the indicated response is position in millimeters. What are possible numerical values of the system parameters and initial conditions that go with this response? Be sure to show your reasoning.

2.3 Second Order Derivation

The supplemental notes handed out in class give results for the position response of a second-order system to an initial condition of $x_0 = 0$ and $v_0 = 1$ for the underdamped, critically damped, and overdamped cases, in Equations (15), (16), and (17), respectively. However, these results are given without showing the derivation. Carry out the detailed calculations to verify these homogeneous responses.

2.4 Second Order Derivation Continued

Repeat the results of Problem 2.3 for the different initial condition $x_0 = 1$ and $v_0 = 0$, showing the details of your derivation, as well as an expression for the position as a function of time for the underdamped, critically damped, and overdamped cases. Use Matlab to plot the resulting position as a function of time for each of the parameter sets of values used in Figures 15, 16, and 17.

3 Higher Order Systems

3.1 Structure of Higher Order System Solutions

As we will see later in the term, the ship roll stabilizer problem can lead to an equation of motion that is 4th order:

$$a_4\frac{d^4x}{dt^4} + a_3\frac{d^3x}{dt^3} + a_2\frac{d^2x}{dt^2} + a_2\frac{d^2x}{dt^2} + a_0x = bu(t)$$

Where x would be the roll angle and u the desired stabilizer fin angle. Right now all we want to do is understand the basic nature and range of possible homogeneous solutions for this type of equation. For this problem we will not use LaPlace transforms, but instead rely on what we already know about homogeneous solutions for linear ordinary differential equations. Assuming that u=0, consider three different sets of parameters . a = a4 a3 a2 a1 a0 [ ]: Case I: . a = 1 10 35 50 24 [ ] Case II: . a = 1 8 42 85 50 [ ] Case III: . a = 1 7 135 550 2500 [ ] Assuming that only x(0) is non-zero:

(a) For each case, write out the basic form of the homogeneous solution where m is the number of complex conjugate root pairs and n-2m is the number of real distinct roots, leaving any real coefficients as undetermined variables.

(b) Describe (Briefly in words!) the resulting characteristic response.

(c) Sketch an approximation of what it will look like, but do not solve for the ci or cj coefficients.

HINT: You might find it useful to use the MATLAB command roots(a), which will find the roots of a polynomial whose coefficients are in a vector a.

3.2 Structure of Higher Order System Solutions

As we will see later in the term, the ship roll stabilizer problem can lead to an equation of motion that is 4th order:

$$a_4\frac{d^4x}{dt^4} + a_3\frac{d^3x}{dt^3} + a_2\frac{d^2x}{dt^2} + a_2\frac{d^2x}{dt^2} + a_0x = bu(t)$$

where x would be the roll angle and u the desired stabilizer fin angle. Right now all we want to do is understand the basic nature and range of possible homogeneous solutions for this type of equation. For this problem we will not use LaPlace transforms, but instead rely on what we already know about homogeneous solutions for linear ordinary differential equations. Assuming that u=0, consider three different sets of parameters. $\mathbf{a} = [a4, a3, a2, a1, a0]$:

Case I: **a** =[1 10 35 50 24]
Case II: **a** = [1 8 42 85 50]
Case III: **a** = [1 7 135 550 2500]

Assuming that only x(0) is non-zero:

(a) For each case, write out the basic form of the homogeneous solution where m is the number of complex conjugate root pairs and n-2m is the number of real distinct roots, leaving any real coefficients as undetermined variables.

(b) Describe (briefly in words!) the resulting characteristic response.

(c) Sketch an approximation of what it will look like, but do not solve for the c_i or c_j coefficients.

HINT: You might find it useful to use the MATLAB command roots(**a**), which will find the roots of a polynomial whose coefficients are in a vector **a**.

4 Mechanical Systems

4.1 Balloon

A string dangling from a helium-filled balloon has it's free end resting on the floor as shown in Fig. 4.1.1. As a result the balloon hovers at a fixed height off the floor and when it is deflected a little from that height it oscillates up and down for a while, eventually returning to the same height. Develop the simplest mathematical model competent to describe the vertical motion of the balloon. You may take all the elements of your model to be linear. Show that your model is competent to describe the observed hovering behavior.

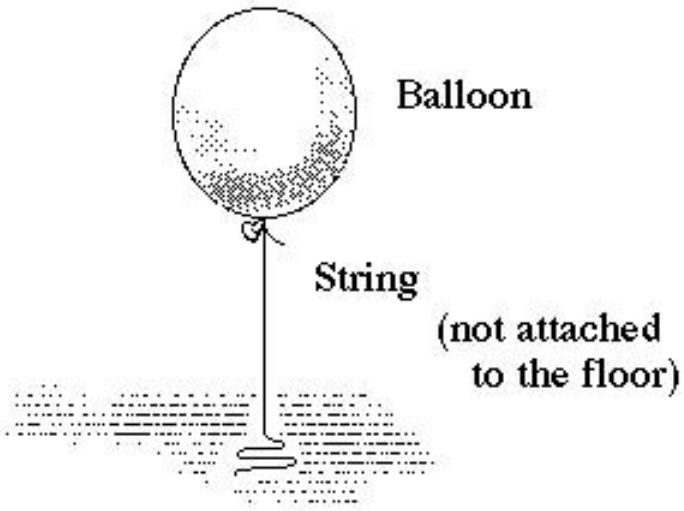


Figure 4.1.1: Hovering Balloon

(a) Show that your model has a steady state.

(b) Show that displacement from the steady state evokes a restoring force.What is the equivalent stiffness parameter?

(c) Show that your model could predict oscillation about the steady state.

(d) Show that your model predicts that the motion will eventually approach the steady state.

The balloon is released from rest at a small distance a above its steady-state height of 3.0 feet. After the first half-cycle of oscillation, which takes 1.5 seconds, the balloon is at a point $0.2a$ below the steady-state height,with an instantaneous velocity of zero. The string is known to weigh 0.5 ounces per foot. Use these data to estimate the following *behavioral* parameters:

(e) the damping ratio ζ;

(f) the undamped natural frequency ω_o;

and the following *model element* parameters:

(g) the effective mass m;

(h) the effective damping coefficient b;

(i) the effective stiffness k.

4.2 Bungee Jumper

A bungee jumper weighs 150 pounds. Bungee cords attached to her ankles have a slack length of 100 feet. She dives off a high tower, the elastic cords extend and instantaneously arrest her motion at a lowest point A (well above the ground). The cords then retract and she bounces through several cycles before finally coming to rest at a point B, where her ankles are 120 feet below the upper attachment point of the bungee cord.

(a) Estimate the location of the low-point A.

(b) During the portion of her jump from the initial take-off to the point A, estimate her maximum *downward* acceleration?

(c) During the portion of her jump from the initial take-off to the point A, estimate her maximum *upward* acceleration.

(d) Briefly explain the assumptions made in obtaining the previous estimates.

Seeking an ever-greater thrill the jumper doubles the slack length of the bungee cords(the new lowest point A' of her jump is still comfortably above the ground). For this second jump

(e) Estimate the location of the low-point A'.

(f) Estimate the location of B' where she finally comes to rest.

(g) During the portion of her jump from the initial take-off to the point A', estimate her maximum *downward* acceleration?

(h) During the portion of her jump from the initial take-off to the point A', estimate her maximum *upward* acceleration?

(i) Make a careful sketch of the time histories of vertical position during the initial downward phase (from initial take-off to point A , or A') for the two jumps.Use the same time and position axes for both curves.

4.3 Elevator Model

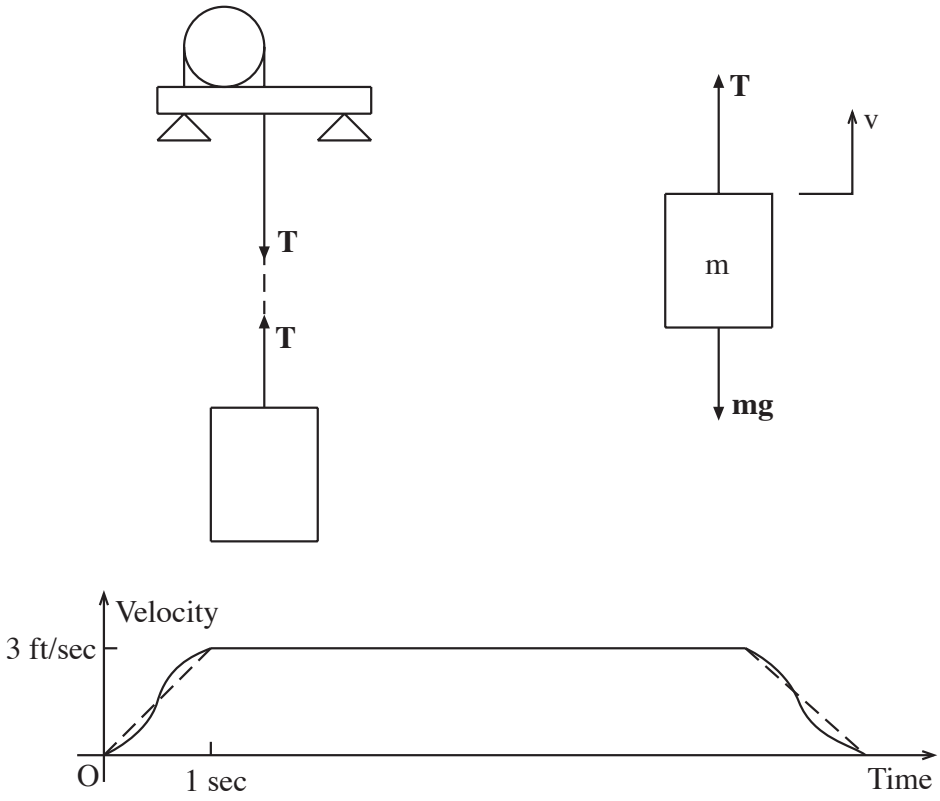


Figure 4.3.1: Elevator Dynamics

An elevator (Fig.4.3.1) is designed to move between floors as follows: starting at rest it accelerates to a speed of 3 feet/second in 1 second, then moves at a constant speed until it decelerates to rest in 1 second. Would it be reasonable to design the structure supporting the winch motor (and hence the elevator) without considering dynamic forces? That is, does dynamics matter in this situation? Provide a quantitative justification for your answer. Work in SI units.

4.4 Hoisting for Engineers

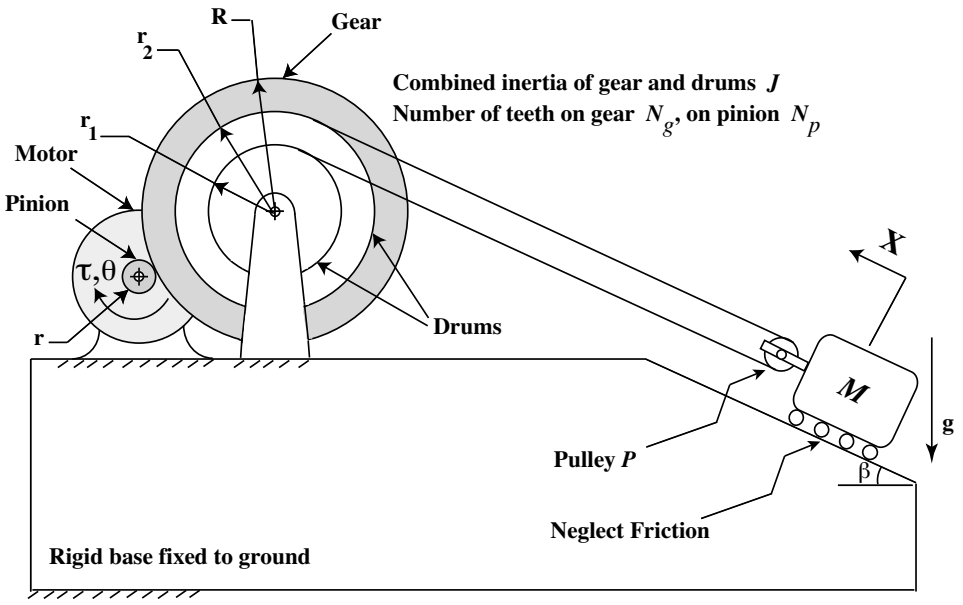


Figure 4.4.1: Hoisting Mechanism

Figure 4.4.1 shows a mechanism for hoisting a mass M up a plane inclined at an angle β to the horizontal. The massless hoisting cable rolls with no slip over a pulley P and over two drums (of radii r_1 and r_2) attached rigidly to a gear of radius R. The combined inertia of the gear and the drums is J.

A motor drives the pinion of radius r with an input torque τ. The transmission ratio of the gear drive is N_g/N_p, where N_g and N_p are the number of teeth on the gear and the pinion respectively. Note that this ratio is the same as R/r.

The system is mounted on a rigid base fixed to the ground. To simplify our model, we neglect (i) friction, (ii) inertia of the motor and the pinion and (iii) mass and inertia of the pulley P. The cable remains under positive tension at all times. X is positive when measured upward along the inclined plane as shown in the figure. Gravity acts on the system in the downward direction.

(a) If the pinion rotates by an angle θ in the clockwise direction as shown in the figure, show that the mass M moves up the inclined plane by a distance X given by

$$X = \frac{1}{2}\,(r_1 + r_2)\,\frac{r}{R}\,\theta \tag{1}$$

Clearly show your reasoning.

(b) Derive the equation of motion for the system. Use and show appropriate free body diagrams showing all the forces and torques acting on

(i) the pinion

(ii) the gear and drum unit

(iii) the pulley P and (iv) the mass M.

Your answer must be a differential equation in X alone, with τ and g as inputs.

(c) Given the values R = 400 mm, r_2 = 300 mm, r_1 = 200 mm, r = 50 mm, M = 50 kg, g = 9.8 m/s^2, $\beta = 30^\circ$, and J = 5 kgm^2, what value of torque τ should be applied to the pinion by the motor so that the mass M has an acceleration of $0.1g$ up the inclined plane?

(d) Now consider the system in Part (d) without the gear transmission drive. The drums are driven directly by the motor in this new system. J, in this case, is the combined inertia of the motor and the drums. If the value of J remains unchanged, calculate the value of torque τ that should be applied to the drums by the motor to achieve the same acceleration of $0.1\ g$ for the mass M.

(e) Assuming that the cost of a motor is proportional to its torque output rating, does your answer in Part (d) support the use of the gear transmission drive? Explain.

4.5 Blocked Springs

Figure 4.5.1 depicts the main components of the suspension for one wheel of an automobile. To change the ride and handling qualities, automobile enthusiasts sometimes insert "blocks" between some of the coils of the spring to prevent that part of it from deflecting. Consider the case where "blocks" are added to immobilize exactly *half* of the coils of each spring. Assume that:

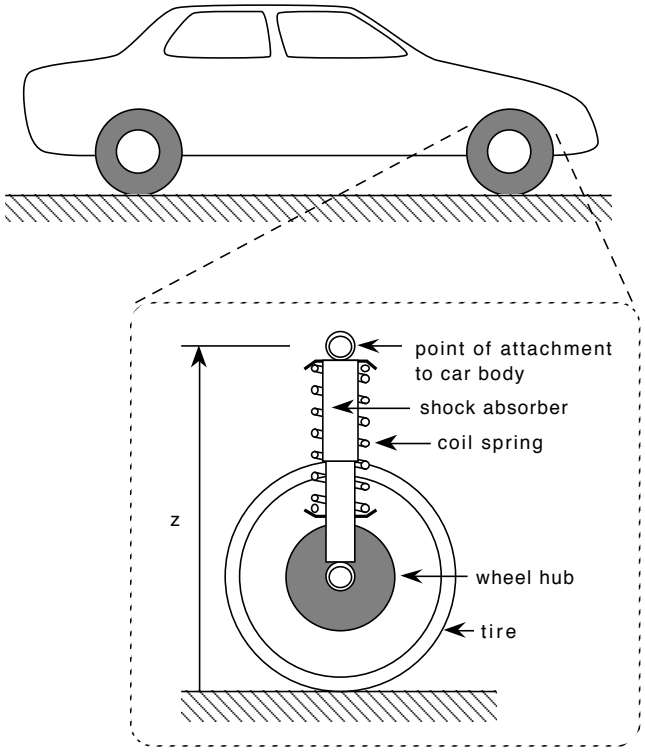


Figure 4.5.1: Automobile Suspension

All four wheels are identical and have identical suspensions.

The car moves vertically as a rigid body.

The tire deflections are negligibly small compared to the spring deflections.

The shock absorbers exhibit linear viscous behavior.

(In practice these assumptions are not especially accurate but they will keep the analysis simple and provide insight to the behavior of the suspension.)
It is known that the vehicle weighs 2,500 pounds, and that before the blocks were added the suspension was critically damped ($\zeta = 1$). After the blocks were added, the ride height was changed by 2.5 inches. Use these data to estimate the following parameters. Work in SI units.

(a) The suspension stiffness before, and after, the blocks were added.

(b) The undamped natural frequency of oscillation before, and after, the blocks were added.

(c) The suspension damping coefficient before the blocks were added.

(d) The suspension damping ratio after the blocks were added.

Write state equations and output equations to compute the response to an abruptly applied vertical load of 1,000 pounds. Adapt the MATLAB scripts below to provide plots of the vertical displacement of the vehicle from its resting height vs. time, for (i) the suspension without blocks, and for (ii) the suspension with blocks added.

4.6 Engine Vibration

A single piston weighs 1 pound and oscillates at frequency Ω with a total stroke (top to bottom of piston travel) of 2 inches.

(a) Assuming that the piston motion is sinusoidal, calculate the amplitude of the sinusoidal force that is required to move the piston at the following crankshaft speeds in revolutions per minute (rpm):

$$\begin{aligned} N &= 1500 \text{ rpm} \\ N &= 3000 \text{ rpm} \\ N &= 6000 \text{ rpm} \end{aligned}$$

The piston in (a) is assumed to be part of an engine, which weighs 200 pounds,and which is supported on a fixed frame by mounts which have an effective stiffness (for vertical motion) of 18,000 pounds/inch, and an effective linear damping coefficient of 2pounds/inch/sec. The oscillating forces considered in (a) are forces acting *on the piston*. By Newton's law of action and reaction, equal and opposite forces act *on the engine block* whenever the crankshaft rotates at the speeds considered. The rest of the problem is devoted to estimating how much vibration of the engine block results from the action of one piston.

(b) Formulate an equation of motion for the engine block in which the output is the displacement of the engine block, and the input is the reaction force from the motion of one piston.

(c) Derive expressions for the amplitude and phase of the steady-state displacement response to piston reaction forces of the form $f(t) = f_a \sin \Omega t$.

(d) Estimate the amplitude of the engine block displacement, due to the motion of one piston, when the engine operates at

$$\begin{aligned} N &= 1600 rpm \\ N &= 1800 rpm \\ N &= 2000 rpm \end{aligned}$$

(e) If these three speeds are generated by starting with the engine block at rest,in equilibrium, and then immediately rotating the crankshaft at the full indicated speed,estimate how long it would take the engine block, in each case, to reach steady state vibration.

4.7 Garage Door

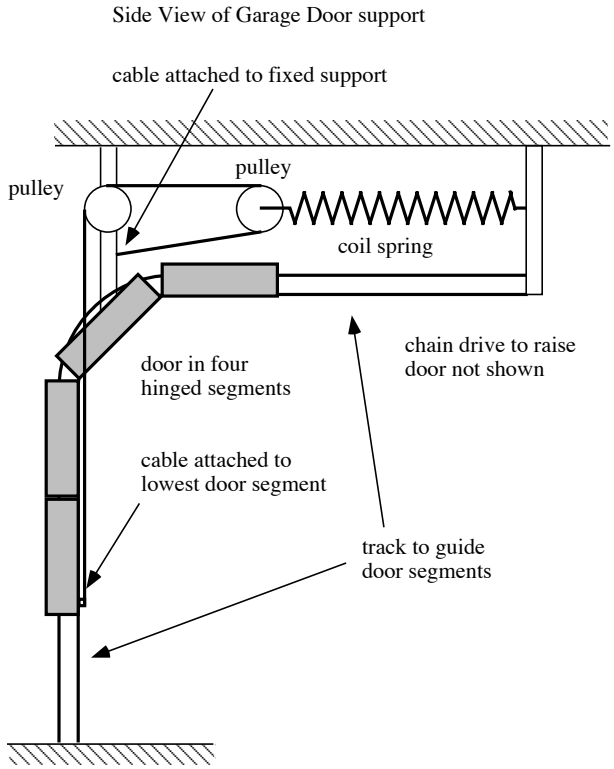


Figure 4.7.1: Schematic of Garage-Door-Support System

Figure 4.7.1 shows a side view of one side of the support mechanism used in many garage-door openers. On each side of the garage door a portion of the weight of the door is supported by a long coil spring attached to the door through a cable and pulley system. It was observed, after the mechanism was recently serviced and freshly lubricated, that the suspended door had a tendency to oscillate up-and-down when disturbed.

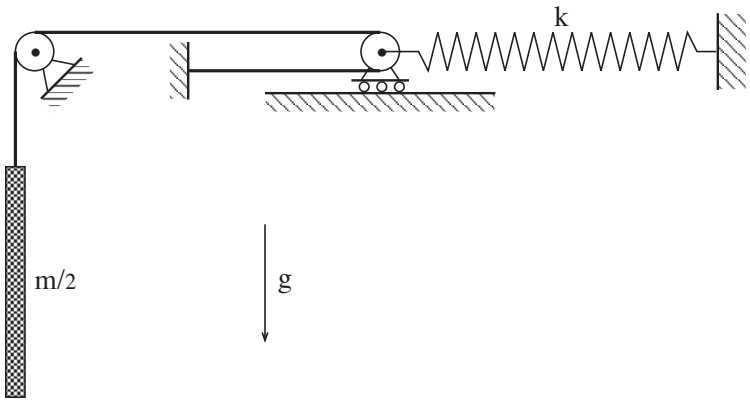


Figure 4.7.2: Simplified Schematic of Garage-Door-Support System

Figure 4.7.2 shows a simplified schematic of one-half of the door support system. Half of the total door inertia is coupled to one of the overhead springs by the pulley system . Take the total weight of the door to be 200 pounds and the stiffness of each spring to be 5 pounds/inch.

(a) Formulate a model to analyze the oscillations of the door.

(b) Estimate the frequency, in Hz, of the oscillations.

(c) List the main assumptions underlying your model.

4.8 Rotating Damped

Consider the rotor with moment of inertia I rotating under the influence of an applied torque T and the frictional torques from two bearings, each of which can be approximated by a linear frictional element with coefficient B (Fig. 4.8.1).

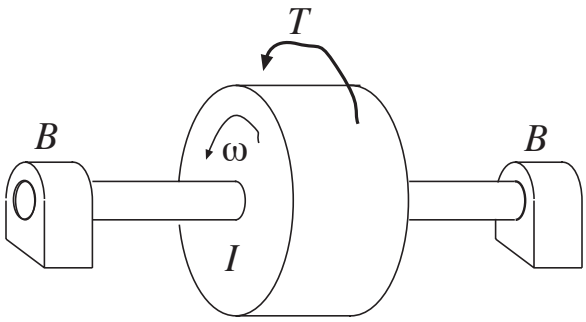


Figure 4.8.1: Rotating Inertia

(a) Formulate the state-determined equation of motion for the angular velocityω as output and the torque T as input.

(b) Consider the case where:

$$\begin{aligned} I &= 0.001 \text{ kg-m}^2 \\ B &= 0.005 \text{ N-m/r/s} \end{aligned}$$

What is the steady-state velocity ω_{ss} when the input is a constant torque of 10Newton-meters?

(c) When the torque T varies sinusoidally at a frequency Ω, the steady-state angular velocity also varies sinusoidally at frequency Ω. Derive a formula for the steady-state angular velocity when $T = T_a \sin \Omega t$. Evaluate the steady-state angular velocity response for the following cases:

(i) $T = 10 \sin(50t)$ Newton-meters

(ii) $T = 10 \sin(5t)$ Newton-meters

(iii) $T = 10 \sin(0.5t)$ Newton-meters

(d) Consider the amplitudes $A(\Omega)$ of the steady-state angular velocity response as the frequency Ω approaches zero. What is the limiting amplitudeA_o as $\Omega \to 0$? Evaluate the ratio $A(\Omega)/A_o$ for $\Omega = 50$ rad/sec, $\Omega = 5$ rad/sec, and $\Omega = 0.5$ rad/sec.

(e) What is the *break frequency* of this system?

(f) Make an accurate, labeled, sketch of the Bode plots for the amplitude ratio$A(\Omega)/A_o$ and the phase angle $\phi(\Omega)$ of this system.

4.9 Car Suspension 1

For the second-order car suspension model shown in Figure 4.9.1, perform the following tasks:

(a) Solve for the unit step response from zero initial conditions. Write an expression for the response as a function of time, and use Matlab to graph this response. For this problem, use the parameters: $m = 500$ kg, $k = 5 \times 10^4$ N/m, and $c = 2 \times 10^3$ Ns/m.

(b) For these parameters, the system will be underdamped. What are the values ofω_n, ζ, ω_d, and σ?

(c) Comment on the quality of the response to the step. Will the passengers like this ride? For what value of c would the system have critical damping? Recalculate the step response for this critical value of damping, and compare the maximum resulting acceleration experienced by the passengers to the underdamped case.

(d) Compare the 5% settling time for both cases.

(e) Use a Laplace transform approach to solve for the time response. Of course, this result should be the same as what you found in part (a).

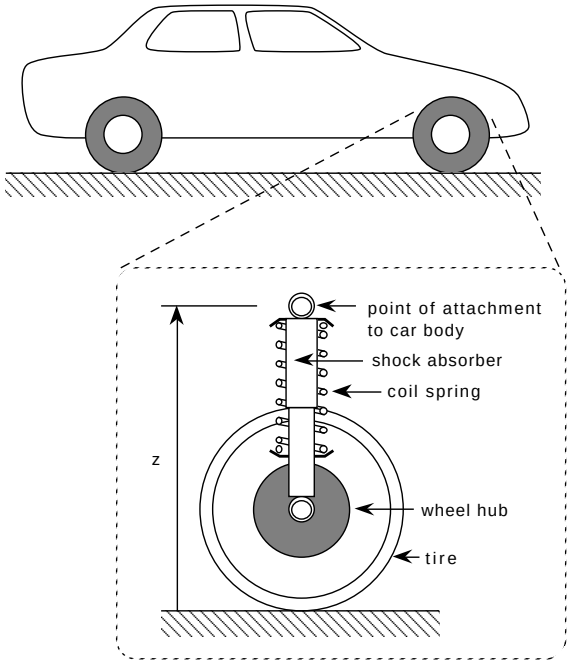


Figure 4.9.1: Car Suspension

4.10 Disk Drive

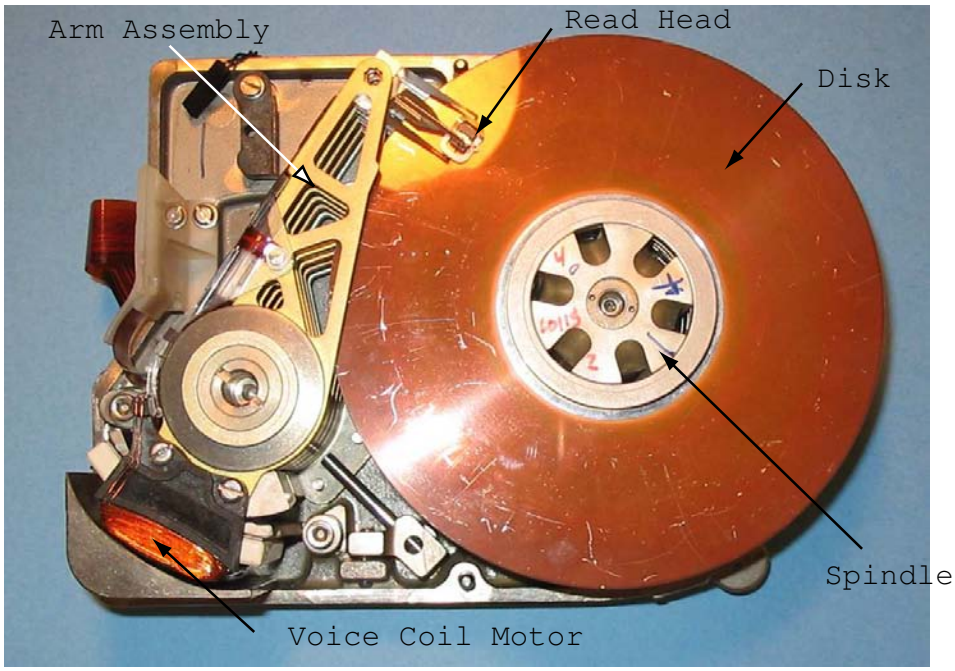


Figure 4.10.1: Picture of Hard Disk drive

A hard disk drive as shown in Figure 4.10.1 has two main components:

1. The head/arm assembly which moves the read/write heads laterally over the disk surface to the desired tracks. The arm is driven by a voice coil motor.
2. The spindle/disk assembly which is driven by a permanent magnet spindle motor which rotates at near constant speed.

This problem studies the spin-up/spin-down transients of the spindle assembly. Assume that the spindle assembly has a rotational inertia J, and that the motor acts as a source of torque τ_m, which is constant independent of speed. Further assume that we can model the air drag acting on the spindle as linearly dependent upon the angular velocity ω, with a rotational damping coefficient b.

(a) For this system draw a free body diagram for the spindle inertia J showing the torques acting on it.

(b) Use this free body diagram to derive a differential equation in terms of the spindle angular speed ω, an input torque τ_m, and using the parameters given above.

(c) Now assume that the spindle inertia is 10^{-3} kgm^2. (This is a big disk drive from an older computer!) We experimentally observe that the disk spins down with a time constant $\tau = 0.5$ sec. What is the numerical value of the associated damping term b?

(d) Finally, assume that the disk is initially at rest at $t = 0$, is spun up by the motor using a constant torque of 0.5 Nm. What is the resulting transient $\omega(t)$? Make a dimensioned graph of this response.

(e) After a long time, at $t = t_1$, the motor is turned off and exerts no torque on the spindle. Write an expression for $\omega(t)$ for $t > t_1$, and make a dimensioned graph of the response.

4.11 Crashworthiness

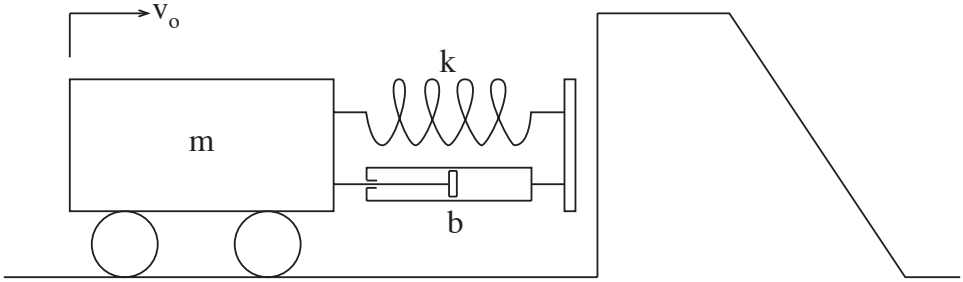


Figure 4.11.1: Model of Vehicle Impacting a Barrier

A vehicle weighing 1 ton is driven into a fixed concrete barrier at 10mph. The vehicle's fender, which strikes the barrier first, is of the type that can deform under this load and return to its original shape (undamaged) when it is unloaded. If the fender were unable to dissipate energy, its maximum deflection would be 6 inches.

(a) Estimate the effective stiffness of the fender.

(b) Estimate the peak deceleration of the vehicle in SI units.

To absorb collision energy, linear dampers are added to the fender. Write state equations and output equations to predict the dynamics of the vehicle-fender system while the fender is in contact with the barrier. The outputs should include:

(i) The deflection of the fender.

(ii) The deceleration of the vehicle.

(iii) The total force exerted on the barrier.

Adapt the MATLAB script "pos2.m" and "eqpos2" shown below to integrate these state equations to find the response of these outputs starting from the moment the fender first contacts the barrier. Make plots of the time histories of these three outputs for the following values of the damping ratio ζ:

(c) $\zeta = 0.25$

(d) $\zeta = 0.50$

(e) $\zeta = 0.75$

(f) $\zeta = 1.00$

(g) In which, if any, of the cases (c) through (f) does the fender remain in contact with the barrier after the impact is over?

(h) Can the peak deceleration of a vehicle with a fender with damping ever be greater than the peak deceleration of a vehicle with an undamped fender? Give a brief physical explanation for your answer.

(i) In which of the cases (c) through (f) is the peak deceleration the greatest?

(j) Estimate the value of the damping ratio ζ which would *minimize* the peak deceleration.

4.12 Nonlinear Rotational System

Figure 4.12.1 shows a rotational inertia and damper system. The moment inertial of the rotor is I. The damping torque from the damper is a nonlinear function of angular velocity,$T_c(\omega) = a\omega^3$. For each of three inputs (i) $\overline{\tau_1} = 1$, (ii) $\overline{\tau_2}$=8 and (iii) $\overline{\tau_3}$=64

(a) Find the equilibrium point $\overline{\omega}$

(b) Derive a linearized model about the equilibrium point

(c) Solve for roots and plot them on the complex place.

(d) Plot the response of ω for step input $\tau = \overline{\tau_i}(1 + 0.01u_s(t))$.

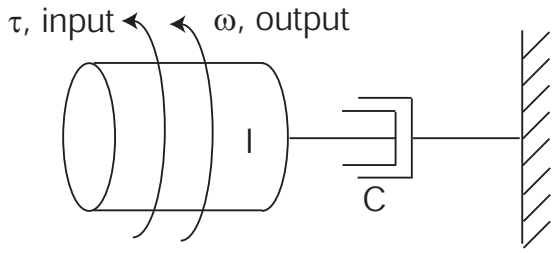


Figure 4.12.1: Nonlinear Rotational System

4.13 Developing Differential Equations

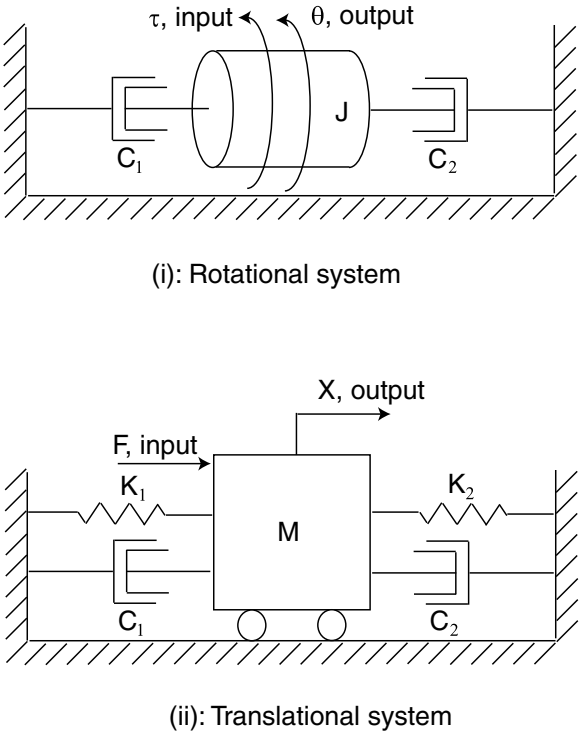


Figure 4.13.1: System Figures

For each of the systems shown in Figure 4.13.1:

(a) Separate the system at a node or nodes into a free body diagram to show the forces acting on each element.

(b) Use the free body diagram to develop a differential equation describing the system in terms of the indicated input and output. For each system, what is the system order?

4.14 Mass Spring System Frequency Response

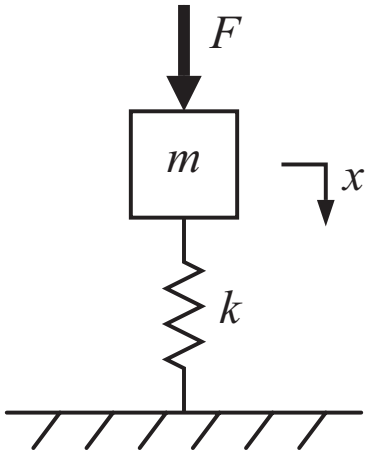


Figure 4.14.1: Mass Spring System

In this problem, $m = 1$ [kg], $k = 100$ [N/m] and $F(t) = \sin \omega t$ [N].

(a) Calculate an expression for the steady-state response $x(t) = M \sin(\omega t + \phi)$, with expressions for $M(\omega)$ and $\phi(\omega)$.

(b) Make hand sketches of $M(\omega)$ versus ω on log-log coordinates and $\phi(\omega)$ versus ω on semi-log coordinates (linear in phase and log in frequency).

4.15 Modeling Practice

This problem concerns the spring-mass-damper system shown in Figure 4.15.1.

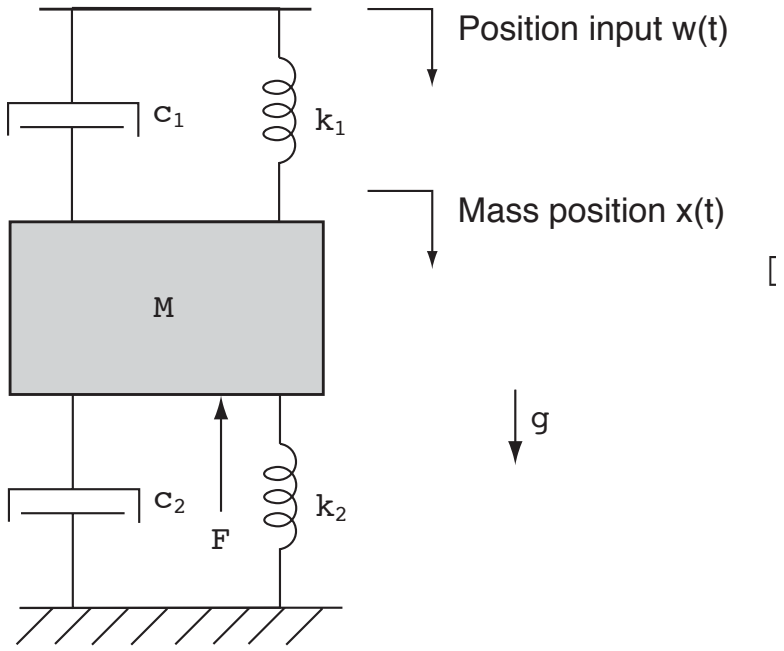


Figure 4.15.1: Sprin-Mass-Damper System

In this figure, gravity acts on the mass in the downward direction as shown. The position of the mass in the downward direction is $x(t)$. A force F acts on the mass in the upward direction. The upper spring and damper are connected to a position source $w(t)$. That is, the position $w(t)$ is specified as in independent input. The position x is defined to be zero when gravity is not acting on the mass, and when the applied inputs are zero.

(a) Draw a free-body diagram for the mass which shows all the forces and associated reference directions acting on the mass. Be sure to label the forces with their dependence on the system position/velocity variables (if any).

(b) Use this free-body diagram to derive a differential equation in terms of x and the system parameters and inputs which describes the dynamics of this system.

4.16 Small Motion Transfer Function

For the system shown in Figure 4.16.1, calculate the transfer function $X(s)/F(s)$, under the assumption of *small* motions. Clearly show the steps in your derivation. The massless linkage has lever arms l_1 and l_2 as shown.

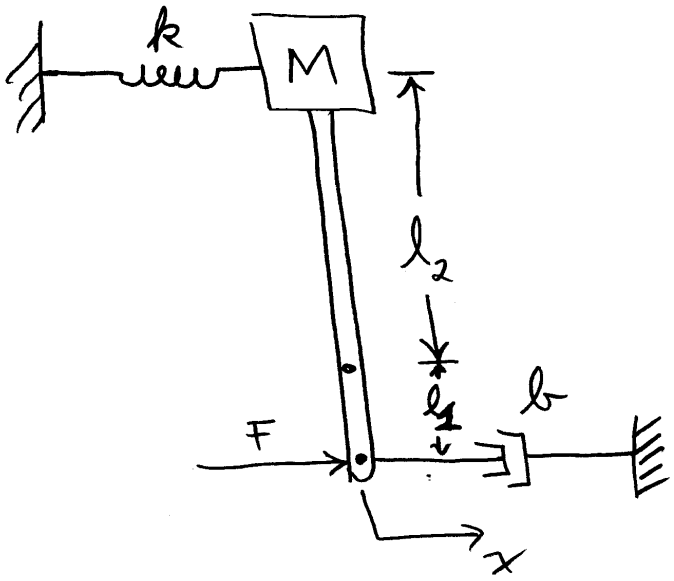


Figure 4.16.1: Small Motion System

4.17 Mass Spring Damper System Frequency Response 1

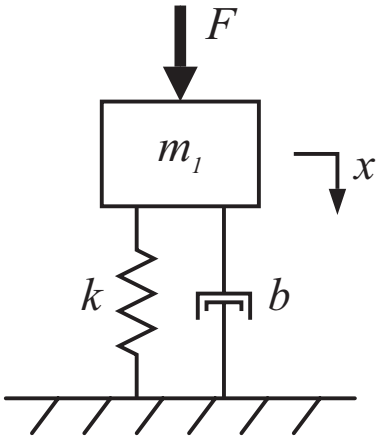


Figure 4.17.1: Mass Spring Damper System

In this problem, $m = 1$ [kg], $k = 100$ [N/m], $b = 1$ [Ns/m] and $F(t) = \sin \omega t$ [N].

(a) Calculate an expression for the steady-state response $x(t) = M \sin(\omega t + \phi)$, with expressions for $M(\omega)$ and $\phi(\omega)$.

(b) Make hand sketches of $M(\omega)$ versus ω on log-log coordinates and $\phi(\omega)$ versus ω on semi-log coordinates (linear in phase and log in frequency).

4.18 Mass Spring Damper System Frequency Response 2

Consider the mechanical system shown in Figure 4.18.1. Note that F acts on m_1 in the direction of x_1.

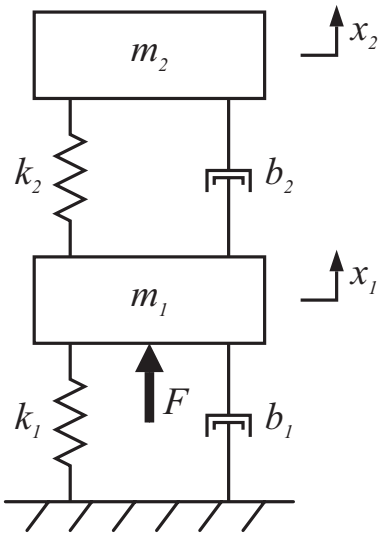


Figure 4.18.1: Mass Spring Damper System

(a) Calculate the transfer functions $H_1(s) = X_1(s)/F(s)$ and $H_2(s) = X_2(s)/F(s)$ in terms of the given parameters.

(b) Now let $m_1 = 25$ [kg], $m_2 = 1$ [kg], $k_1 = 100$ [N/m], $k_2 = 10^4$ [N/m] and $b_1 = b_2 = 1$ [Ns/m]. Use MATLAB to plot the Bode plots for H_1 and H_2. Also plot the poles/zeros for both transfer functions. Relate features on the Bode plots to the pole/zero locations and the damping ratio of the poles and zeros. For a unit sinusoidal input $F(t)$, at what frequency is the motion on m_1 a relative maximum? What is the magnitude of motion on m_2 at this frequency? For a unit sinusoidal input $F(t)$, at what frequency is the motion on m_1 a relative minimum? What is the magnitude of motion on m_2 at this frequency?

4.19 Propeller Shaft Vibration

This is a different kind of vibration problem for the light aircraft engine we have been considering. Previously, we considered uniaxial translation of the engine block due to the inertia loading from accelerating pistons. We now consider oscillations in the rotational speed of the propeller due to torsional vibration of the short elastic coupler shaft connecting the propeller to the crankshaft. The source of the oscillation is the fluctuating speed generated by the reciprocating engine.
Periodic firing of the cylinders in an internal combustion engine causes its rotational speed to vary periodically. One stroke of a piston is one move from top dead center to bottom dead center (or from bottom to top). In a four-stroke engine, three strokes are used to clear out the products of combustion from the previous firing, let in fresh air and fuel, and compress the mixture prior to firing. It is only in the fourth stroke that the explosion occurs and a very large force on the piston exerts torque around the axis of the crankshaft. Ina four-cylinder engine the resulting torque on the crankshaft is smoothed out considerably by arranging it so that one cylinder fires on every stroke. The remaining fluctuation in torque, when applied to the inertia of the crankshaft, results in a fluctuation in the output speed of the engine, which varies in an approximately sinusoidal manner at the firing frequency, which is half the rotational frequency of the crankshaft. It is this periodic engine speed fluctuation which excites the torsional vibration.
Consider the case of a four-cylinder 150 horsepower engine which operates between 500 and 2700 rpm. The moment of inertia of the two-bladed propeller can be estimated to be the same as that of a uniform solid rod of aluminum, six feet long and two inches in diameter (the density of aluminum is 2.72grams/cc). It is observed that the steady-state oscillations of propeller speed at the firing frequency reach a peak amplitude when the engine runs at2200 rpm. Furthermore the magnitude of the oscillation at 2200 rpm is four times larger than the magnitude at 500 rpm.

(a) Develop a model to describe the steady-state fluctuations in propeller speed(output) in response to the fluctuations in engine speed (input). To keep the analysis simple, assume that the amplitude of the engine-speed fluctuations delivered to the coupler shaft are independent of the engine speed, so that in the steady state the angular position θ_{eng} and the angular speed $\omega_{eng}$of the engine can be assumed to take the form

$$\begin{aligned} \theta_{eng} &= \Omega t + \epsilon \sin \frac{\Omega}{2} t \\ \omega_{eng} &= \Omega(1 + \frac{\epsilon}{2} \cos \frac{\Omega}{2} t) \end{aligned}$$

(b) Use your model to estimate the torsional stiffness K of the elastic coupler shaft.

4.20 Safe Packaging

A packing crate was designed to protect a fragile instrument during shipment. Assuming that the packing material can be modeled as an ideal linear spring of stiffness, k , in parallel with an ideal linear damper, b , and that the instrument and crate are of mass, m_1 and m_2 , respectively, the system can be modeled as shown in Figure 4.20.1A.

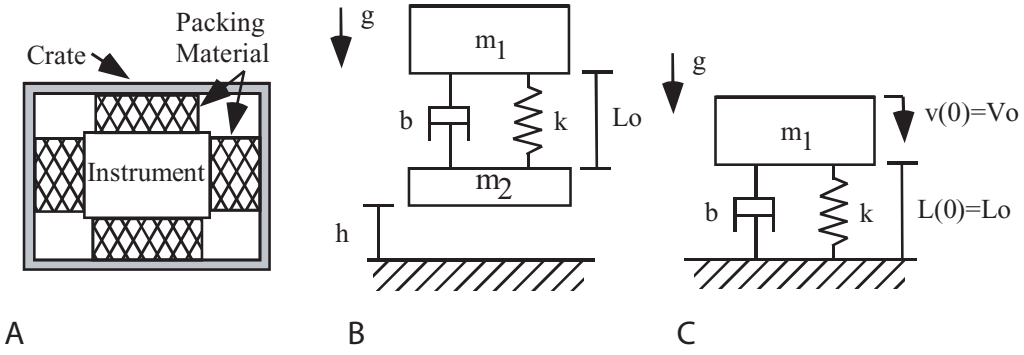


Figure 4.20.1: Shows actual system and two models of the situation.

The packing crate (with instrument inside) is dropped from a height, h , as shown in Figure 4.20.1B. The height is sufficiently large that by the time the crate hits the ground, the spring is fully extended to its unloaded length, L_o , as shown in Figure 4.20.1C. Note that the crate hits the ground with velocity, V_o , and in the presence of gravity.

(a) Derive the differential equation for the system. Clearly indicate the initial conditions, and any inputs present.

(b) For what values of, b , will the instrument oscillate?

(c) Assuming that the instrument does oscillate, derive an analytical expression for the complete solution.

4.21 Sliding Damped

Consider the mass m sliding horizontally under the influence of the applied force f and a friction force which can be approximated by a linear friction element with coefficient b (Fig. 4.21.1.

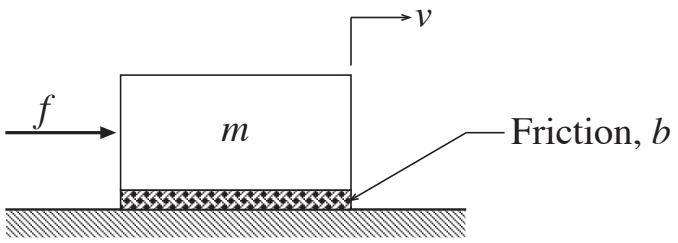


Figure 4.21.1: Sliding Mass

(a) Formulate the state-determined equation of motion for the velocity$v$ as output and the force f as input.

(b) Consider the case where:

$$\begin{aligned} m &= 1000 \text{ kg} \\ b &= 100 \text{ N/m/s} \end{aligned}$$

What is the steady-state velocity v_{ss} when the input is a constant force of 10Newtons?

(c) When the force f varies sinusoidally at a frequency Ω, the steady-state velocity also varies sinusoidally at frequency Ω. Derive a formula for the steady-state velocity when $f = f_a \sin \Omega t$. Evaluate the steady-state velocity response for the following cases:

(i) $f = 10 \sin(0.5t)$ Newtons

(ii) $f = 10 \sin(0.05t)$ Newtons

(iii) $f = 10 \sin(0.005t)$ Newtons

(d) Consider the amplitudes $A(\Omega)$ of the steady-state velocity response as the frequency Ω approaches zero. What is the limiting amplitude A_o as $\Omega \to 0$? Evaluate the ratio $A(\Omega)/A_o$ for $\Omega = 0.5$ rad/sec, $\Omega = 0.05$ rad/sec, and $\Omega = 0.005$ rad/sec.

(e) What is the *break frequency* of this system?

(f) Make an accurate, labeled sketch of the Bode plots for the amplitude ratio$A(\Omega)/A_o$ and the phase angle $\phi(\Omega)$ of this system.

4.22 Toy Flywheel

A toy consists of a rotating flywheel supported on a pair of bearings as shown in Figure 4.22.1. The flywheel is connected to a pulley , around which is wrapped a flexible but inextensible cable connected to a spring. In operation, the flywheel is initially at rest, the string made taut, and at $t = 0$, the input $x_s(t)$ undergoes a step change in position of magnitude x_0.

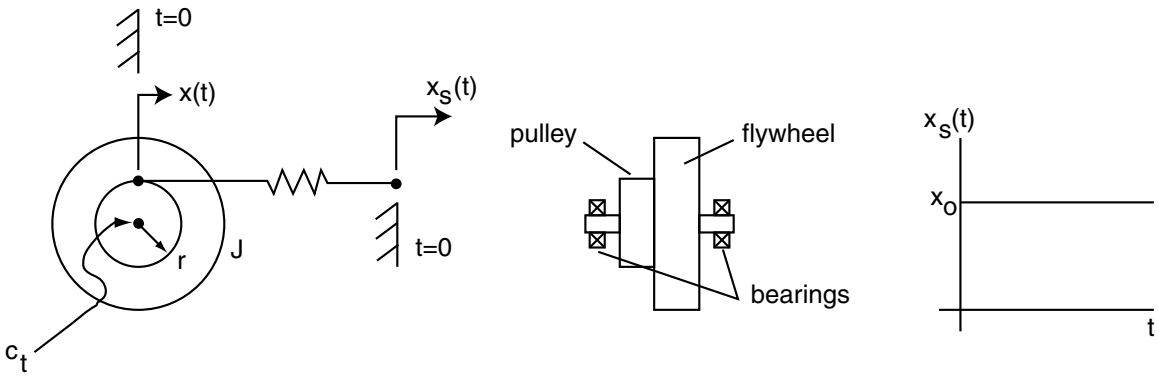


Figure 4.22.1: Flywheel Diagrams

Assume that the flywheel-shaft-pulley unit has rotational inertia, J. The bearings can be modelled as a viscous rotational damper of coefficient, c_t. the pulley is of radius, r. The spring is an ideal linear spring with spring constant, k.

(a) Write the system equation as a differential equation in $x(t)$, the length of cable unwound from the pulley, as well as system parameters, J, r, and k. [Note that $x(0) = 0$]

(b) For what range of values of c_t (expressed in terms of system parameters J, r, and k) will the cable never go slack?

(c) Assuming that c_t has some non-zero value such that the cable **does** go slack, write an expression (in terms of system parameters J,r, and k) for the response $x(t)$, i.e. the length of cable unwound from the pulley. Sketch the response, x(t). Carefully indicate the time over which the expression and the sketch are valid.

(d) Assume that c_t is zero. Write an expression (in terms of system parameters J, r, and k) for t, the time at which the cable first goes slack.

4.23 Truck Shocks

At a tailgate party, it was observed that when Uncle Massive, a sprightly 250-pounder,hopped up on the back of his pick-up truck, the truck suspension bounced for more than 5clearly defined cycles (it has been some time since Uncle Massive has replaced his shock-absorbers) at a frequency of 2 Hz. When the vibration stopped it was noted that the back of the truck was one inch lower than it was before Uncle Massive sat down.

(a) Estimate the effective mass of the back end of the truck.

(b) Briefly explain any assumptions you have made in arriving at your estimate.

4.24 Car Suspension 2

A simplified model for a car suspension is shown in Figure 4.24.1. Here, the road position is $r(t)$, and the car body position is $x(t)$. A force F is applied to the body mass. Assume $r(t) = 0$ throughout this problem. We also let the force be a step function $F(t) = 500u_s(t)$ [N].

(a) Solve for the resulting step response $x(t)$ from zero initial conditions. Write an expression for the response as a function of time, and use Matlab to graph this response. For this problem, use the parameters: $m = 500$ kg, $k = 5 \times 10^4$ N/m, and $b = 2 \times 10^3$ Ns/m.

(b) For these parameters, the system will be underdamped. What are the values of ω_n, ζ, ω_d, and σ?

(c) Comment on the quality of the response to the step. Will the passengers like this ride? For what value of b would the system have critical damping? Recalculate the step response for this critical value of damping, and compare the maximum resulting acceleration experienced by the passengers to the underdamped case.

(d) Compare the 5% settling time for position $x(t)$ for both cases.

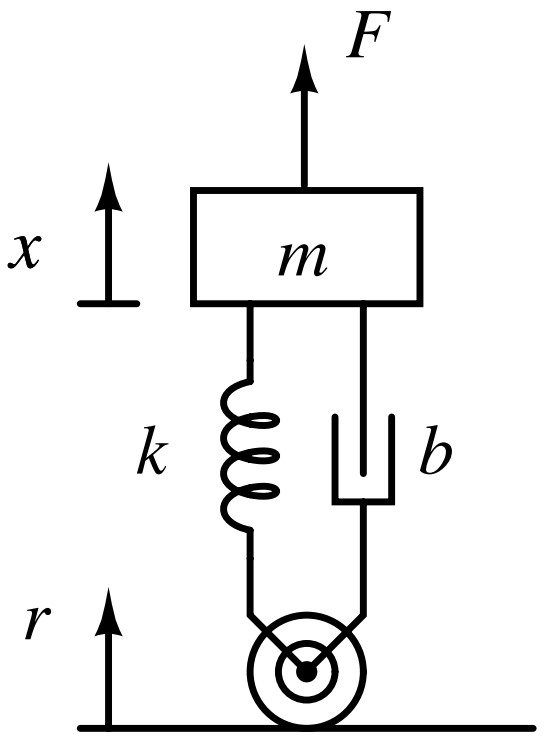


Figure 4.24.1: Car suspension quarter model

4.25 Kid-Skid

You have a great new business idea! Sick and tired of seeing kids injured when their friends jump off the see-saw (teeter-totter, etc), you invent the Kid-Skid®. The Kid-Skid® is simply a spring damper mechanism that insures a soft landing if the friend on the other end jumps off. A schematic is shown in Figure 4.25.1:

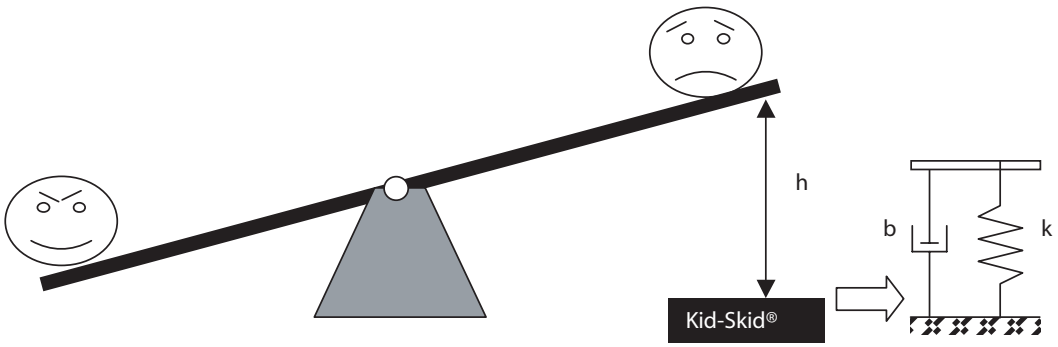


Figure 4.25.1: Kid-Skid® Schematic

Assuming that the see-saw has an length L, and a mass m_s, and that the kid on the right hand side is of mass m_k:

(a) How would you model this system, for the case where the nasty kid on the left jumps off while the poor kid on the right is at the full height h above the ground. Since we do not know how to model impact(wait until 2.004 for that), assume that the Kid-Skid® is massless.

(b) If you want to write the equation of motion for the kid-seesaw–Kid-Skid® system once contact has been made, what would the equation be, and what would be the initial conditions. (Be specific about the initial conditions, and express them in terms of the system parameters given.)

(c) If the Kid-Skid® has a full range of motion = y_{max}, how would you ensure that it was not exceeded given the parameters above?

4.26 Rolling Machine

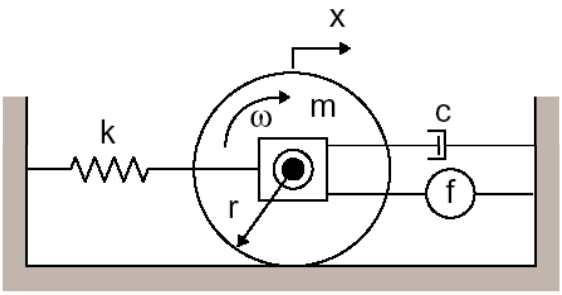


Figure 4.26.1: Rolling Machine Schematic

A model of a commercial rolling machine (for metals processing) has been created and is shown above. The model comprises a cylinder with a mass, m, with a radius, $r = 0.5$ m, that spins about an axle. The cylinder rolls without slip on the lower surface. Attached to the axle housing are a damper, b, a spring, $k = 200$ N/m, and a force source, f. In testing the machine, to impose a step on the force source f, and measure the following response for the horizontal motion of the axle, x.

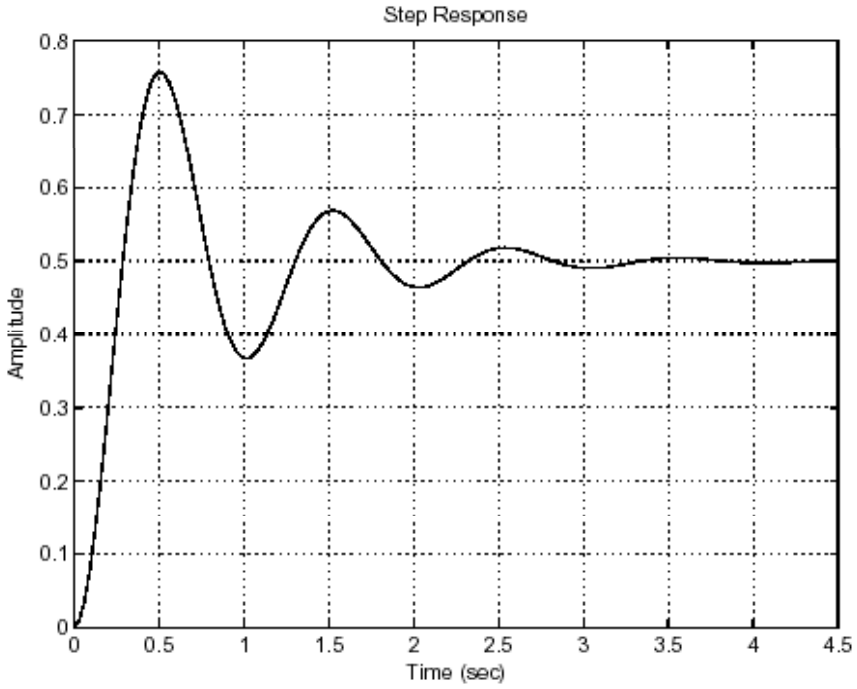


Figure 4.26.2: System Step Response

Using the given parameters and the measured step response in Figure 4.26.2,

(a) Show the a free body diagram of the mass/inertia, and determine the equation of motion for x(t)

(b) From the data determine the damping constant, b, and the equivalent mass, m_{eq}, where m_{eq} is mass equivalent of the combined rotational inertia and linear mass.

(c) If the cylinder has a $m = 3$ kg, determine the inertia of the cylinder.

4.27 Sunday Bike Ride

A couple is out for a Sunday bike ride. The guy (weighing 200 lbs) is on a 30 yr old, all steel (very heavy) 10 speed, and his wife (weighing 100 lbs) is on a lightweight racing bike with high-pressure tires, aluminum rims; etc. They are riding alongside each other when they reach a significant downhill. They agree to coast and not pedal or brake on the hill. At the bottom, the guy ends up very far ahead of his wife. They are both confounded; why spend $1000 on her fancy bike if this is going to happen?

You must explain it using system dynamics.

(a) Create a model for the bike that will lead to a first order equation of motion (even though it has two rotating wheels with inertia + the inertia of the frame and rider[1]). Consider all sources of friction or drag on the bike as it is pulled downhill.

(b) Draw the free body diagram for the bike that will show all forces acting on it.

(c) Write the equation of motion and justify the assumption that the system is linear.

(d) Based on the description of the bikes and the riders, use the model to show why the hefty guy on a clunky bike ends up going faster at the bottom of the hill.

[1]Remember we said we could only deal with "fixed axes of rotation" using 2.003

4.28 Compound Mass Spring Damper System 1

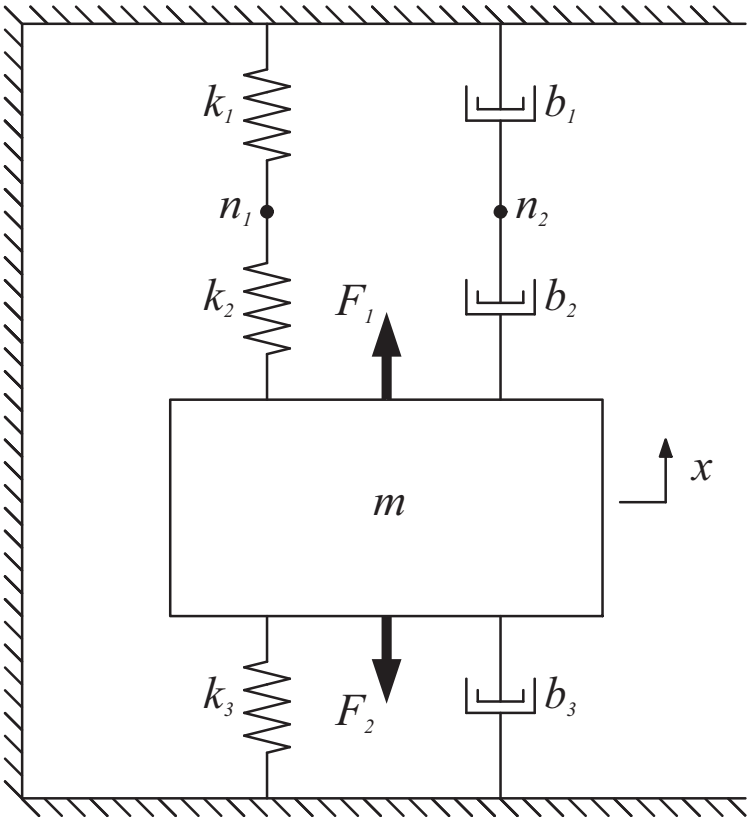


Figure 4.28.1: Mass Spring Damper System

Refering to Figure 4.28.1 and neglecting gravity, please do the following:

(a) Draw free body diagrams for the mass m, and each of the nodes n_1 and n_2.

(b) Derive the 2^{nd} order differential equation for the position $x(t)$ of the mass m.

(c) Using the following parameters

$$\begin{aligned} m &= 1[kg] \\ k_1 = k_2 &= 10000 \text{ [N/m]} \\ k_3 &= 5000 \text{ [N/m]} \\ b_1 = b_2 &= 10 \text{ [Ns/m]} \\ b_3 &= 5 \text{ [Ns/m]} \end{aligned}$$

give values for the damped natural frequency ω_d and the damping ratio ζ. Plot the poles of this system on the complex s-plane.

(d) Using MATLAB, plot the response of this system for the following inputs and initial condi-

tions:

$$
\begin{aligned}
x(0) &= 10^{-2}\ [\mathrm{m}] \\
\dot{x}(0) &= 0\ [\mathrm{m/s}] \\
F_1 = F_2 &= 0\ [\mathrm{N}]
\end{aligned}
$$

4.29 Compound Mass Spring Damper System 2

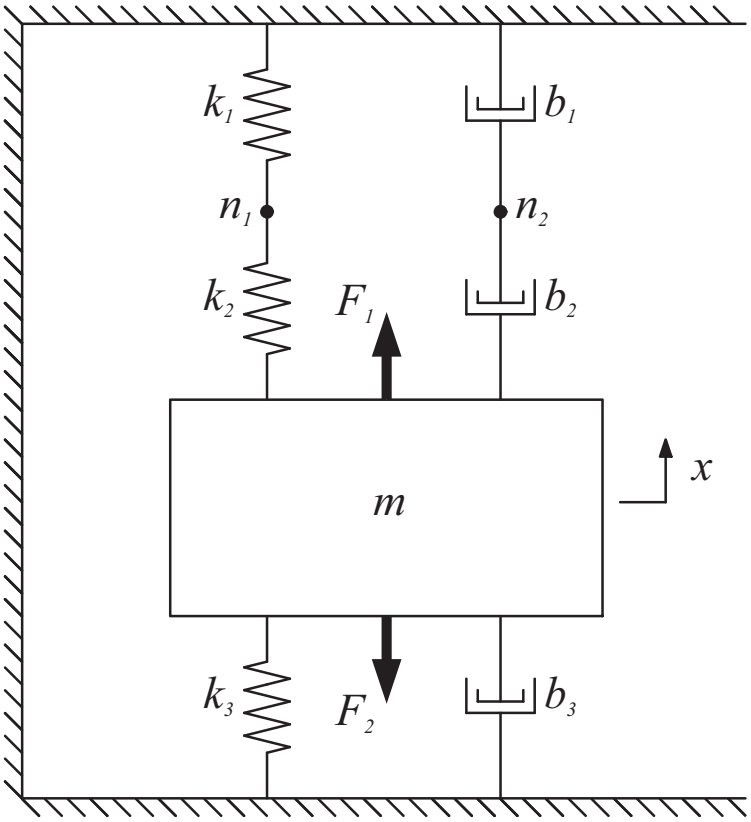


Figure 4.29.1: Mass Spring Damper System

This problem is a continuation of Problem 4.28. You should use the solutions for that problem to aid you in this problem. Use the same parameters for the mass, damping coefficients and spring constants. Referring to Figure 4.29.1 and neglecting gravity, solve $x(t)$ and sketch the solution by hand using the following parameters:

$$F_1 = 10 \text{ [N]}$$
$$F_2 = 0 \text{ [N]}$$
$$x(0) = \dot{x}(0) = 0$$

4.30 Wind Induced Building Vibrations

Tall buildings are often subject to noticeable "sway" or horizontal motion at the upper floors. This sway is driven by wind pressure variations on the building. The building itself is an interconnected set of columns and beams, all with continuously distributed mass and elasticity. Assume that we want to create a simple, lumped parameter model of this building to capture the sway motion dynamics.
A measurement was made of this motion along with wind velocity, and the data is shown in Figures 4.30.1 and 4.30.2.

(a) Propose a mechanical model to describe the motion that would lead to the equation:

$$\ddot{x} + a\dot{x} + bx = cu$$

where x is the sway motion of the building and u is the force from the wind. Be sure to provide a simple sketch of your equivalent mechanical model and the resulting free body diagram.

(b) From the data, which of the parameters of the equation (a, b and c) could you determine? (Or in other words, why can't you determine all three?)

(c) For this data, determine: ω_n, ζ, σ, and ω_d.

(d) Plot the roots of the characteristic equation for this system on an s-plane. A highly paid consultant has said that she can double the damping of the building with a feedback control system.

(e) On the s-plane plot from d) show where the new roots would end up if she succeeds.

(f) Describe how the building would have reacted to the increased damping, and sketch the response on a plot with scales similar to Fig. 2. Please make your description in terms of 2nd order system transient characteristics.

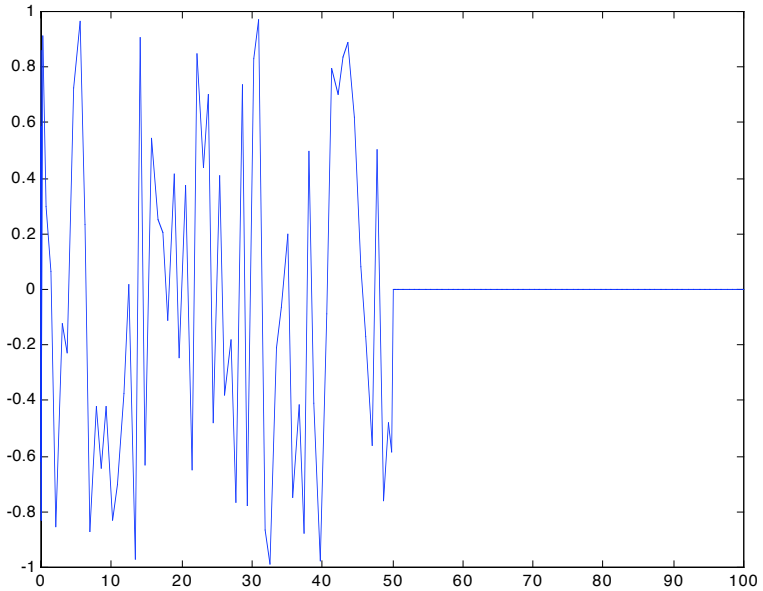


Figure 4.30.1: Wind pressure versus time. (Note the wind stops suddenly at $t = 50$.)

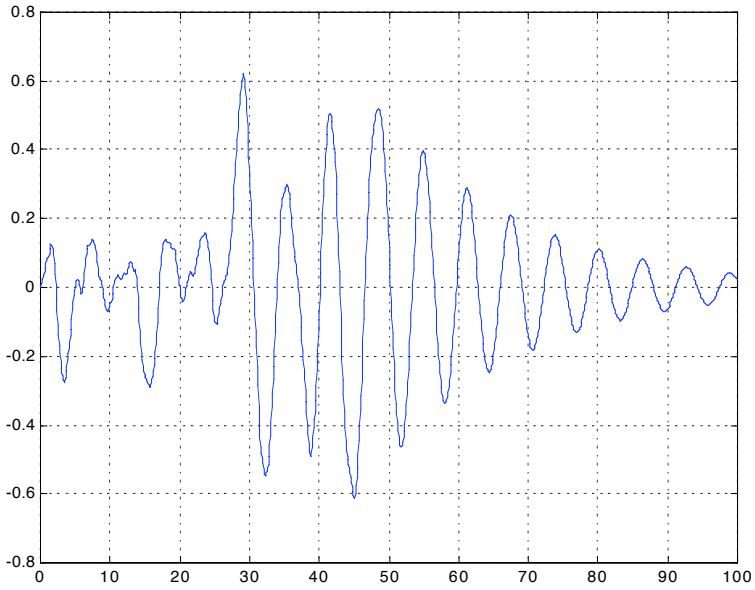


Figure 4.30.2: Building Motion ($x(t)$) vs. time.

5 Electrical Systems

5.1 Camera Flash Circuit

This problem considers further the flash circuit described in class. Suppose that the capacitor has a value $C = 120\mu$F, and the battery voltage is $V_{bat} = 1.2$V. Also suppose that the largest inductor you can use is $L = 1$ mH, and that this inductor has a maximum current limit of 0.1 A.

(a) How much energy can be transferred from the inductor to the capacitor in one cycle?

(b) How many cycles would be required to charge the capacitor from 0 to 300 volts? What frequency of charging (in inductor charge/discharge cycles per second) would be required to do this in one second?

(c) What is the maximum charging frequency in cycles per second which is possible given the parameters for L, C, and V_{bat} above? Explain your reasoning. What changes could you make to the components to allow faster charging?

5.2 Batteries and Capacitors

With terminals shorted, a battery has a short circuit current defined as I_{sc}. With its terminals open, it has an open-circuit voltage defined as V_{oc}. For example, a 1.5V AA cell as shown in the figure 5.2.1

Figure 5.2.1: AA Battery

can be modeled internally as a 1.5 V voltage source in series with an internal resistance R_b as shown in the figure 5.2.2

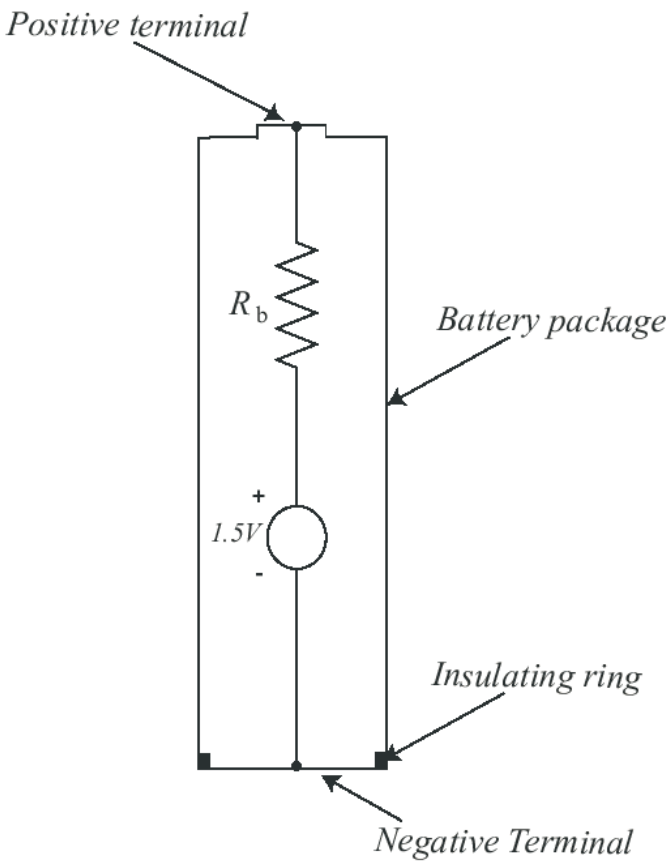


Figure 5.2.2: AA Battery Schematic

If we eliminate the packaging details, this circuit appears more simply as figure 5.2.3, where V_b is the voltage appearing at the battery terminals.

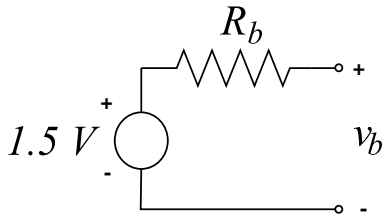


Figure 5.2.3: Simplified battery circuit

(a) With a short briefly applied to such a 1.5 V battery, I measured a current out of the + terminal and back into the - terminal of $I_{sc} = 5.7A$. What is the value of R_b?

(b) In the experiment above, how much power is the battery voltage source supplying? Where is it being dissipated? (Batteries are really not happy having a short-circuit applied to them for any significant time, as they can dangerously overheat due to the internal dissipation.).

An electrolytic capacitor, as shown in the picture 5.2.4, allows a relatively large amount of capacitance in a small package as compared with other types of capacitors. These capacitors

are polarized, in the sense that they should only have voltage applied of the indicated polarity. Reversal of voltage can be hazardous, and cause failure or explosion.

Figure 5.2.4: Electrolitic capacitor of a single use flash camera

The capacitor shown is used to provide a pulse of current used to fire a flash lamp in a single use camera.

After the capacitor is charged, we observe that the capacitor voltage bleeds away to zero over the course of about 30 minutes. This internal loss can be modeled by including a leakage resistor R_l in parallel with the capacitor as shown in figure 5.2.5.

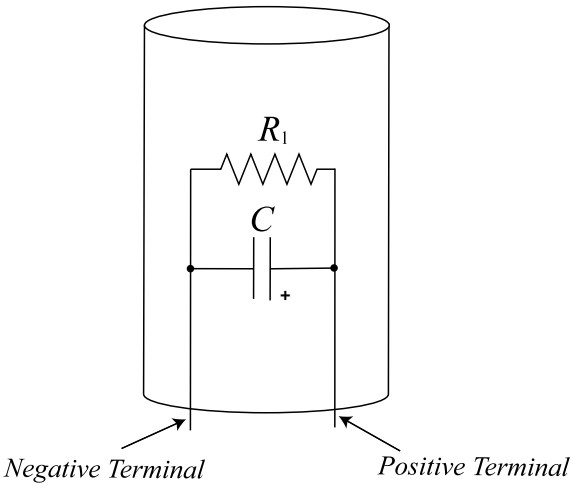


Figure 5.2.5: Electrolitic capacitor model

If we eliminate the packaging details, this circuit appears more simply as in figure 5.2.6.

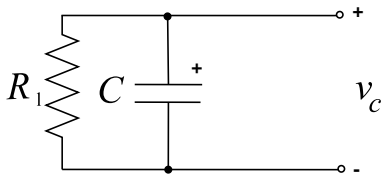


Figure 5.2.6: Electrolitic capacitor circuit

The + sign is usual to indicate that this is an electrolytic capacitor, as well as its associated allowed polarity.

(c) Suppose that $C = 120uF$, and that we observe that from an initial charge it takes $10min$ for the capacitor voltage to fall to 37 percent of its initial value. What is the value of the leakage resistor R_l?

Now, we want to study using the battery to charge the capacitor. At $t = 0$, the battery is connected to the initially uncharged capacitor.

(d) Draw a circuit diagram showing this connection.

(e) Write a differential equation which discribes the behavior of the circuit in this connection. The differential equation should be written in terms of the capacitor voltage v_c.

(f) From the initially uncharged state at $t = 0$, write a solution $v_c(t)$ for this charging process. What is the associated time constant? Make a dimensioned graph of $v_c(t)$ and show the time constant on this graph. Where is the system pole located? Make an s-plane sketch showing this pole location.

After charging for a long time, the capacitor voltage reaches a final, or steady state value. Then, at time t_1, the battery is disconnected from the capacitor.

(g) Write a differential equation which describes the behavior of the capacitor voltage $v_c(t)$ in this disconnected condition. Solve the differential equation for $v_c(t)$, $t > t_1$, and make a dimensioned sketch of $v_c(t)$. Where is the system pole located? Make an s-plane sketch showing this pole location.

5.3 Loaded Motor

This problem considers the motor system shown below.

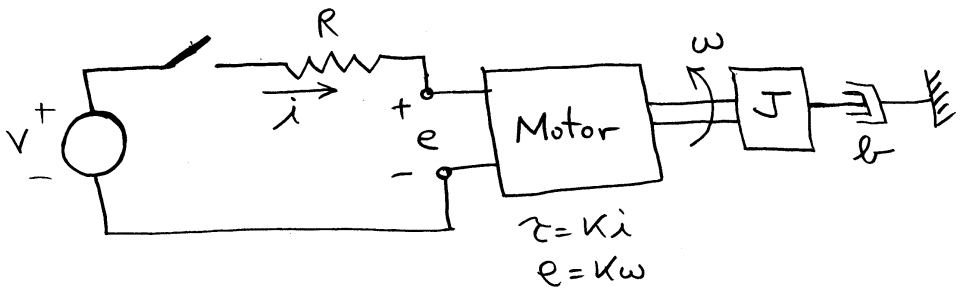


Figure 5.3.1: Loaded Motor

The motor in Figure 5.3.1 is driven by a voltage source V through a switch, which can be opened and closed. The motor internal coil resistance is modeled by the resistor R as shown. The motor itself is ideal and described by the relations $\tau = Ki$, and $e = K\omega$, where τ is the torque applied to the load, and ω is the load speed. The load consists of a rotary inertia J, and a rotational damper b.

(a) Suppose the system is initially at rest. At $t = 0$ the switch is closed. Solve for the resulting rotor speed as a function of time, and write an expression for $\omega(t)$ in terms of the system parameters. Clearly show the steps in your solution. Make a labeled and dimensioned sketch of $\omega(t)$.

(b) After the switch has been closed for a long time, the switch is opened. Let's call this opening time t_1. Solve for the resulting rotor speed as a function of time for $t > t_1$, and write an expression for $\omega(t)$ in terms of the system parameters. Clearly show the steps in your solution. Make a labeled and dimensioned sketch of $\omega(t)$.

(c) Solve for the value of the coil current i as a function of time throughout the motions determined above, and write an expression for $i(t)$ in terms of the system parameters. Clearly show the steps in your solution. Make a labeled and dimensioned sketch of $i(t)$.

6 Fluid Systems

6.1 Rotational Fluid System

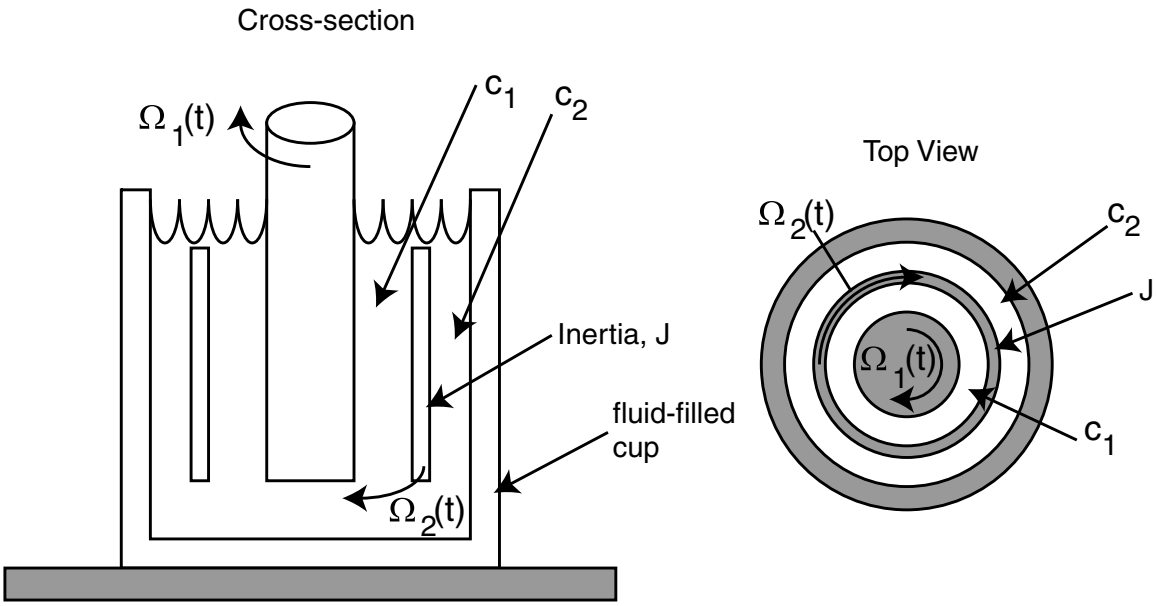


Figure 6.1.1: Cross-Sectional and Top view

This problem considers the rotational mechanical system shown in Figure 6.1.1.

The setup is similar to the one used in Lab 2. The central shaft rotates at an arbitrary velocity $\Omega_1(t)$ in a fluid filled cup, which is fixed. Unlike Lab 2, there is an intermediate ring, of inertia J. This ring is supported on bearings which are not shown in the figure. The ring rotates with angular velocity $\Omega_2(t)$ as shown.

The fluid-filled annuli create a damper c_1 between the shaft and the ring, and a damper $c2$ between the ring and the cup. Assume that any other damping is negligible.

(a) Draw a free-body diagram for the ring showing the torques acting on the ring.

(b) Use this free-body diagram to derive a differential equation in terms of $\Omega_1(t)$ and $\Omega_2(t)$ which describes this system. Note that $\Omega_1(t)$ is an arbitrary velocity which is externally specified.

(c) Assume that $\Omega_1(t)$ is a step, i.e., $\Omega_1(t) = u_s(t)$, and that $\Omega_2(0) = 0$. solve for the resulting motion $\Omega_2(t)$ for $t \geq 0$.

(d) In steady-state, what torque must be exerted on the input shaft? Why?

6.2 Fluid Leak

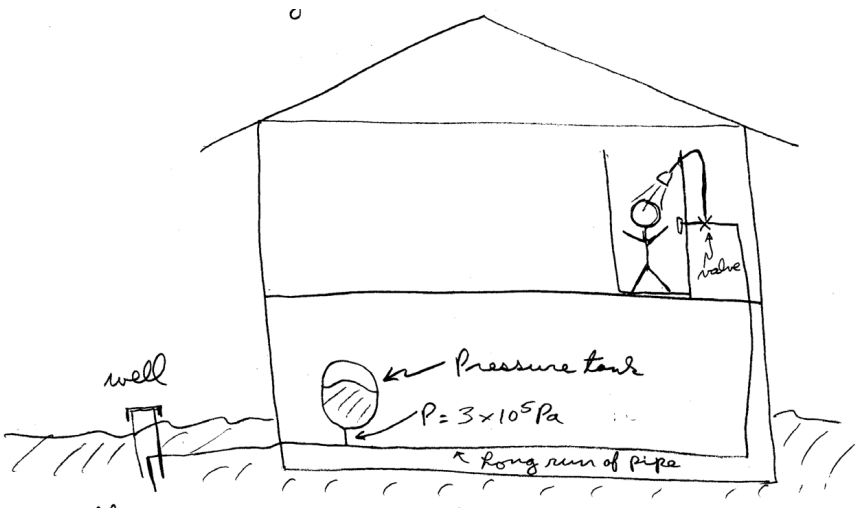


Figure 6.2.1: Diagram of Fluid Problem

You must read the Fluid Example on the webpage before starting this problem. Imagine that in the system in Figure 6.2.1 the valve is shut so that there is no flow at $t = 0$ and that $P_s(t)$ leaks down from $3 \cdot 10^5$ Pa very rapidly after the valve is closed, ie. $P_s(0^-) = 3 \cdot 10^5$ and $P_s(0^+) = 0$.

(a) Solve for $P_c(t)$.

(b) At what time is $P_c(t)$ at it's minimum value?

(c) How long does it take this system to settle withen $P_c(t) = 300$ Pa?

6.3 Where'd the water go?

This problem extends the plumbing example presented in the supplementary notes. The figure from the notes is repeated below.

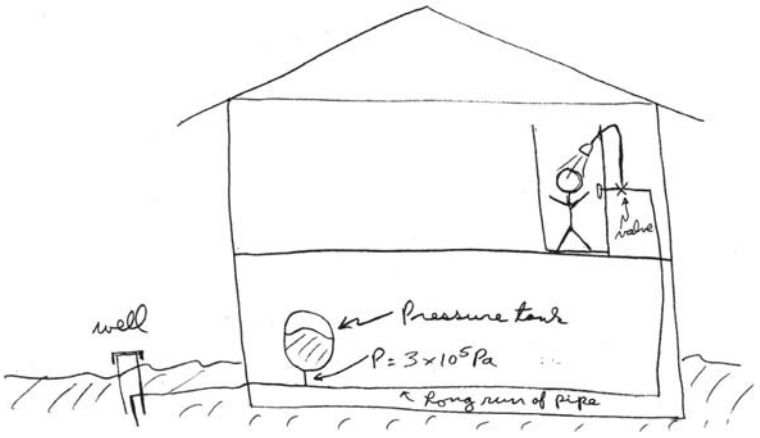


Figure 6.3.1: Shower Figure

You are upstairs taking a shower while your 9 year old daughter is downstairs skating madly around in the basement on her in-line skates with a hockey stick. (Why indoors? Why in the basement? Why with a hockey stick? Why didn't she ask if this was a good idea? If these questions haven't occurred to you, you aren't a parent yet.)

Anyway, taking an especially sharp turn, she slides out skates first into the pipes where they connect to the pressure tank which breaks them off essentially instantaneously. With calm engineering detachment, we will model this "event" as taking the pressure rapidly to zero, i.e., as $P_s(t) = 3 \times 10^5(1 - u_s(t))$ [Pa].

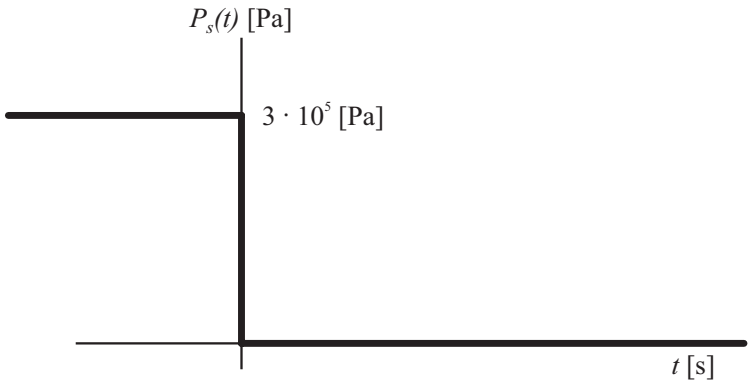


Figure 6.3.2: Pressure function

(a) Use the model and parameters from the example problem. Develop an analysis which predicts the resulting pressure and flow at the shower nozzle. Sketch these two functions. About how long does it take for the shower flow to go to zero?

(b) So now you're standing there naked in the shower with soap in your hair and no water flow wondering what happened, when you hear your kid screaming that the basement is starting to fill up with water. (The water pump is valiantly trying to restore pressure to $P_s = 3 \cdot 10^5$ Pa). We leave the next steps to your imagination...

(c) If all of this seems a bit rigorous, perhaps you lack the stern constitution required for parenthood?

(d) (Adapted from a true story).

7 Thermal Systems

7.1 Thermal Block Question

A block of copper is brought out of an oven at an initial temperature of 1000 K, and allowed to cool to the ambient temperature of 300 K. The graph below shows data which might represent this experiment. The copper block has a mass of 0.1 kg. Given that copper has a specific heat of 385 $\frac{J}{kg^oC}$, propose a thermal model of this experiment, and write the equation of motion. Given the experimental data, determine the value of thermal resistance to ambient. Clearly show the steps in your derivation.

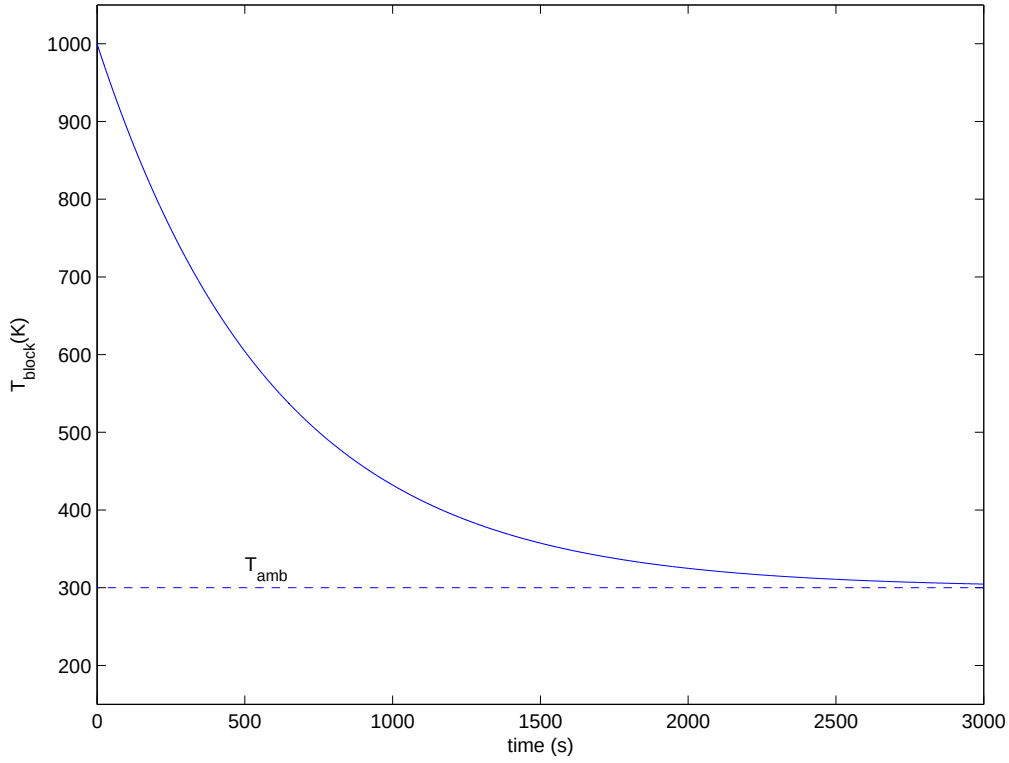


Figure 7.1.1: The block temperature (T_{block}) is plotted against time (t). The ambient temperature is given by T_{amb}.

7.2 The Hot Copper Block 1

This problem considers the thermal dynamics of water and a copper block inside a vacuum bottle. A diagram for the system is shown below in Figure 7.2.1.

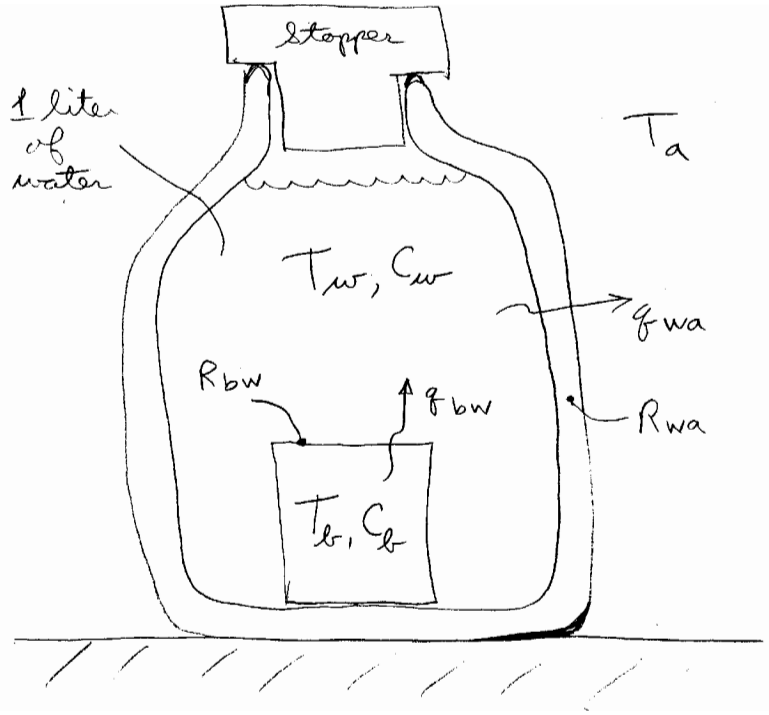


Figure 7.2.1: Copper block and water thermal system

Initially, a 100 gm copper block is heated outside the bottle in an oven to a temperature of 95°C. We are not concerned with the heating process. The water in the vacuum bottle has been there for a long time and is equilibrated to the ambient temperature $T_a = 25$°C.

At $t = 0$, the stopper is removed, the copper block is lowered into the bottle, and the stopper is replaced. Now we would like to develop a model which can describe the temperature dynamics as a function of time.

The system parameters are as follows: The copper block has a mass of 100 gm; copper has a specific heat of 0.385 J/gm°C= 385J/kg°C. The bottle contains 1 liter of water; water has a specific heat of 4.2 J/gm°C= 4200J/kg°C.

The temperature of the block is defined as T_b. The water is assumed uniform in temperature, with a value of T_w.

The heat flow from the block to the water is defined as q_{bw} [W]. The heat flow from the water to ambient is defined as q_{wa} [W]. These flows are modeled as passing through thermal resistances $R_{bw} = 0.1$°C/W and $R_{wa} = 17$°C/W, respectively. **Note: Take a look at the 2^{nd} order thermal example on the course web page before attempting this problem.**

(a) Write the governing differential equations for the system in *state-space* form as

$$\frac{dT_w}{dt} = f_1(T_w, T_b, T_a)$$
$$\frac{dT_b}{dt} = f_2(T_w, T_b)$$

where f_1 and f_2 are linear functions.

(b) Convert this representation to a 2^{nd} order differential equation in $T_w(t)$. You may assume that the ambient temperature is constant, and thus $\frac{dT_a}{dt} = 0$.

(c) Solve for the response $T_w(t)$ for $t \geq 0$. What are the system natural frequencies? Plot the system poles on an s-plane plot.

(d) Use Matlab to plot $T_w(t)$, and its two constituent modes, over an interesting time range. You may need to use two plots with different time scales to capture the dynamics. Explain qualitatively why the plot looks as it does. Can you show us the effect of the system time constants?

(e) Suppose the block were initially at only 60°C. Make a sketch on your time response plot of the resulting $T_w(t)$. Explain your reasoning. (Hint: If you exploit linearity, this should require no new calculations.)

7.3 The Hot Copper Block 2

This problem is a continuation of Problem 7.2. Reconsider the system of Problem 7.2, but with an electrical heater added to the inside of the copper block. The heater provides an input power $q_{in} = 4$ W. This new system is shown in Figure 7.3.1. The system starts at $t = 0$ in a rest state with $T_b = T_w = T_a = 25°\text{C}$. At $t = 0$, the heater is turned on.

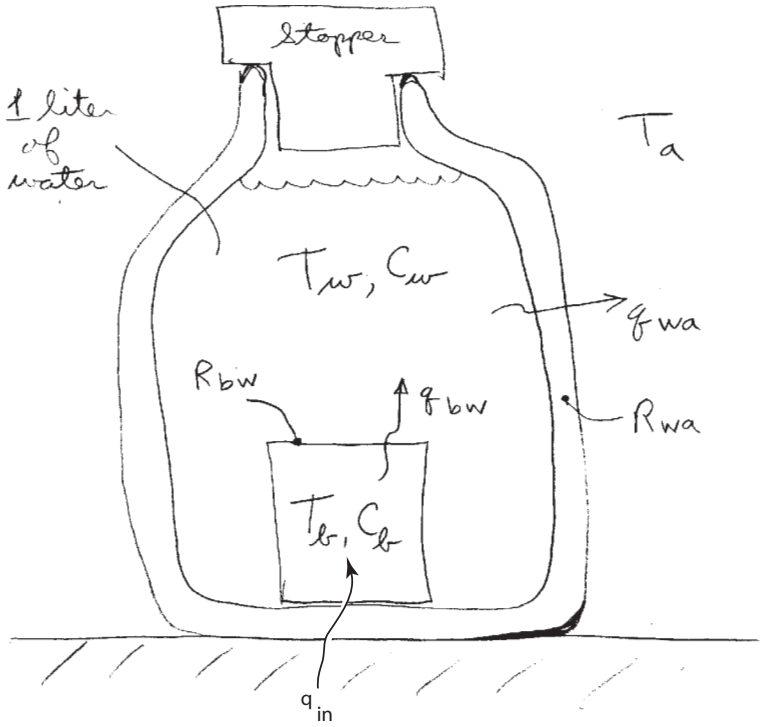


Figure 7.3.1: Copper block and water thermal system with heater

(a) Write the governing differential equation in terms of water temperature T_w.

(b) Solve the differential equation for $T_w(t)$, with the numerical values given in Problem 7.2.

(c) Use MATLAB to plot $T_w(t)$ over both a short and long time interval to show the interesting parts of the graphs. What is the steady-state value of T_w? What is the 10% to 90% rise time t_r? How long does the response take to settle within 1% of the final value?

7.4 Transistor on heat sink

Figure 7.4.1 is a schematic cross-section of a power transistor mounted on a heat sink. The transistor itself is fabricated on a thin piece of silicon, perhaps 5mm x 5mm x 0.5mm. This device is mounted within a package (case) which provides mechanical protection as well as electrical and thermal connections to the outside world. The case is mounted on a finned heat sink in order to transfer heat dissipation in the transistor into the ambient air.

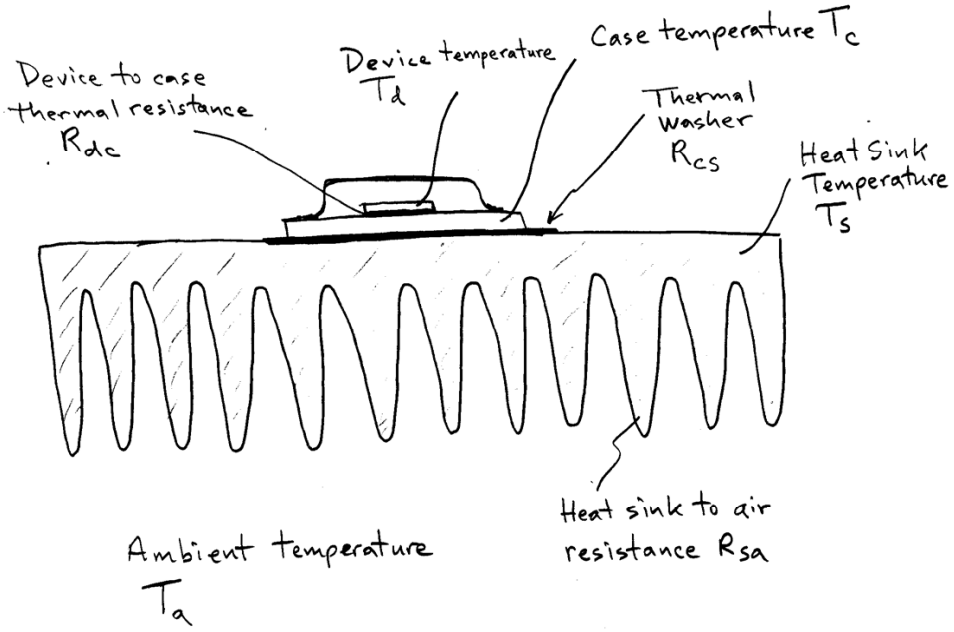


Figure 7.4.1: Transistor Schematic

The case of most transistors needs to be electrically insulated from the heat sink. This is accomplished with the indicated thermal washer, which has the conflicting requirements of thermal conduction and electrical insulation.

We define the device temperature as T_d, the case temperature as T_c, and the heat sink temperature as T_s. The ambient temperature is $T_a = 25°\text{C}$. The thermal resistance between the device and case is R_{dc}, between the case and heat sink is R_{cs}, and between the heat sink and air is R_{sa}.

The transistor is used as an electrical switch or amplifier. In the process electrical power is converted to heat. The heat flow needs to be managed without allowing the device to become too hot (generally the device temperature should stay below 100°C).

(a) Assume $R_{dc} = 0.1°\text{C/W}$, $R_{cs} = 0.2°\text{C/W}$, and $R_{sa} = 0.5°\text{C/W}$. Further assume that in steady-state the transistor is dissipating 50W. What are the steady-state values of T_d, T_c, and T_s?

(b) Now assume the heat sink is constructed with 500gm of aluminum, which has a specific heat of 0.90 J/g°K. The system is operating in the steady-state condition above when, at $t = 0$ the transistor is pulsed with 500W of power dissipation for 1 second, and returns to dissipating

50W. Write expressions for $T_d(t)$, $T_c(t)$, and $T_s(t)$ for $t \geq 0$, and make plots of these functions. Assume that the device and case have zero thermal capacitance. What is the peak device temperature in this linear model. Will it likely survive the transient? Would a bigger heat sink help?

8 Circuits

8.1 RC Transfer Function

This problem considers the circuit shown in Figure 8.1.1.

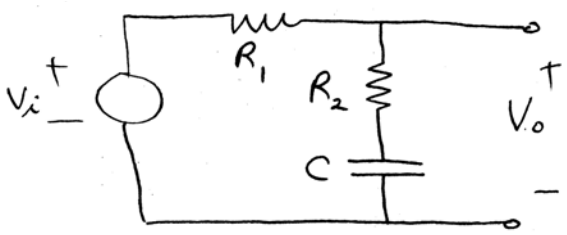


Figure 8.1.1: RC Circuit

(a) Calculate the transfer function $V_o(s)/V_i(s)$.

(b) For the values $R_1 = 3k\Omega$, $R_2 = 12k\Omega$, and $C = 10$ μF, solve for the pole and zero of the system, and plot these on the s-plane.

(c) For the values given in part (b), calculate the transfer function magnitude and phase as a function of frequency for a sinusoidal input. Make an accurate hand-sketch of the Bode plot for this transfer function, i.e., the magnitude and phase as a function of the input frequency ω.

8.2 RLC Transfer Function

This problem considers the circuit shown in Figure 8.2.1.

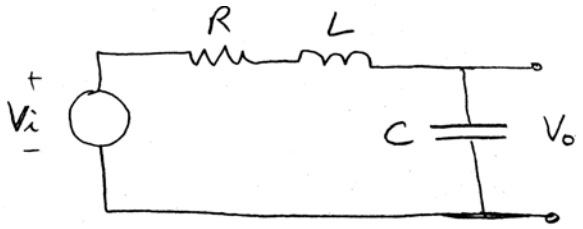


Figure 8.2.1: RLC Circuit

(a) Calculate the transfer function $V_o(s)/V_i(s)$.

(b) Let $L = 0.01$ H. Choose the remaining parameters of the circuit such that $\omega_n = 10^5$ and $\zeta = 0.05$. Plot the poles of the system, indicating the pole positions in rectangular coordinates and polar coordinates.

(c) For these parameters, make a carefully-dimensioned sketch of the response to a unit step in input voltage. Specifically, show the overshoot value, the time to the peak, and the time-scale for each of the next 3 peaks in the response. What is the time required to settle to 5%?

(d) For these parameters, make an accurate sketch of the Bode plot for the transfer function in part (a).

8.3 Bandpass Filter

The circuit shown in Figure 8.3.1 acts as a bandpass filter, in that input frequencies will only significantly propagate to the output if they are near the filter resonant frequency. Input components away from the resonant frequency will be attenuated; the degree of attenuation will depend upon the separation between the driving and resonant frequencies.

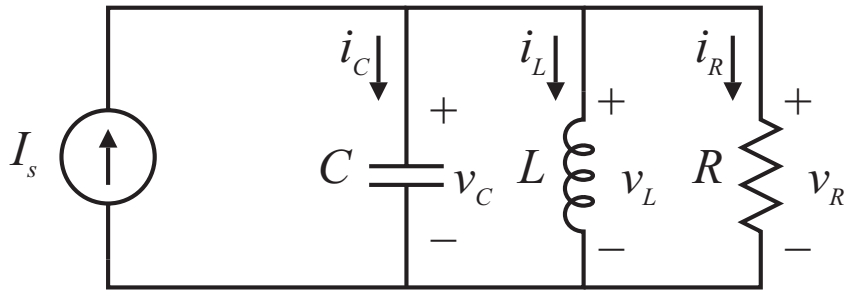


Figure 8.3.1: Bandpass Filter Circuit

An example of the use of such a filter would be at the front end of a radio. Here the filter is used to select a particular radio station carrier frequency for listening, and to attempt to reject other radio station carrier frequencies. In this situation, the current i would represent current from the antenna attached to the radio, and the output signal v_R would represent a voltage used in the radio for further processing.

In this problem, the numerical values have been selected to tune the filter to a Boston-area FM radio station. We ask you to work out the frequency response function for the filter and then consider how much an adjacent station's signal is attenuated. For the purposes of this problem, assume that radio stations emit a pure sinusoidal signal at their carrier frequency. So, for example, a station such as WBCN operates at 104.1 MHz (104.1×10^6 Hz). Although the real signal consists of a band in the vicinity of this frequency, we will assume that the station puts out a pure sine wave at 104.1 MHz, and that this is what the filter is processing. In this problem, be careful to keep track of what frequencies are in Hz and what frequencies are in rad/sec.

(a) Solve for the differential equation which relates the input current I_s to the output voltage v_R. Show the steps in your derivation.

(b) Use the results of a) to derive the transfer function from input $I(s)$ to output $V_R(s)$.

(c) In terms of the circuit parameters, what are the values of the natural frequency ω_n and the damping ratio ζ? What are the locations of the system poles and zeros?

(d) In the following parts of the problem assume the components take the values:$R = 50$ kΩ, $C = 10^{-11}$ F $= 10$ pF, and $L = 3.14815 \times 10^{-7}$ H. (Caution: The values chosen must be used with all given digits to guarantee a sufficiently accurate answer.) For these parameter values, what are the numerical values of ω_n and ζ?

(e) Use the MATLAB `bode` command to plot the frequency response magnitude and phase for this filter. Can you see why it is called a bandpass filter? What is the center frequency of the filter in MHz? What Boston-area radio station operates at this frequency?

(f) In a given area, radio stations are generally spaced in frequency by at least 400 kHz. Use the frequency response function to determine how much attenuation will be experienced by a signal at 400 kHz above the filter resonant frequency. Show this frequency and the resulting attenuation on a Bode plot zoomed in on the relevant frequency region.

8.4 LC Circuit Differential Equations

Consider the circuit in Figure 8.4.1

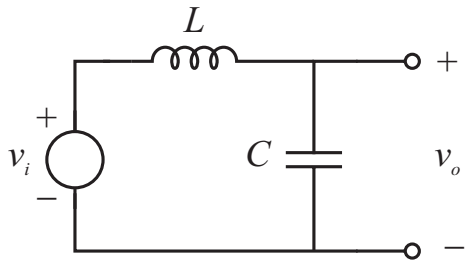


Figure 8.4.1: LC Circuit

(a) Write the governing differential equation for this circuit.

(b) The input is a unit step $v_i(t) = u_s(t)$. Calculate $v_o(t)$, $t > 0$, assuming initial rest. Write an expression for $v_o(t)$, and make a properly labeled sketch of $v_o(t)$.

8.5 Equivalent Impedance

Calculate the equivalent impedance, Z_{eq}, at the indicated terminal pair in Figure 8.5.1.

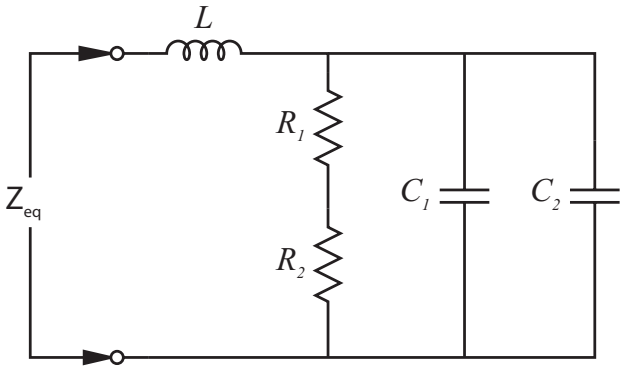


Figure 8.5.1: LRC Circuit

8.6 Circuit Response

For each of the circuits below, derive the differential equation which relates input to output, and solve for the unit step response for the listed initial conditions. Note that some of the circuits use a current source input. In each case, make a sketch of the output vs. time.

a) Solve for i_c as output

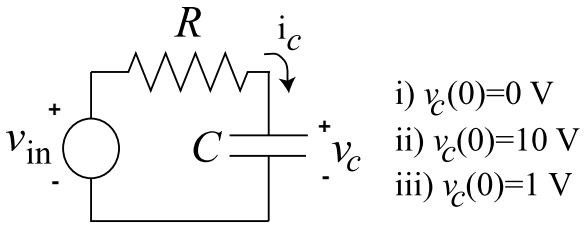


Figure 8.6.1: Circuit for Part (a)

(b) Solve for v_L as output

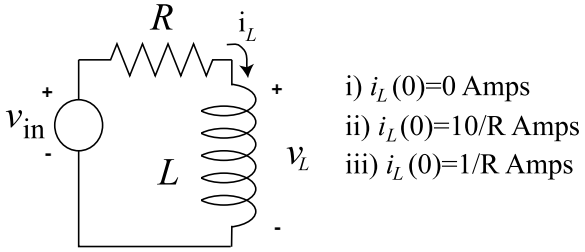


Figure 8.6.2: Circuit for Part (b)

(c) Solve for v_L as output

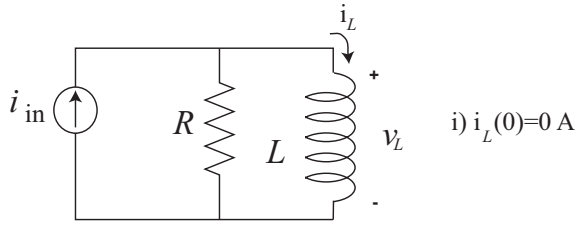


Figure 8.6.3: Circuit for Part (c)

(d) Solve for v_c as output

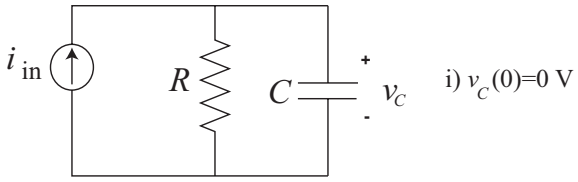


Figure 8.6.4: Circuit for Part (d)

(e) For parts $(a) - (d)$, use impedance techniques to directly find the appropriate transfer functions.

8.7 LRC Circuit 1

This problem studies the circuit shown below.

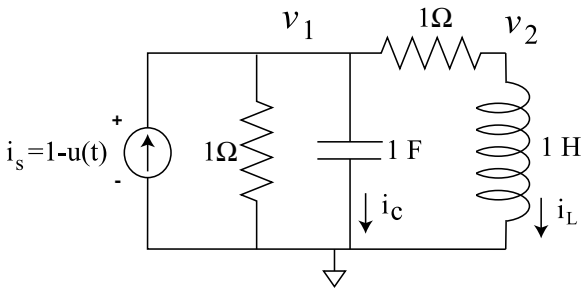


Figure 8.7.1: LRC Circuit

(a) For t<0, what are the values of v_1, v_2, i_L, and i_c

(b) What are the initial conditions at $t = 0^+$, i.e., the values of $v_1(0^+)$ and $\frac{dv_1}{dt}(0^+)$? Be sure to show how you derived these results.

(c) For $t \geq 0$ a differential equation describing this circuit in terms of v_1 is

$$\frac{d^2v_1}{dt^2} + 2\frac{dv_1}{dt} + 2v_1 = 0.$$

Write the solution of this equation as a function of time for $T \geq 0$. Show carefully how you derived the solution and satisfied the initial conditions from the previous section. If you were not able to derive the initial conditions, you can write your solution in terms of $v_1(0^+)$ and $\frac{dv_1}{dt}(0^+)$.

8.8 LRC Circuit 2

This problem studies the circuit shown below. Note that the circuit is driven by an "unstep" function $i_s(t) = 1 - u_s(t)$, and thus the input is 1 [A] for $t < 0$ and 0 [A] for $t > 0$.

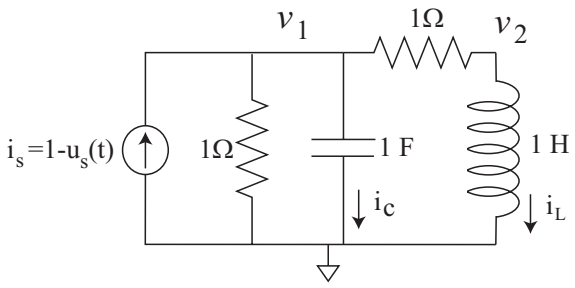


Figure 8.8.1: LRC Circuit

(a) Write node equations at v_1 and v_2 to derive a differential equation describing the circuit. The differential equation should be in terms of v_1 and i_s (that is v_2 should be eliminated).

(b) For $t < 0$, what are the values of i_s, v_1, v_2, i_L, and i_c?

(c) Use Laplace transform techniques to develop a solution $v_1(t)$ for this system for $t \geq 0$.

(d) Make a hand sketch of this solution.

(e) Calculate $\frac{V_1(s)}{I_s(s)}$ and $\frac{I_L(s)}{I_s(s)}$

8.9 LR Circuit Step

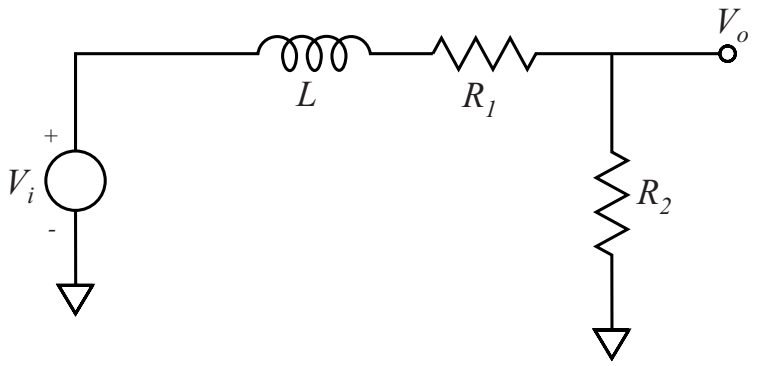


Figure 8.9.1: Circuit Diagram

(a) Derive a differential equation which describes the circuit behavior in terms of the variables $V_o(t)$ and $V_i(t)$. Plot the system pole on the complex plane.

(b) For this section of the problem, let $R_2 = 1\text{k}$. The output $V_o(t)$ in response to a unit step input, from zero initial conditions is shown below. Determine the values of L and R_1 that correspond to this response. Be sure to show your reasoning.

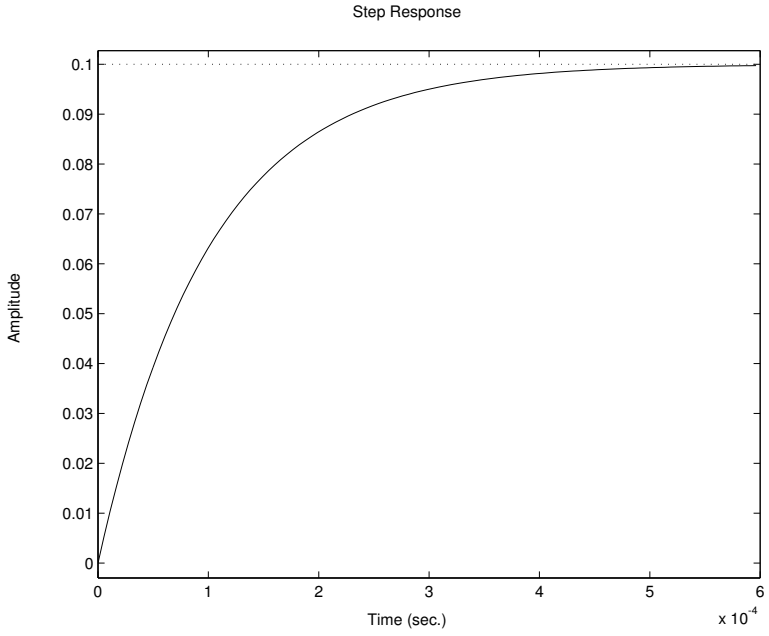


Figure 8.9.2: Step Input Response from zero initial conditions

(c) Now let the input the to circuit be a sinusoid $V_i(t) = \sin \omega t$. In steady-state, the output will have the form $V_o(t) = M \sin(\omega t + \phi)$. Using the numerical values of the circuit elements you derived above, calculate expressions for M and ϕ as a function of ω.

(d) Make plots by hand of $M(\omega)$ and $\phi(\omega)$. For the plots, use logarithmic axes for ω, log axes for M, and linear axes for ϕ. That is, the magnitude is displayed on a log-log plot, and the phase is displayed on a semi-log plot. What is the DC gain of this circuit? What is the high-frequency asymptotic gain?

9 Op Amps

9.1 Opamp Arithmetic

Design an opamp circuit that realizes the following input-output relationship

$$v_o = 5v_1 + 3v_2 - 10v_3$$

where v_1, v_2, and v_3 are three indepdent output voltages. Use the minimum number of opamps that you find possible, and assume that the opamps follow the ideal infinite gain model. Explain your thinking.

9.2 Inverting Op-Amp

The circuit shown in Figure 9.2.1 is an inverting op amp connection, with a capacitor C' connected between the inverting input and ground. As we saw in lecture, adding a capacitor in this location can have a significant effect on the stability of the op amp connection. This problem explores such an effect.

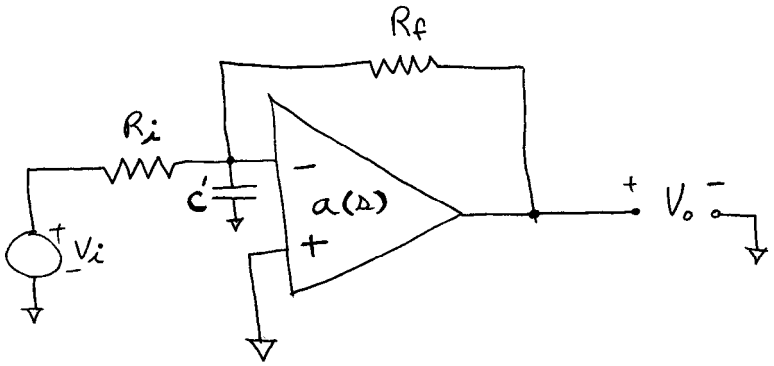


Figure 9.2.1: Inverting Op-Amp Circuit

(a) For this connection, draw a block diagram representing the circuit.

(b) Let the components take the values $R_i = R_f = 10$ kΩ, $C = 10$ μF, and $a(s) = 2\pi 10^6/s$. Calculate the transfer function V_o/V_i. What are the closed-loop damping ratio and natural frequency?

9.3 Opamp Transfer Function

Design an opamp circuit that realizes the following input-output transfer function

$$\frac{V_o(s)}{V_i(s)} = 100\frac{10^{-3}s+1}{s(10^{-4}s+1)}.$$

Clearly show your thinking. You can design this circuit using only resistors, capacitors, and op amps. Assume that the opamps follow the ideal infinite gain model. Explain your thinking.

9.4 Op-Amp Differentiator 1

The circuit in Figure 9.4.1 acts approximately as a differentiator.

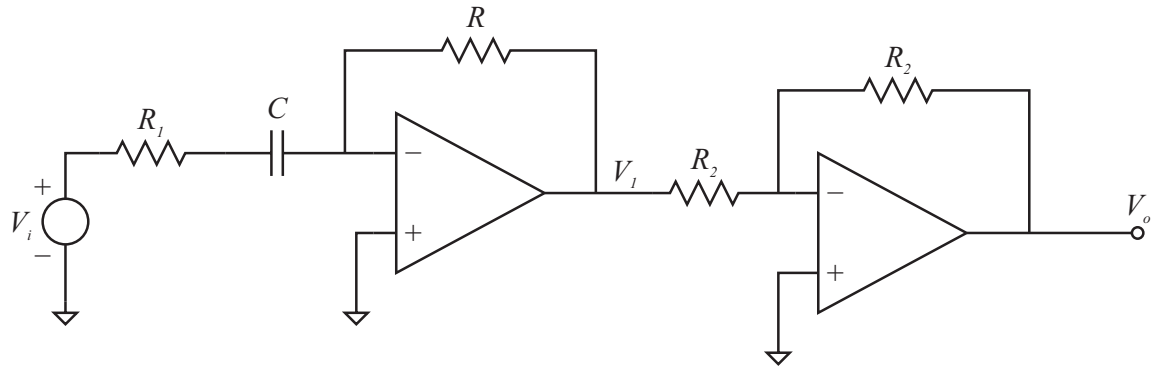


Figure 9.4.1: Op-Amp Differentiator

The second amplifier just acts as an inverter to cancel out the inverter associated with the first stage. For the values $R_1 = R_2 = 100$ kΩ, $R = 10$ kΩ, and $C = 20$ μF, and assuming the op-amp is modeled as ideal with infinite gain,

(a) Calculate the pole and zero locations.

(b) Calculate the magnitude and phase of the transfer function $V_o(j\omega)/V_i(j\omega)$ as a function of ω.

(c) Use MATLAB to plot the step response of this circuit. Does it act as a differentiator? How is the response non-ideal?

(d) Use MATLAB to generate the Bode plot for this transfer function. Indicate the breakpoint frequency on the Bode plot. How is this related to the pole location?

9.5 Op-Amp Proportional plus Integral Controller

The circuit in Figure 9.5.1 can be used as a proportional plus integral controller.

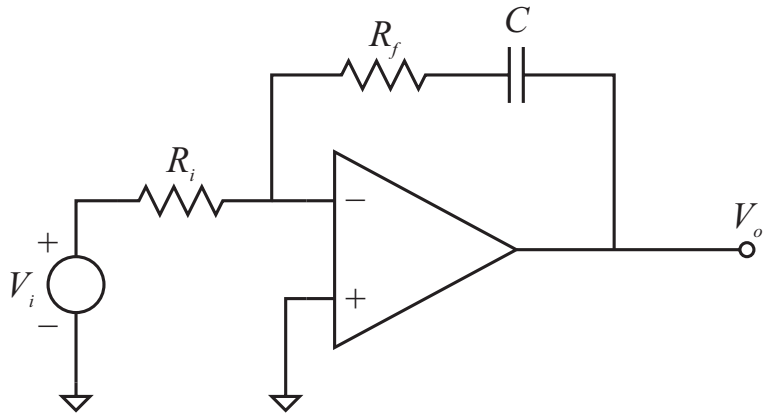


Figure 9.5.1: Op-Amp PI Circuit

For the values $R_i = 10$ kΩ, $R_f = 100$ kΩ, and $C = 10$ μF, and assuming the op-amp is modeled as ideal with infinite gain,

(a) Calculate the pole and zero locations.

(b) Calculate the magnitude and phase of the transfer function $V_o(j\omega)/V_i(j\omega)$ as a function of ω.

(c) Use MATLAB to plot the step response of this circuit. Does it act as a proportional plus integral (PI) filter? How so?

(d) Use MATLAB to generate the Bode plot for this transfer function. Indicate the breakpoint frequency on the Bode plot. How is this related to the zero location?

9.6 Op-Amp Circuit Design 1

Design an op-amp circuit that realizes the following input-output relationship

$$v_o = 5v_1 + 3v_2 - 10v_3$$

where v_1, v_2, and v_3 are three independent output voltages. Use the minimum number of op-amps that you find possible, and assume that the op-amps follow the ideal infinite gain model. Explain your thinking.

9.7 Op-Amp Block Diagram

In the op-amp circuit shown in Figure 9.7.1, the op-amp has a transfer function $a(s)$ from the differential input to the output voltage.

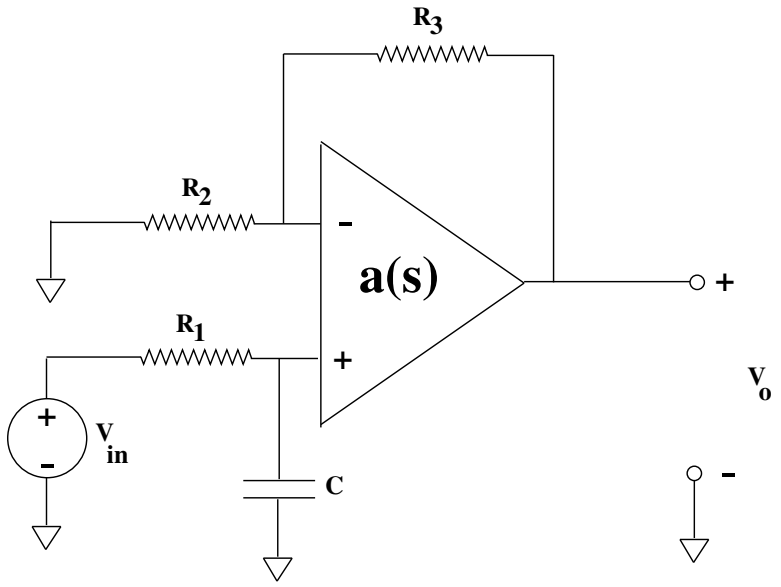


Figure 9.7.1: Op-Amp Circuit

(a) Draw a block diagram representing the circuit. Your block diagram should show the variables V_{in}, V_o, v_+, and v_-. Be sure to show how the block diagram was derived.

(b) Let $a(s) = g/s$. What is the transfer function from input V_{in} to output V_o, in terms of the given system parameters?

9.8 Op-Amp Circuit Design 2

Design an opamp circuit that realizes the following input-output transfer function

$$\frac{V_o(s)}{V_i(s)} = 100\frac{10^{-3}s+1}{s(10^{-4}s+1)}. \tag{1}$$

Clearly show your thinking. You can design this circuit using only resistors, capacitors, and op amps. Assume that the opamps follow the ideal infinite gain model. Explain your thinking.

9.9 Op-Amp Differentiator 2

The circuit in Figure 9.9.1 is the circuit from Problem 9.4.

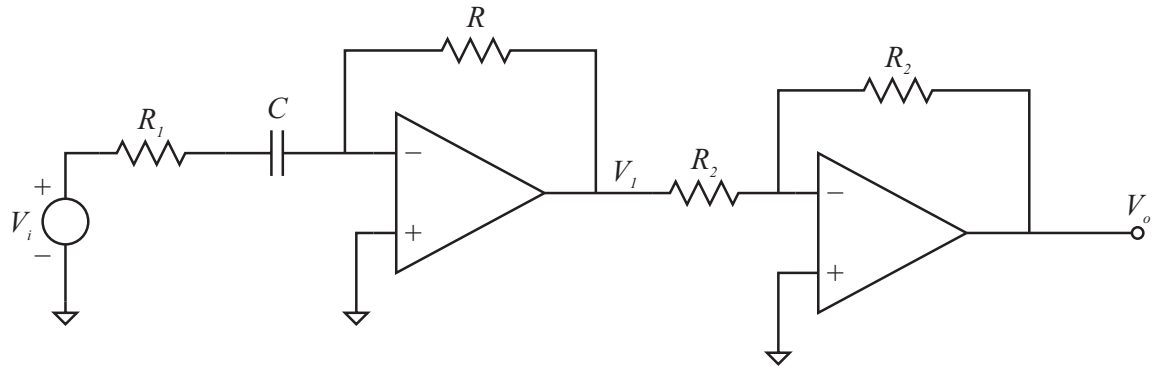


Figure 9.9.1: Op-Amp Differentiator

The second amplifier just acts as an inverter to cancel out the inverter associated with the first stage. For the values $R_1 = R_2 = 100$ kΩ, $R = 10$ kΩ, and $C = 20$ μF, and assuming the op-amp is modeled as ideal with infinite gain,

(a) Draw a block diagram for the circuit of Figure 9.9.1 in which the op amp is modeled as having dynamics. That is, the op-amp output voltage is $V_{out} = A(s)(v_+ - v_-)$.

(b) Solve for the input-output transfer function for this circuit configuration in the case where $A(s) = g/s$ and $g = 2\pi 10^7$, and using the numerical values of the parameters given above.

(c) Plot the poles and zeros of the transfer function from part (b). What dynamics can you estimate will dominate the step response? Make a sketch of the step response that would result from these dominant dynamics.

(d) Enter the exact transfer function into MATLAB, and thereby plot the exact step response. How does this differ from the response calculated in Problem 9.4? How are the dominant dynamics of part (c) evident in this step response?

9.10 RC Op-Amp Frequency Response

(a) For the circuit shown below calculate the transfer function V_o/V_{in} under the assumption that the op amp has infinite differential gain as shown.

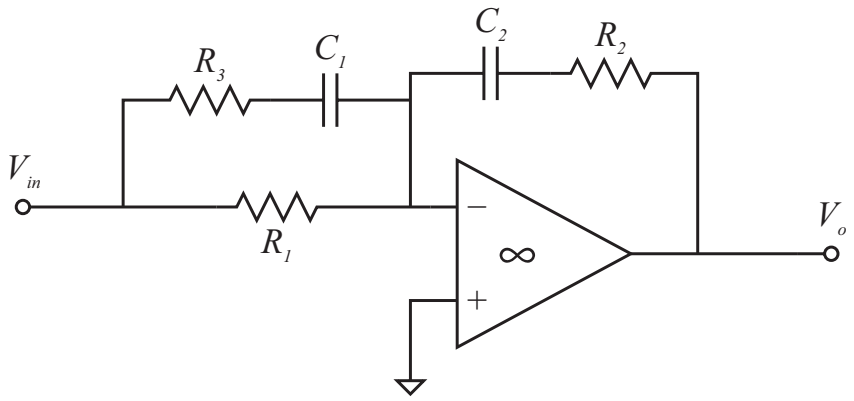


Figure 9.10.1: RC Op-Amp Circuit

(b) Now let the resistor $R_1 = 10$ [kΩ]. Choose the other components such that the input/output transfer function Bode plot has the asymptotic magnitude shown in Figure 9.10.2. Explain your calculations. With the indicated slopes, it is possible to determine the value of A as shown on the plot. What must this value be?

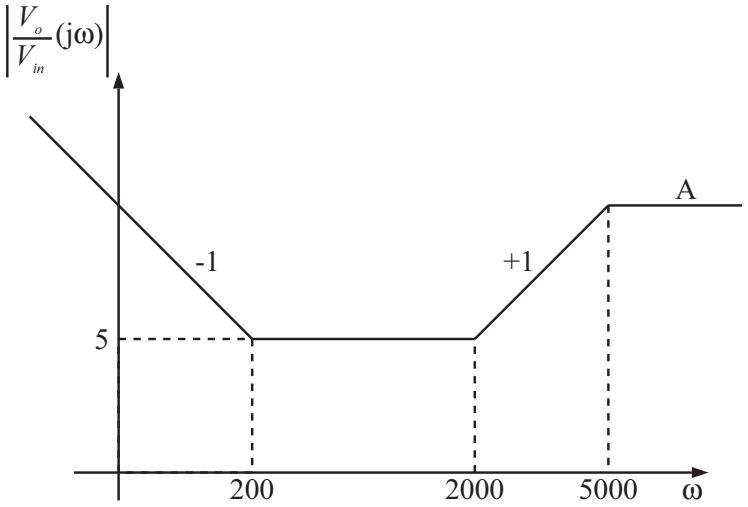


Figure 9.10.2: Asymptotic Magnitude of Bode plot

10 Differential Equations

10.1 Differential Equations 1

Here we consider a system described by the differential equation

$$500\dot{y} + 1000y = 0.$$

(a) Plot the system pole on the complex plane.

(b) Suppose that $y(-3) = 10$. Write an expression for $y(t)$ valid for all time. Show your reasoning. Sketch this response.

(c) At what time does the response fall below a value of 10^{-6}?

(d) Use Matlab to plot the response in the range $-4 \leq t \leq 1$.

(e) Design a mechanical system which will have the differential equation given above. Clearly indicate the physical variable that y represents, as well as the numerical values of the mechanical system parameters. Note that there are many possible systems which will have the same differential equation.

10.2 Differential Equations 2

Here we consider a system described by the differential equation

$$\ddot{y} + 10\dot{y} + 10000y = 0.$$

(a) What are the values of the natural frequency ω_n, the damping ratio ζ, and the damped natural frequency ω_d? Plot the system poles on the complex plane.

(b) Suppose that $y(0.1)$ = -3, and that $\dot{y}(0.1) = 0$. Write an expression for $y(t)$ valid for all time. Show your reasoning. Sketch this response.

(c) Use Matlab to plot this response in the range $-0.5 \leq t \leq 1$. Indicate on your plot that the auxilary conditions are met.

(c) At what time is the value of the response guaranteed to be below 10^{-3}? Do not calculate the exact time value when this occurs, but provide a bound using the exponential envelope of the waveform.

(d) Design a mechanical system which will have the differential equation given above. Clearly indicate the physical variable that y represents, as well as the numerical values of the mechanical system parameters.

10.3 Differential Equations 3

Here we consider a system described by the differential equation

$$\ddot{y} + 200\dot{y} + 10000y = 0.$$

(a) What are the values of the natural frequency ω_n, the damping ratio ζ, and the damped natural frequency ω_d? Plot the system poles on the complex plane.

(b) Suppose that $y(0) = 1$, and that $\dot{y}(0) = 1$. Write an expression for $y(t)$ valid for all time. Show your reasoning. Sketch this response.

(c) Use Matlab to plot this response over an interesting range of time values. Indicate on your plot that the auxilary conditions are met.

10.4 Euler's Theorem 1

(a) Use Euler's theorem to demonstrate the following equivalence

$$ce^{j\omega t} + c^* e^{-j\omega t} = A\cos\omega t + B\sin\omega t$$

where $c = \alpha + j\beta$ is a complex number and A and B are real. What is the resulting relationship between c and A and B?

(b) Given $A = 7$, $B = 2$, and $\omega = 100$, what is the corresponding value of c?

(c) Use Matlab to plot three cycles of the vaveform defined in part b), and hand in the resulting plot.

10.5 Euler's Theorem 2

(a) Use Euler's theorem to demonstrate the following equivalence

$$A\cos\omega t + B\sin\omega t = M\cos(\omega t + \phi)$$

where A, B, M, and ϕ are real. What is the resulting relationship between A,B and M, ϕ?

(b) What is the relationship between M, ϕ, and c as defined in Problem 10.4? How are these related to the amplitude of the waveform you plotted in Problem 10.4?

10.6 Force Steps

Assume that we apply a unit step in force separately to a mass m, a dashpot c, and a spring k. The mass moves in inertial space. The spring and dashpot have one end connected to inertial space (reference velocity = 0), and the force is applied to the other end. Solve for and plot the resulting velocity and position of the mass, and the velocity and position of the free ends of the spring and damper under this step in force.

Recall that the unit step function $u_S(t)$ is defined as $u_S(t) = 0;\ t < 0$ and $u_S(t) = 1;\ t \geq 0$. We will also find it useful to introduce the *unit impulse* function $\delta(t)$ which can be defined via

$$u_S(t) = \int_{-\infty}^{t} \delta(\tau)\ d\tau$$

This means that we can also view the unit impulse as the derivative of the unit step:

$$\delta(t) = \frac{du_S(t)}{dt}$$

10.7 Input for Zero Output

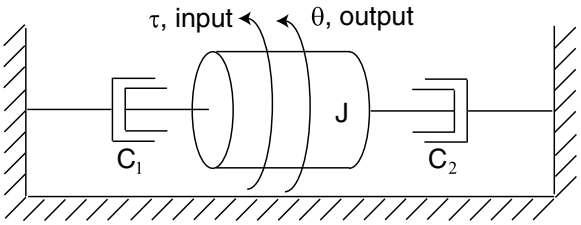
A system with an input u and an output y is described by

$$y = \frac{d^2u}{dt^2} + 2\frac{du}{dt} - 15u \tag{1}$$

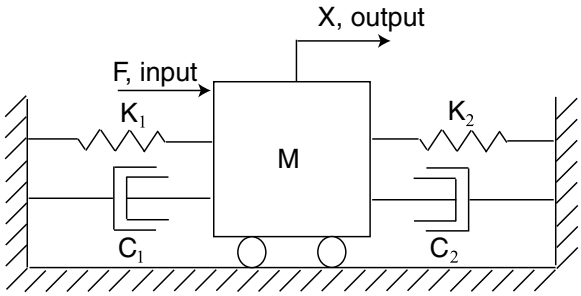
What is the most general input $u(t)$ applied for all t which yields $y(t) = 0$? Show your reasoning, and verify by direct substitution into (1).

10.8 Zero Input Response

For each of the systems shown in Figure 10.8.1, suppose that the indicated input is set equal to zero. That is, we will solve for the zero input response(ZIR). Don't use Laplace techniques to find the requested solution.



(i): Rotational system



(ii): Translational system

Figure 10.8.1: System Diagrams

(a) For the rotational system, solve for $\theta(t)$ with the initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \omega_0$. Write expressions for the system time constants.

(b) For the translational system, solve for $x(t)$ with the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$, under the assumption that the system is under-damped. Write expressions for the system natural frequency, damped natural frequency and damping ratio.

10.9 Zero Step Response 1

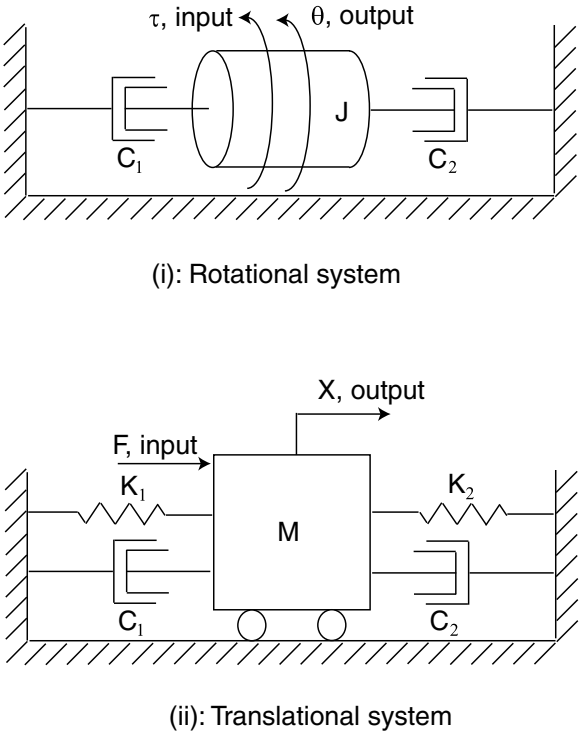


Figure 10.9.1: System Diagrams

For each of the systems shown in Figure 10.9.1, suppose that the initial state is equal to zero, and that the input is active. That is, solve for the zero state response(ZSR). Specifically assume that the input is a step, i.e.,

(i) $\tau(t) = u_s(t); \quad \theta(0) = \dot{\theta}(0) = 0.$

(ii) $F(t) = u_s(t); \quad x(0) = \dot{x}(0) = 0.$

Now solve for the resulting output variable as a function of time. Again, in the case of the translational system, you may assume that it is under-damped.

10.10 Zero Step Response 2

Now assume the following numerical parameters for the systems in Problem 10.9

(i) $J = 0.1$ kg-m^2; $C_1 = 10^{-3}$ N-m-s/rad; $C_2 = 3 \times 10^{-3}$ N-m-s/rad.

(ii) $M = 10$kg; $K_1 = 10^5$N/m; $K_2 = 8 \times 10^5$N/m; $C_1 = C_2 = 300$N-s/m.

(a) For these parameters, write expressions for $\theta(t)$ and $x(t)$ for the zero state response of Problem 10.9, and make hand-sketches of these responses.

(b) Use Matlab to plot the responses of Part a.

10.11 Nonlinear String System

Figure 10.11.1 shows a mass M supported by the constant force string, which has constant tension force no matter how much it is stretched. The mass is attached to the center point of the string. $L = 0.5$ m. T=100 N. For each of three cases (i) $M_1 = 10$ kg, (ii) $M_2 = 14.5$ kg and (iii) $M_3 = 17.7$ kg.

(a) Find the equilibrium point $\overline{X}$.

(b) Derive a linearized model about the equilibrium point.

(c) Solve for roots and plot them on the complex place.

(d) Plot the zero input response from zero initial velocity and the initial position is 1cm above the equilibrium point, i.e. $X(0) = \overline{X} - 0.01$m.

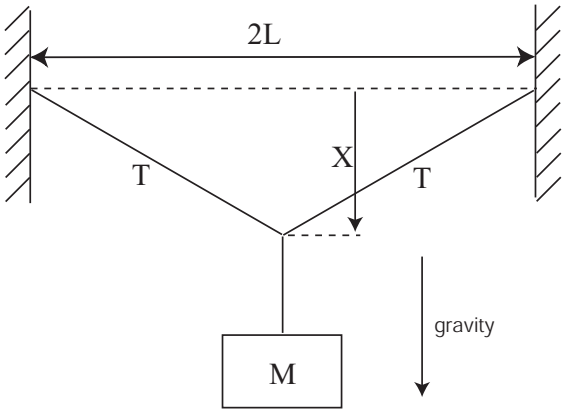


Figure 10.11.1: Nonlinear String System

10.12 Homogenous and Particular Solutions

For this problem, we use the rotational system of Lab 2, seen in Figures 10.12.1 and 10.12.2. Refer to Prelab 2 for more details. In the prelab, you are asked to calculate the moment of inertia for the shaft with and without the flywheel, and to use the video shown on the course web page to extract an estimate of the damping coefficient c resulting from the honey in the cup at the bottom of the shaft. Note that in the video the flywheel is attached to the shaft, and so for this problem you will use the inertia value with the flywheel present. Use the following values in this problem:

$$K = 10^{-2}\text{Nm} \ , \ I = 0.000317\text{kg} - \text{m}^2 \text{ and } c = 0.0011\text{N-m-sec/rad}$$

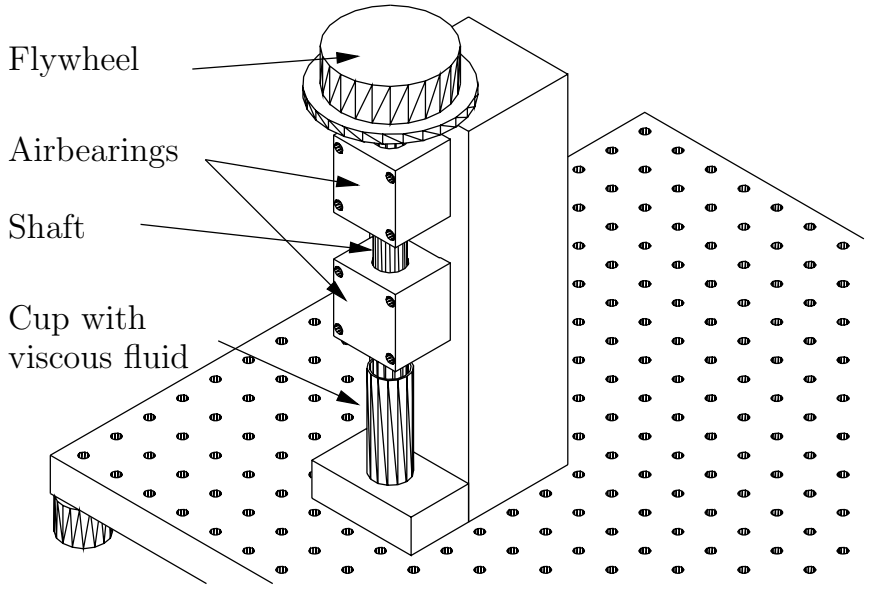


Figure 10.12.1: Rotational System

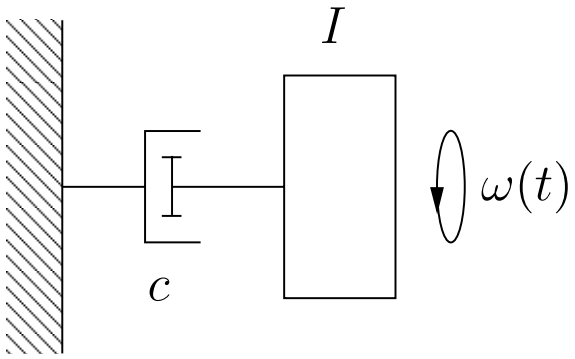


Figure 10.12.2: Lumped System Model

(a) Assume that the Lab 2 system is driven by an input torque T_{in} acting on the rotating shaft, and has a resulting angular velocity ω. Write a differential equation which describes the system with the given torque input.

(b) Now assume that the torque is a step of magnitude K, *i.e.*, $T_{in} = Ku_s(t)$, and that the shaft has an initial angular velocity $\omega(0) = \omega_0$. Find the particular and homogeneous solutions and combine these to write an expression for the total solution. Given the input magnitude K, for what initial velocity will the solution reach final value in zero time, *i.e.*, there will be no transient?

(c) Accurately sketch the solutions for $K = 10^{-2}$ Nm, and for each of the following initial angular velocities: $\omega_0 = 0$ rad/sec; $\omega_0 = 40$ rad/sec.

(d) Now we change the torque source to a sinusoidal input $T_{in} = u_s(t)K\cos\Omega t$. Again the system has an initial angular velocity $\omega(0) = \omega_0$. Note that the step function $u_s(t)$ which multiplies the cosine function "turns on" the input at $t = 0$.The homogeneous solution will have the same form as in part b), so you will not need to repeat this calculation. We ask you to find the particular solution in two ways. The first is by using only trigonometric functions, and no complex variable representations. The second approach is by writing the input as the real part of a complex exponential.

(i) Solve for the particular solution $\omega_p(t)$ to the cosinusoidal drive without making use of complex notation. You'll need to work through some trigonometric identities to work out the solution.

(ii) Solve for the particular solution $\omega_p(t)$ to the cosinusoidal drive by using complex notation. That is, we recognize that the input can be expressed as $T_{in} = \mathrm{Re}\{u_s(t)Ke^{j\Omega t}\}$. Then because the system is linear, we can first solve for the particular solution for a complex-valued input $T_{in} = u_s(t)Ke^{j\Omega t}$, taking the real part of the solution at the end to find the particular solution for the original real-valued input. The solution should be the same as you found in part i). As you'll see, the math is simpler via this path than for the solution taken in i) by using trigonometric identities. As the system order grows larger than this first-order example, this simplification becomes more significant.

(e) Assume that the initial shaft velocity is zero, *i.e.*, $\omega_0 = 0$. For the particular solution to the sinusoidal drive that you found above, what homogeneous solution is required to satisfy this initial condition? Use Matlab to plot the total solution for $K = 10^{-2}$, and for the following two values of input frequency: $\Omega = 0.5$ rad/sec; $\Omega = 20$ rad/sec. How do these two solutions differ in appearance? Specifically, how significant is the homogeneous solution in these two? Also, what effect does the system have on the output sinusoidal magnitude and phase relative to the input?

11 Frequency Response

11.1 Circuit Bode Plots

For each of the circuits shown in Figure 11.1.1

(a) Determine the associated transfer function.

(b) Accurately sketch the associated Bode plots (magnitude and phase).

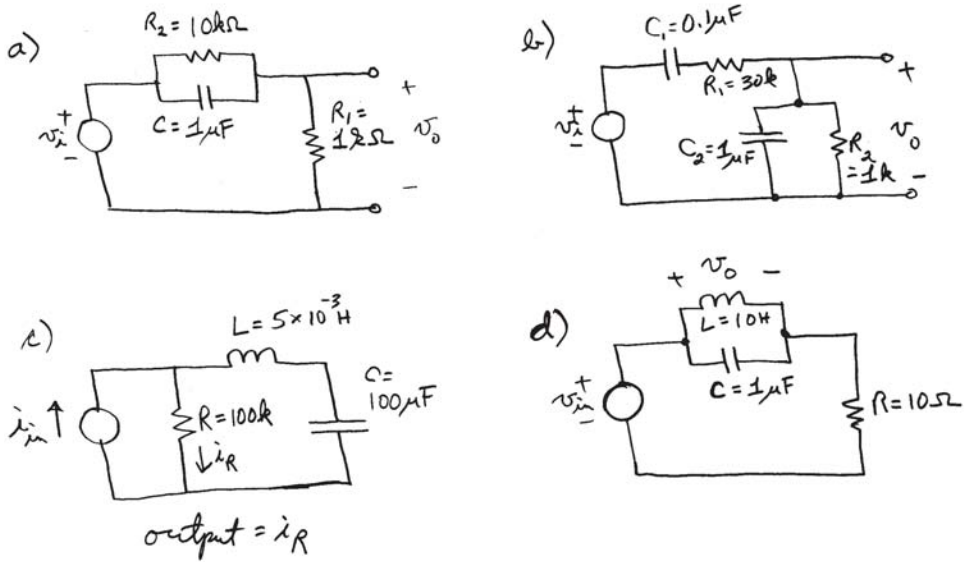


Figure 11.1.1: Circuits for analysis

11.2 Sketch Bode Plots

Using the appropriate graph paper, sketch the Bode plots for the following transfer functions:

(a) $G(s) = \dfrac{5s+1}{s+10}$

(b) $G(s) = \dfrac{5s+1}{s^2+3s+1}$

(c) $G(s) = \dfrac{s+10}{(s+2)(s^2+10s+100)}$

11.3 LC Circuit Frequency Response

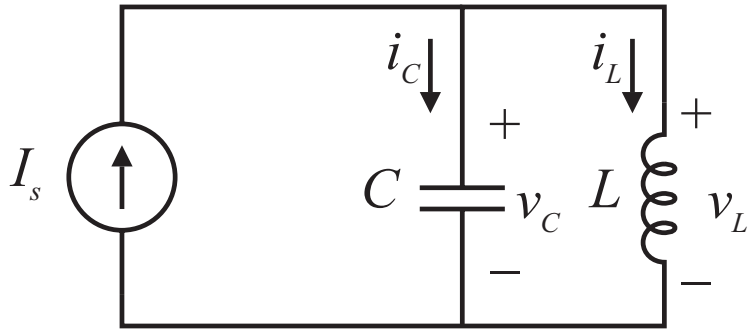


Figure 11.3.1: LC Circuit

(a) Derive a differential equation which describes the circuit behavior in terms of the input $I_s(t)$ and taking the output as $V_c(t)$. What are the values of ω_n, ω_d, σ, and ζ for this circuit? Plot the poles/zero on the complex plane.

(b) Calculate and make a carefully labeled graph of the response to a unit step current input $I_s(t) = u_s(t)$, from non-zero initial conditions $v_c(0) = 0$ [V] and $i_L(0) = -1$ [A].

(c) Now let the input the to circuit be a sinusoid $I_s(t) = \sin \omega t$. In steady-state, the output will have the form $V_o(t) = M \sin(\omega t + \phi)$. Calculate expressions for M and ϕ as a function of ω.

(d) Make plots by hand of $M(\omega)$ and $\phi(\omega)$. For the plots, use logarithmic axes for ω, log axes for M, and linear axes for ϕ. That is, the magnitude is displayed on a log-log plot, and the phase is displayed on a semi-log plot. You should find that magnitude response has an infinite peak at $\omega = 1/\sqrt{LC}$. Thus this circuit can be used as a band-select filter, predominantly passing a single frequency. Such a filter is useful for instance in a radio tuner to select a particular station.

11.4 LRC Circuit Frequency Response

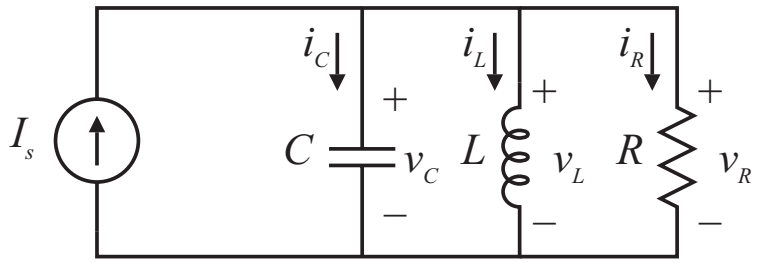


Figure 11.4.1: LRC Circuit

(a) Derive a differential equation which describes the circuit behavior in terms of the input $I_s(t)$ and taking the output as $V_c(t)$. What are the values of ω_n, ω_d, σ, and ζ for this circuit? Plot the poles/zero on the complex plane.

(b) Now let the circuit parameters have the values $L = 5$ [mH], $C = 0.02$ [μF], and $R = 2.5$ [kΩ]. Calculate and make a carefully labeled graph of the response to a unit step current input $I_s(t) = u_s(t)$, from zero initial conditions $v_c(0) = 0$ [V] and $i_L(0) = 0$ [A].

(c) Now let the input the to circuit be a sinusoid $I_s(t) = \sin \omega t$. In steady-state, the output will have the form $V_o(t) = M \sin(\omega t + \phi)$. Calculate expressions for M and ϕ as a function of ω.

(d) Make plots by hand of $M(\omega)$ and $\phi(\omega)$. For the plots, use logarithmic axes for ω, log axes for M, and linear axes for ϕ. That is, the magnitude is displayed on a log-log plot, and the phase is displayed on a semi-log plot. You should find that magnitude response has a finite peak. What is the peak frequency? How is the peak magnitude related to the circuit parameters?

11.5 Pole-Zero Plots

For the pole-zero plots shown in Figure 11.5.1:

(a) Determine the associated transfer function, under the assumption that the magnitude of the Bode plot passes through 1 at $\omega = 1$.

(b) Accurately sketch the associated Bode plots (magnitude and phase), under the assumption from (a).

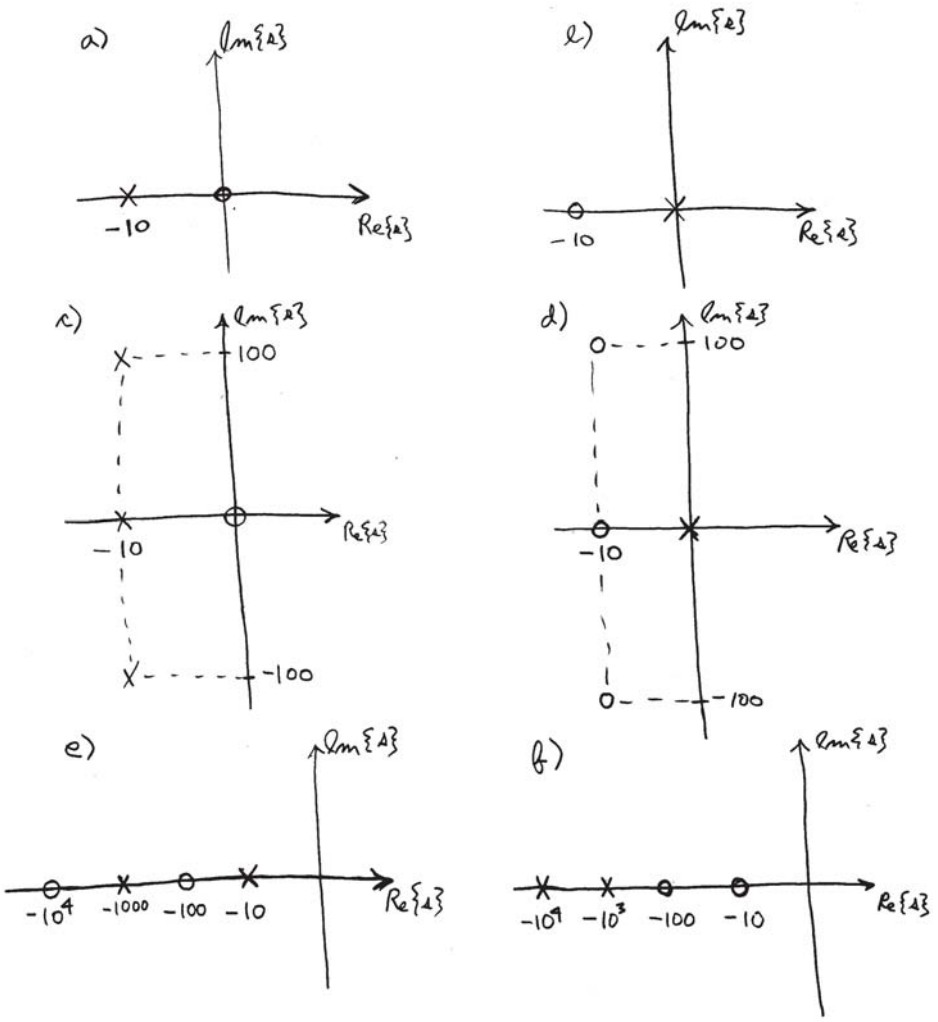


Figure 11.5.1: Pole Zero Plots

12 Signals, Transforms, and Transfer Functions

12.1 First Order Zeros

This problem considers the effect of zeros on the step response of a first-order system. Specifically,consider a system with input u and output y described by the differential equation

$$\tau\dot{y} + y = a\dot{u} + u$$

Note that this system has a pole at $s = -1/\tau$ and a zero at $s = -1/a$.

(a) Write an analytical solution for the response of this system to a unit step, assuming zero initial conditions. Use impulse matching to work through the transient at $t = 0$, and use the approach of particular and homogeneous solutions to solve the problem. Do not use Laplace transform techniques.

(b) We now want to study the step response for a number of values of the parameter a. Let $\tau =$ 0.1 sec,and sketch the step response for each the following values: $a = 0.05, 0.09, 0.11, 1, -1, -.1$. You should be able to sketch the response by hand, without any need for computational assistance. Along with the step response, include a pole-zero plot showing the locations of the pole and zero on the$s$-plane.

(c) Collocated poles and zeros in some sense cancel each other. Argue that if we let $a = \tau =$ 0.1 sec, then the system reverts a simple gain of 1 from input to output, that is, from a condition of initial rest, the output at all times equals the input. What is the step response for this choice of parameters? Show that your response in part b) above approaches this response in the limit $a = \tau$. Can you see this asymptotic behavior in the step responses you've plotted in part b)?

(d) Another way to think about zeros is that they allow properly chosen exponential inputs with no corresponding output. Specifically, assume that the input takes the form$u(t) = e^{s_1 t}$. Show that by the proper choice of s_1, the driven output is identically zero. How is this value of s_1 related to the zero location?

(e) Solve for the step response from a zero initial condition via Laplace techniques. This should yield the same result as you found in part (a).

12.2 Laplace Practice

A signal $w(t)$ is defined as

$$w(t) = u_s(t) - u_s(t - T)$$

where T is a fixed time in seconds and $u_s(t)$ is the unit step. Compute the Laplace transform $W(s)$ of $w(t)$. Show your work.

12.3 Step-Zero

This problem studies the effect of a real-axis zero on the step response of a second-order system. Specifically, consider the transfer function

$$H(s) = \frac{s/\alpha\zeta + 1}{s^2 + 2\zeta s + 1}.$$

Note that we have normalized by choosing $\omega_n = 1$. This is the same as allowing a non-unity value of ω_n and making all plots with respect to $\omega_n t$. For this transfer function:

(a) Plot on the s-plane the poles and zero as a function of α for the values $\alpha = 0.1,\ 1,\ 10$,and for $\zeta = 0.01,\ 0.1, 0.7$. Make a separate plot for each value of ζ, showing the effect of α on the zero location. Clearly label the pole and zero locations on the real and imaginary axes. Note that you cannot use a linearly scaled pole/zero plot if you want to show the pole and zero locations with adequate resolution.

(b) Use Laplace techniques to derive an analytical expression for the response of this system $y(t)$ to a unit step $u(t) = u_S(t)$ from initial condition $y(0) = \dot{y}(0) = 0$. Carefully show all steps. Be sure to include an expression which gives the output transform $Y(s)$ in terms of the input transform $U(s)$. Expand this expression in terms of given entries in a table of Laplace Transforms, performing inversion from the Laplace Transform to a function of time.

(c) Use Matlab to plot the response function you have derived versus time and for the values of α and ζ listed in part a). Include separate plots for each value of ζ, which overlay the variation with α. How does the zero location affect the step response? Can you see that the term $s/\alpha\zeta$ contributes a derivative effect in the step response? Explain.

12.4 Laplace to Time

For each Laplace Transform $Y(s)$, find function $y(t)$:

(a)

$$Y(s) = \frac{1}{(s+a)(s+b)}$$

(b)

$$Y(s) = \frac{s}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1}$$

12.5 Time to Laplace

For each of the functions $y(t)$, find the Laplace Transform $Y(s)$:

(a) $y(t) = e^{-at}$

(b) $y(t) = e^{-\sigma t} \sin \omega_d t$

(c) $y(t) = e^{-\sigma t} \cos \omega_d t$

12.6 Laplace Transform

A signal has a Laplace transform

$$X(s) = b + \frac{a}{s(s+a)}$$

where a, $b > 0$, and with a region of convergence of $|s| > 0$. Find $x(t)$, $t > 0$.

12.7 Signal in Time and Frequency Domain

A signal $x(t)$ is given by

$$x(t) = (e^{-t} - e^{-1})\big(u_s(t) - u_s(t-1)\big)$$

(a) Sketch this signal as a function of time.

(b) Calculate $X(s)$. Clearly show the steps in your calculation.

12.8 Transfer Function and Time Constants

(a) Calculate the transfer function $v_o(s)/v_i(s)$ for the circuit shown in Figure 12.8.1

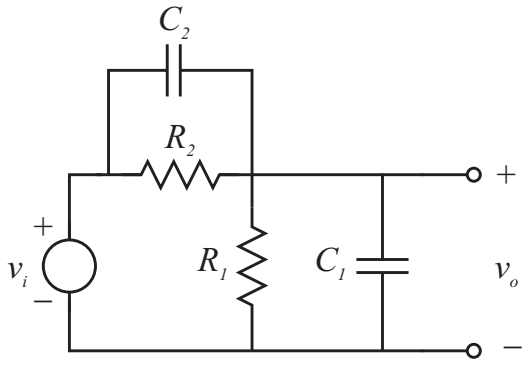


Figure 12.8.1: Electrical Circuit

Write this transfer function in the form

$$K\frac{\tau_1 s + 1}{\tau_2 s + 1}$$

Write expressions for K, τ_1, and τ_2. Use Laplace initial and final value theorems to calculate $v_o(0^+)$ and $v(\infty)$. Plot the pole and zero of this system on the s-plane. What happens when $R_1C_1 = R_2C_2$?

(b) Let $v_i(t) = u_s(t)$, a unit step. Assuming initial rest conditions, calculate an expression for the step response $v_o(t)$. Make sketches of $v_o(t)$ for the following three sets of parameter values:

(i) $R_1 = 1k\Omega$, $R_2 = 9k\Omega$, $C_1 = 1\mu F$, $C_2 = 9\mu F$

(ii) $R_1 = 1k\Omega$, $R_2 = 9k\Omega$, $C_1 = 9\mu F$, $C_2 = 1\mu F$

(iii) $R_1 = 9k\Omega$, $R_2 = 1k\Omega$, $C_1 = 9\mu F$, $C_2 = 1\mu F$

Also plot the pole/zero diagram for each set of parameters.

12.9 Fourier Series Input

A common periodic signal is a triangle wave similar to that shown below. It can be thought of as a series of ramps repeating at a frequency of ω_o. The purpose of this problem is to see how different dynamic systems, each with a different transfer function $G(s)$, will react to that signal. To do this, we can make use of the Fourier series, that states that any periodic function $f(t)$ can be described by an infinite series:

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty} \left[a_n \cos\left(n\omega_o t\right) + b_n \sin\left(n\omega_o t\right)\right]$$

where:

$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t) cos\left(n\omega_o t\right) dt$$

$$b_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t) sin\left(n\omega_o t\right) dt$$

where ω_o is the fundamental frequency of the waveform.

(a) Using references you might have from 18.01/2/3, determine the appropriate Fourier Series for the triangle wave shown below

(b) For each of the transfer functions below, sketch (not using MATLAB) the Bode diagram.
$G_1(s) = \frac{K}{\tau s+1}$ with $K = 10$, $\tau = 20$.
$G_2(s) = \frac{K}{(\tau_1+1)(\tau_2+1)}$ with $K = 1$, $\tau_1 = 5$, and $\tau_2 = 10$.
$G_3(s) = \frac{s\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ with $\zeta = 0.5$ and $\omega_n = 50$.
$G_4(s) = \frac{K}{\tau s+1}\frac{s^2+2\zeta_z\omega_{zn}s+\omega_{zn}^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ with $K = 1$, $\tau = 0.01$, $\zeta = 1.0$, $\omega_n = 100$, $\zeta_z = 0.05$, and $\omega_z n = 10$.

(c) Using the diagram only, write the steady-state output $y(t)$ that would result if the triangle wave $u(t)$ of Figure 12.9.1 were applied to the input of each transfer function. For this equation, consider only the first 5 terms of the series in your answer.

(d) Using MATLAB, plot 2 full cycles of the following:

(i) u(t)

(ii) u(t) with only 3 term in the series included

(iii) y(t) for G1 AND G2 based on the series solutions from part C)

(e) Now consider that we would like to have the output of the transfer function, $y(t)$, be a "pure" sine wave of frequency $3 * \omega_o$. (In other words we want to "tune in" the frequency $3 * \omega_o$ and "tune out" all the others.) Propose a transfer function that will come close to this and then check how it does using the above procedure.

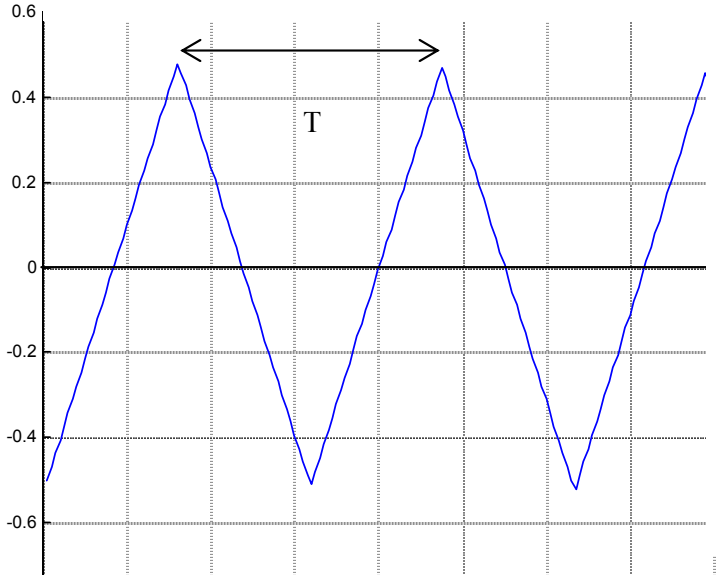


Figure 12.9.1: Triangle wave $u(t)$ with $T = 0.628$ sec.

12.10 Spring Mass System

Consider the simple spring mass system shown in Figure 12.10.1 subjected to a force F(t) that is a "pulse" as shown in Figure 12.10.2.

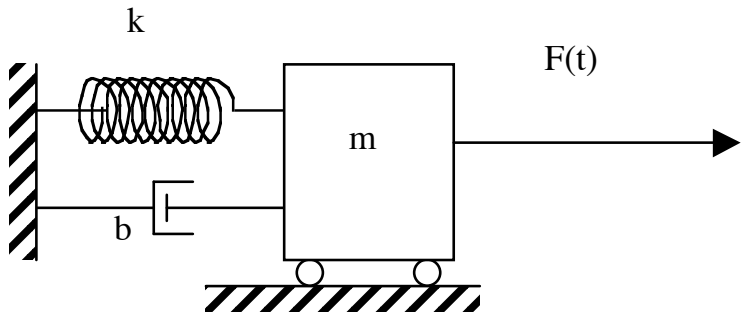


Figure 12.10.1: Spring Mass System

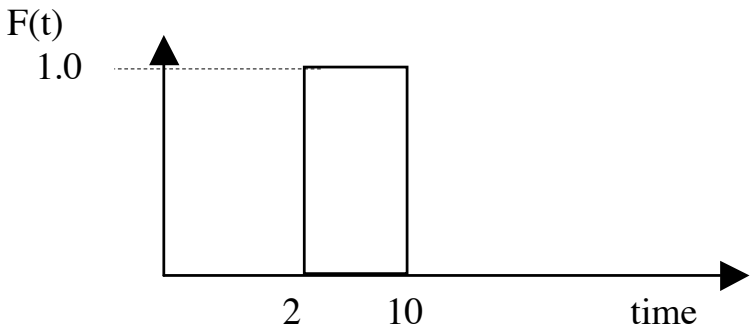


Figure 12.10.2: Force Pulse

If m= 1; k = 1 and b = 1:

(a) Show how you would model the F(t) using Laplace transforms. (i.e. find F(s).)

(b) Using Laplace Transforms, find the solution of x(t) in response to F(t) for $0 \leq t \leq 20$ sec.

(c) Using MATLAB, plot x(t).

(d) Describe why the response look as it does.

(e) How would the response change if the pulse were 2 seconds wide instead of 8? Just describe qualitatively and make a sketch of what you'd expect. Do not recompute the response.

12.11 Transfer Function

Given the transfer function

$$G(s) = \frac{X(s)}{U(s)} = \frac{s^3 + 110s^2 + 1100s + 1000}{s^4 + 2s^3 + 52s^2 + 51s + 50} \tag{1}$$

(a) Find the poles and zeros (you can use the MATLAB *roots()* command for this).

(b) Plot these poles on a s-plane.

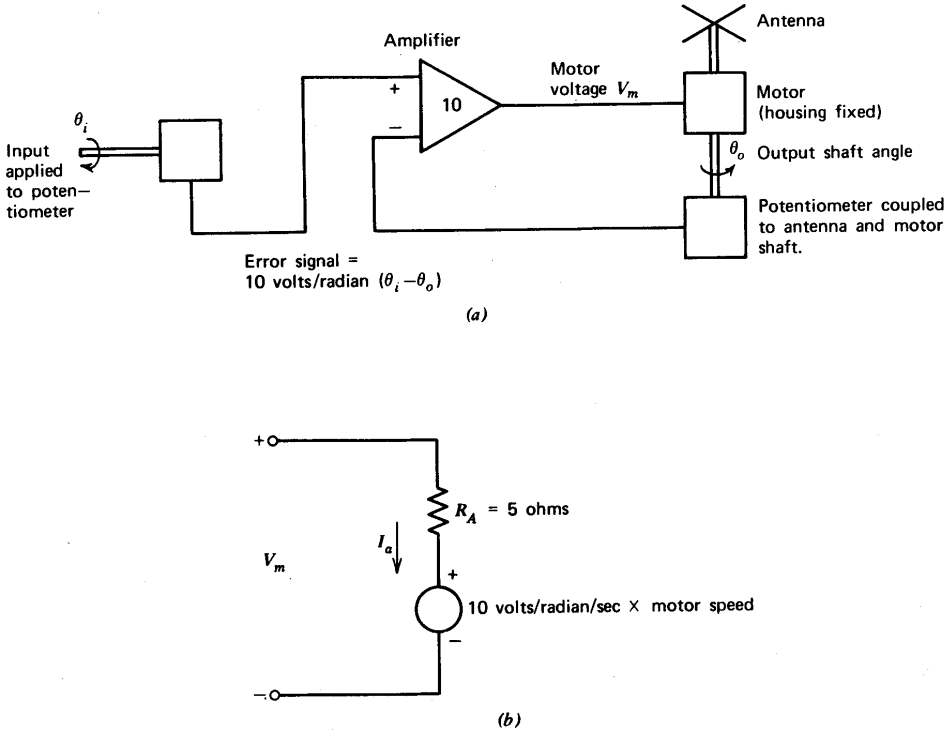
(c) Based on the location of these poles, *describe* the expected response to a step input. Please explain your reasoning.

(d) Using the MATLAB *step* command, create a plot of x vs. t to see if your prediction was OK, and compare.

13 Controls

13.1 Rooftop Antenna

(The following problem is adapted from Roberge, J.K., *Operational Amplifiers*, Wiley and Sons, 1975, problem P2.9.) A negative-feedback system used to rotate a rooftop antenna is shown in Figure 13.1.1(a). The total inertia of the output member (antenna, motor armature, and potentiometer wiper) is 2kg-m^2. The motor can be modeled as a resistor in series with a speed-dependent voltage generator as shown in Figure 13.1.1(b).



Antenna rotator System. (*a*) System configuration. (*b*) Model for motor.

Figure 13.1.1: Antenna Diagram

The torque provided by the motor that accelerates the total output-member inertia is 10 Nm per ampere of motor current I_a. The polarity of the motor dependent generator is such that it tends to reduce the value of I_a as the motor speed increases. The scale factor is such that I_a becomes zero for a motor shaft velocity equal to $V_m/10$ rad/sec.

(**Note:** There are a number of repeated analyses in this problem. You will want to use Matlab to automate these. The problem shows some of the key tradeoffs associated with feedback control, so please do think carefully about what you observe as the controller gain is changed through the values 10, 100, and 1000.)

(a) Draw a block diagram that relates θ_o to θ_i. You may include as many intermediate variables

as you wish, but be sure to include V_m and I_a in your block diagram. Calculate the transfer function θ_o/θ_i.

Also include in your block diagram a torque disturbance T_d applied to the antenna shaft. For instance, this torque disturbance might represent the effect of wind on the antenna. Calculate the transfer function θ_o/T_d which represents the response to such a disturbance.

(b) Plot the system poles in the s-plane. Calculate the response in θ_o to a 1 rad step input θ_i. Plot this response showing relevant time scales.

(c) Generate the Bode plot for θ_o/θ_i.

(d) Calculate the unit step response and Bode plots for the disturbance transfer function θ_o/T_d.

(e) Calculate the transfer function of the negative of the loop transmission for this feedback loop. Plot a Bode plot for this transfer function. What are the crossover frequency and phase margin for this loop? (These terms have been discussed in lecture.)

(f) Now change the gain of the differential amplifier in the loop from 10 to 100. (This will increase the loop transmission magnitude by a factor of 10.) Repeat parts (b), (c), (d), and (e) with this new value of loop gain. Comment on the effect of this larger loop gain on the system closed-loop dynamics exhibited in the speed and damping of the step response, Bode plots, command following ability, and the ability to reject a disturbance torque. How do the poles move as the gain is increased?

(g) Repeat the analysis of part (f) for an amplifier gain of 1000.

13.2 Lead Controller

We now extend the laboratory system to consider position control, and the design of a *lead* controller. The system transfer functions used in the lab all refer to the motor output shaft angle θ_m. We will define the potentiometer shaft angle as θ_p. The potentiometer is geared down from the motor by 6:1. That is, $n_d = 6n_p$, and thus $6\theta_p = \theta_m$. Further, the potentiometer outputs $V_{pot} = \pm 10$ V for a potentiometer shaft rotation of $\theta_p = \pm\pi$ rad.

(a) Show that the relationships above imply that $V_{pot} = K_{pot}\theta_p$, where $K_{pot} = 10/\pi$ V/rad. Note that we have defined K_{pot} as depending linearly upon θ_p.

(b) Modify the block diagram of Fig. 2 of the prelab to include the motor angle θ_m, the potentiometer angle θ_p, and the potentiometer gain K_{pot}. Use this modified block diagram to calculate the transfer function V_{pot}/V_{in}. (Recall that V_{in} is the amplifier input voltage as shown in the prelab.) Sketch the poles of this transfer function as well as its Bode plot.

(c) Now assume that the motor position controller is a lead controller of the form

$$\frac{V_{in}(s)}{E(s)} = K\frac{10\tau s + 1}{\tau s + 1}.$$

Here $E(s) = R(s) - V_{pot}(s)$, where $R(s)$ is a position reference voltage, and $E(s)$ is thus the position error. The controller is embedded in the position feedback loop as shown in Figure 13.2.1.

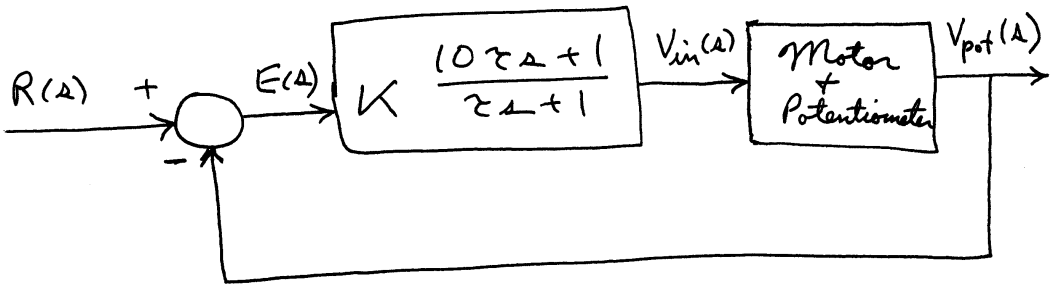


Figure 13.2.1: Block Diagram

Choose the controller parameters K and τ to achieve a loop crossover frequency $\omega_c = 100$ rad/sec, and phase margin $\phi \geq 45°$. Explain how you developed a design satisfying these design values. Include a Bode plot of the negative of the loop transmission which shows that you have achieved these design values. Also include a plot of the closed-loop pole locations.

13.3 Third Order

Consider the mechanical drive of Problem 14.1 with the load *free* to move. The model parameters have the following values:
I_{rotor} = 5e-5 kg-m^2
B_{motor} = 1e-4 N-m-s/r
$B_{coupler}$ = 2e-5 N-m-s/r
$K_{coupler}$ = 1.24e-2 N-m/r
I_{load} = 5e-5 kg-m^2
B_{load} = 1e-4 N-m-s/r
This is a system with fairly special symmetry. Note that $I_{motor} = I_{load} = I$,and that $B_{motor} = B_{load} = B$. For this special case, develop a model to predict the angular velocity of the load in response to an applied motor torqueT_m.

(a) Use the variables:

$\Delta\theta$, relative angular displacement of motor with respect to load,

ω_m, motor angular velocity, and

ω_l, load angular velocity,

as the state variables for this model in a state-determined system. Construct the state equations for this model.

(b) Consider a change of variables to the following three variables:

$$\Delta\theta, \qquad \Delta\omega = \omega_m - \omega_l, \qquad \text{and} \qquad \omega_{avg} = \frac{\omega_m + \omega_l}{2}$$

Express the variables ω_m and ω_l in terms of the variables$\Delta\omega$, and ω_{avg}.

(c) Re-write the state-determined system using $\Delta\theta$,$\Delta\omega$, and ω_{avg} as state variables. Show that the dynamic behavior of the system may be viewed as a combination of a first-order system with state variable ω_{avg} and a second-order system with state variables $\Delta\theta$, and $\Delta\omega$. Note that both sub-systems are excited by the motor torque T_m but are otherwise independent of one another.

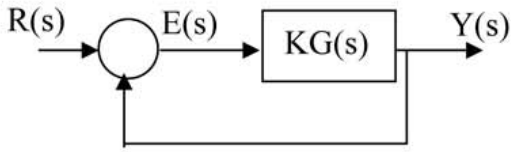
(d) Evaluate the first-order behavioral parameter: the decay time-constantτ. Estimate the steady-state speed ω_{ss} when T_m = 6.0e-3 N-m.

(e) Evaluate the second-order behavioral parameters: the undamped natural frequency ω_o, and the damping ratio ζ.

13.4 Error Constants

We have not formally treated the subject of error and error constants, and it becomes important when doing loop shaping with compensators. To consider it here, we will construct a classical table of error as a function of K_{loop} for different input types and controller types.,

Consider first a canonical control system:



Where G(s) is the loop transfer function and K is the DC loop gain.

a) Derive the error transfer function E(s)/Y(s)

Now consider three types of input for R(s)

Step	$R(s) = 1/s$	(also referred to as an *position* input)
Ramp	$R(s) = 1/s^2$	(referred to as a *velocity* input)
Parabola	$R(s) = 1/s^3$	(referred to as an *acceleration* input)

b) For each of these find an expression for E(s) given R(s) and the loop transfer function G(s)

Now we consider 3 "types" of systems

Type 0	G(s) has no integrators: G(0) = constant = K
Type 1	G(s) has one integrator (i.e. a single pole at the origin)
Type 2	G(s) has two integrations (i.e. a double pole at the origin)

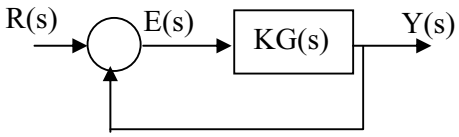
c) Based on a) and b) and the definition of "type", fill in the following table:

Steady State errors

	Type 0	Type 1	Type 2
Step Input			
Ramp			
Parabola			

Figure 13.4.1:

13.5 System Type



Where G(s) is the loop transfer function and K is the DC loop gain.

Using the same controller form as above, you are given a transfer function

$$G(s) \quad \frac{400}{s(s^2 \quad 10s \quad 400)}$$

By the above definitions, this is a type 1 system, and should have zero error to a step (or "displacement") input. However, the velocity error (that resulting from a ramp input) will not be zero. Indeed our goal is to achieve a velocity error of < 2%.

For this we can explore two different approaches.

a) First plot a bode diagram of G(s)

b) From that diagram, what will be the velocity error if the closed-loop system has a phase margin of 50º? (That is, after you have found a loop gain to achieve the phase margin)

c) Since the error is too large, a former 2.003 student tells you simply to increase the system type from 1 to 2 and all your problems will be over: just look at the error table!

Good advice or not? - Please explain by looking at the resulting open loop bode diagram.

Now look at the alternative of using a lag compensator.

d) Given that we want a velocity error < 2%, design a lag compensator that will achieve that result without changing the phase margin at the crossover frequency you found in part b). Show the final Bode plot to confirm the result

e) Now plot the resulting step response of the system designed in part d) How does it compare to the expected values?

f) Looking at the closed-loop system response to a ramp input, confirm that you have achieve a steady-state velocity error of 1%

Figure 13.5.1:

13.6 Unity Feedback

Consider the feedback system of Figure 13.6.1 in block diagram form where $G(s) = 1/s$.

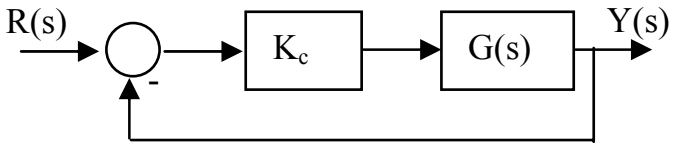


Figure 13.6.1: System with unity feedback

(a) Determine the closed-loop transfer function $T(s) = Y(s)/R(s)$ for this system.

(b) Find the gain K_c that will result in a closed-loop step response with a settling time of 1 millisecond.

(c) On a single Bode magnitude plot, show the following:

- $G(s)$
- $T_1(s)$ for $K = 1$
- $T_2(s)$ for $K = 10$
- $T_3(s)$ for $K = 100$

Be sure to label each plot clearly.

(d) If we define bandwidth as the range of frequencies over which the magnitude is > 0.5, what are the corresponding bandwidths of the three $T(s)$ plots in part (c)? Recall that we said a true model of an op amp is: $G_a(s) = K/s$ where $K \approx 10^6$

(e) Given this information, what is the expected step response of a unity-gain op-amp?

(f) What is the corresponding bandwidth of that op-amp?

13.7 P and PI Controllers

In the lab this week, you are designing P and PI controllers for the motor velocity. This problem refers to the designs you've developed for the lab. Please use the results from your Prelab 9 calculations to work this problem.

(a) For the proportional velocity loop, write an expression for the transfer function of the negative of the loop transmission. Plot the Bode plot for this transfer function. What are the crossover frequency and phase margin for this loop? What is the closed-loop pole location? How does this compare with the crossover frequency?

(b) For the proportional-integral velocity loop, write an expression for the transfer function of the negative of the loop transmission. Plot the Bode plot for this transfer function. What are the crossover frequency and phase margin for this loop? Also plot the closed-loop poles.

(c) For the proportional-integral velocity loop, the controller has the form

$$K\frac{\tau s + 1}{s}.$$

In the prelab, you have calculated values of K and τ to locate the poles with $\zeta = 0.5$ and $\omega_n = 30$. In part b) above, you have looked at this design from a loop transmission perspective. Now we want to consider the design as it is affected by the loop phase margin.

In this section, modify the PI controller so that you achieve a crossover frequency of 200 rad/sec, with a phase margin of 15 degrees. What values of K and τ result? Be sure to explain how you arrived at these parameter values. Plot the closed-loop poles that result. What are the values of ζ and ω_n associated with these new root locations? How do these root locations compare with the values in part b)? What root parameter seems to be associated with the crossover frequency, and what parameter seems to be associated with the phase margin?

14 Motors and Transformers

14.1 Locked Load

The sketch (Fig. 14.1.1) depicts a common mechanical drive configuration discussed in class. To identify parameters of the motor and coupler it is often useful to conduct a test of the locked-load dynamic response. The load is immobilized; a step change in motor torque is applied; and the resulting motion of the motor is observed.

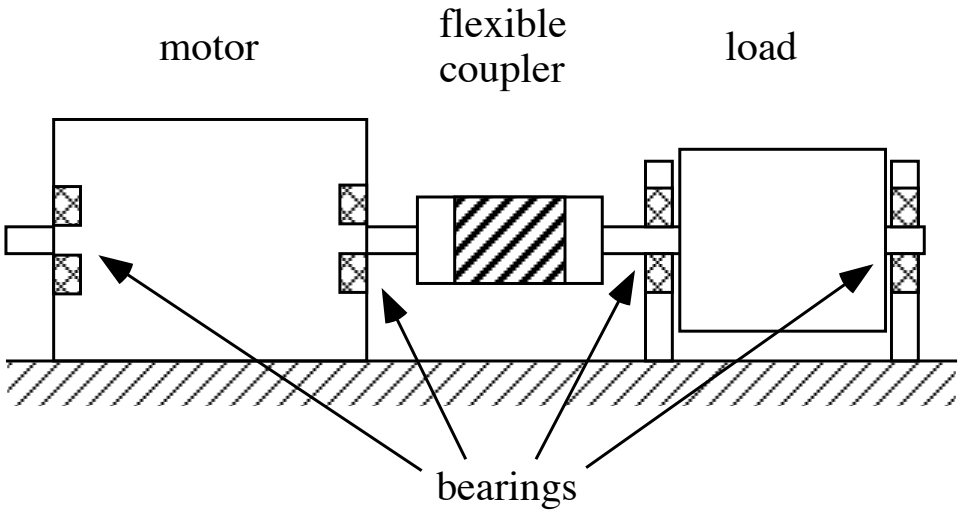


Figure 14.1.1: Mechanical Drive

(a) Assuming that all model elements have linear constitutive equations, develop a model to predict the angular position of the motor in response to an applied motor torque T_m. Express this model as a state-determined system.

(b) Calculate

(i) the damped natural period, and

(ii) the decrement ratio for the oscillatory response, when the model parameters have the following values:
$Irotor$ = 5e-5 kg-m^2
B_{motor} = 1e-4 N-m-s/r
$B_{coupler}$ = 2e-5 N-m-s/r
$K_{coupler}$ = 1.24e-2 N-m/r
T_m = 4e-3 N-m abruptly applied constant torque
$\omega(0) = 0$ initial angular velocity
$\theta(0) = 0$ initial displacement

(c) A proportional *velocity* feedback controller is added to the uncontrolled system described in (b) above. The motor angular speed is sensed and used in the control algorithm

$$T_m = G_1(r_1 - \omega_{sensed}) \tag{1}$$

where r_1 is a reference *speed* parameter, and G_1 is a gain parameter. In what units should G_1 be expressed? Analyze the closed-loop system and construct a state-determined system which describes its dynamic behavior. Calculate the controller gain, G_1, required to yield a critically-damped locked-load response.

(d) A proportional *position* feedback controller is added to the uncontrolled system described in (b) above. The motor angular position is sensed and used in the control algorithm

$$T_m = G_2(r_2 - \theta_{sensed}) \tag{2}$$

where r_2 is a reference *position* parameter, and G_2 is a gain parameter.In what units should G_2 be expressed? Analyze the closed-loop system and construct a state-determined system which describes its dynamic behavior. Derive algebraic expressions which show how the following closed-loop behavioral parameters depend on the feedback gain, G_2:

(i) The locked-load undamped natural frequency ω_O.

(ii) The locked-load damping ratio ζ.

(e) The usual implementation of the control algorithms (1) and (2) is with electronic signal processing and amplification. It is however theoretically possible to implement these algorithms mechanically by interpreting the control algorithms as constitutive equations for elements connected between the motor rotor and a reference rotor which is driven with the desired motion. Can you identify the type of mechanical element which is represented by:

(i) The algorithm (1)?

(ii) The algorithm (2)?

14.2 Motoring with a Capacitor

This problem concerns the motor connection shown in Figure 14.2.1. In this system, we assume that the motor torque is related linearly to current $\tau_m = Ki_m$, and thus that the motor voltage is given by $v_m = K\omega_m$. We further assume that the motor coil has zero resistance $R = 0$ and zero inductance $L = 0$. The capacitor C is connected to the motor terminals as shown. The motor rotor has a moment of inertia J.

Assume that the motor is driven with a torque source, i.e., that τ_{in} is specified as an external input independent of the shaft speed. The shaft of the torque source is rigidly connected to the motor output shaft as shown in the figure. The torque source is massless.

Note that we have defined the reference direction for motor speed ω_m to be the same as that for the source speed ω_{in}. Further note that with these definitions the motor torque is applied to the rotor in the same direction as the reference for ω_m.

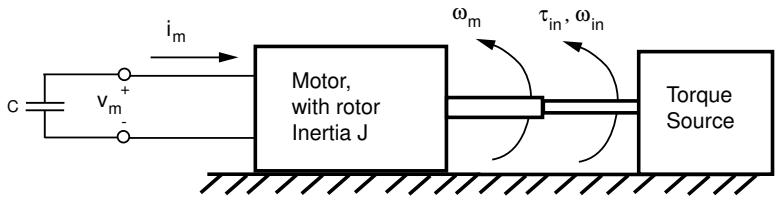


Figure 14.2.1: Motor Diagram

(a) Draw a free body diagram for the motor rotor showing the torques acting on it. Write Newton's second law for this free body. Note that continuity requires that $\omega_{in} = \omega_m$.

(b) How are the motor voltage and current related in light of the capacitor connection?

(c) Use the information gathered above to write a differential equation for this system with τ_{in} as the input and the resulting speed ω_{in} as the output. Be sure to show the steps you used to develop this differential equation. The electrical variables should be eliminated in your differential equation.

(d) What is the equivalent mechanical model seen by the input torque source? Why?

(e) Let the torque take a step from 0 to A Nm at $t = 0$ with zero initial rotor velocity. What is the resulting velocity ω_{in} as a function of time?

14.3 Leadscrew with Translating Stage

The system shown in Figure 14.3.1 has a translational stage of mass M which moves on frictionless bearings. The stage is coupled to a lead-screw via the nut shown; the lead screw has a pitch of 10^{-2} m/revolution, and is fixed in the x-direction by the bearings shown, but is free to rotate about its axis. The lead screw has a moment of inertia J about this axis. The mass of the nut is included in M. The stage is connected to translational spring K_2 and damper c_2. The lead screw is connected to rotational spring K_1 and rotational damper c_1.

Write the transfer function $X(s)/F(s)$ where F is an input force driving the stage, and x is the stage position.

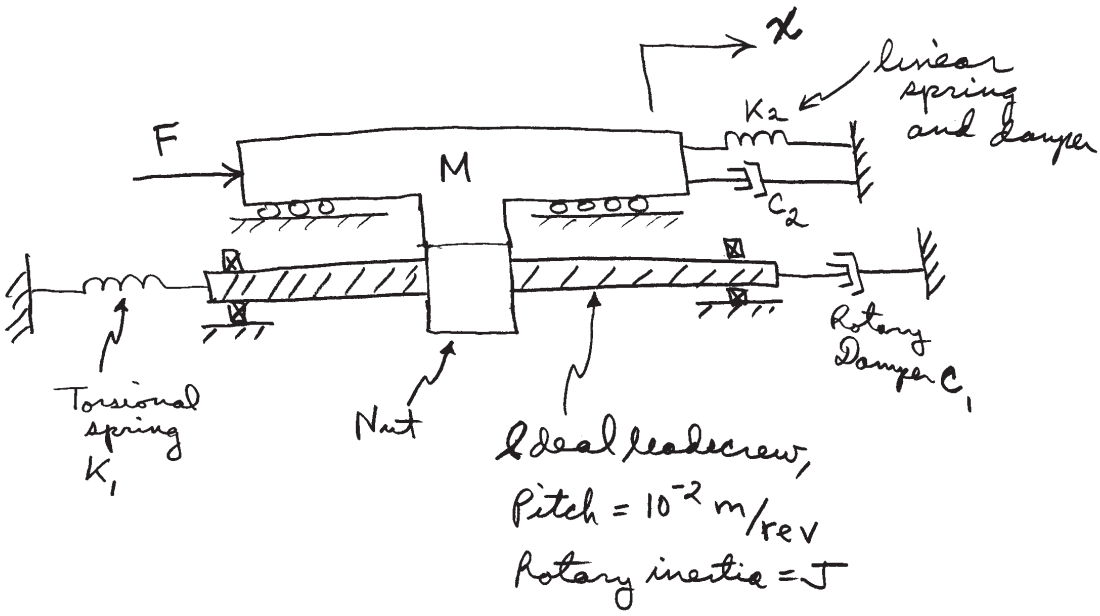


Figure 14.3.1: Leadscrew

14.4 Mass Pulley System

This problem concerns the mechanical system shown below.

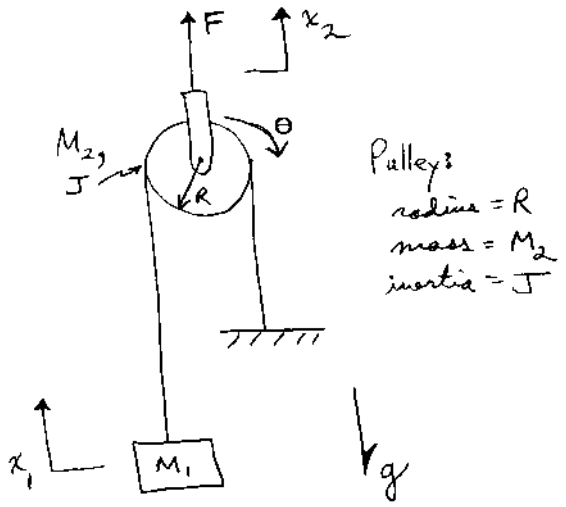


Figure 14.4.1: Pulley System

As shown in the figure, the system has a force input F which acts on the pulley carrier. The position of the carrier is x_2. The pulley has a radius R, moment of inertia J, and mass M_2. The carrier is included in M_2. The angular rotation of the pulley is θ, as shown.
A mass M_1 hangs on a string which is wrapped around the pulley as shown, and anchored to ground. The position of M_1 is x_1. Note also that gravity acts on the system in the downward direction. We assume that the motions of the system are such that the string remains under positive tension.

(a) What is the relationship between x_2 and θ? Show your reasoning.

(b) What is the relationship between x_2 and x_1? Show your reasoning.

(c) Now we ask you to develop a dynamical relationship between the applied force F, the gravity load(s), and the resulting motion x_2. That is, write a differential equation that describes the motion of the system with F and g as inputs, and with only x_2 as the dependent variable. The variables x_1 and θ should not appear in your equation. Clearly show your reasoning in developing this model. Be sure to include relevant free-body diagrams, and other figures that explain the steps in your development.

14.5 DC Motors

Consider the DC motor shown below. It is connected to a rudimentary circuit with only a single resistor, which follow Ohm's Law : $e = Ri$ where R is the circuit resistance.

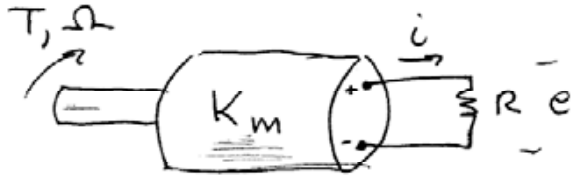


Figure 14.5.1: DC Motor with resistor

(a) Given the motor constant K_m and the circuit as shown find the relationship between T and Ω as seen at the motor shaft. Be careful with signs and look at your answer critically to be sure you got it right.

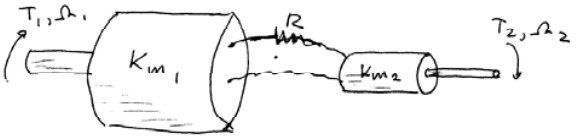


Figure 14.5.2: DC Motors connected together

(b) Now consider the system shown in Figure 14.5.2, where two motors are connected together as shown. Assuming negligible mechanical friction and inertia in the motors, find the overall transformer relationship θ_1/θ_2 and T_1/T_2 as a function of K_{m_1}, K_{m_2} and R.

(c) What would be a purely mechanical equivalent of this?

14.6 Gear Train

For the system shown below with $k = 10$ [Nm/rad], $J = 1$ [Nm s^2/rad] and $b = 1.25$ [Nms/rad]:

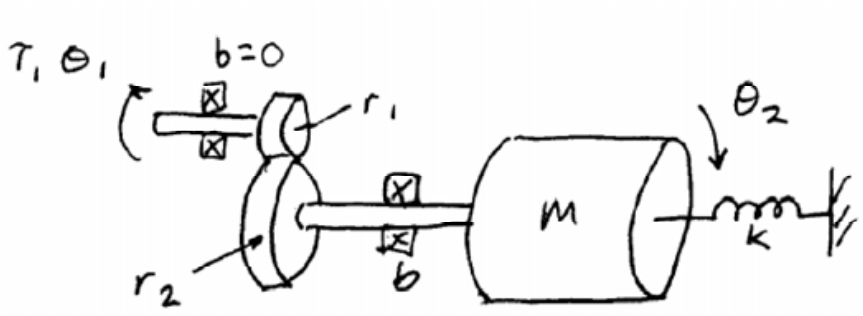


Figure 14.6.1: System Diagram

(a) Find the basic dynamic properties ζ and ω_n as seen at θ_2.

(b) Now reflect these properties back to the input shaft. What are the basic properties ζ and ω_n as seen at θ_1.

(c) Sketch the response of θ_1 and θ_2 to a unit step in T_1 in the same plot.

(d) What effect does the gear train have on the characteristic dynamics of this system? Please explain your answer.

14.7 Non-Ideal Transformer

For the system shown in the figure below, we cannot ignore the dynamic properties of the transformer and must try to account for them in our model.

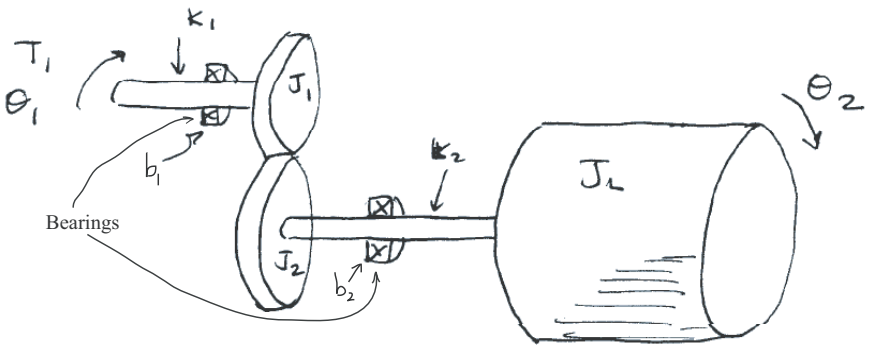


Figure 14.7.1: Diagram of Non-Ideal Transformer

Assuming that k_1 is very large, the load inertia is large but of the same order of magnitude as J_1 and J_2, and that the bearings exert a viscous drag on the shaft.

(a) Write the equation of motion for each important mass in this system.

(b) Express each of those equations in terms of appropriate input/output transfer functions.

(c) Using a block diagram, show how these transfer functions combine to give us the overall transfer function $G(s) = \theta_2/T_1$.

(d) Now using block diagram algebra, derive the overall transfer function for this system explicitly.

14.8 Equivalent Mass and Inertia

(a) Derive an expression for the equivalent mass at the point of application of the force in Figure 14.8.1.

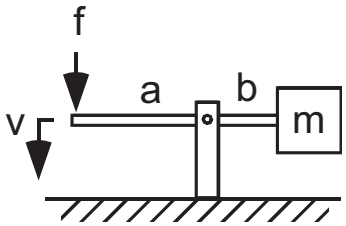


Figure 14.8.1: Mass on cantilever.

(b) Derive an expression for the equivalent mass at the point of application of the force in Figure 14.8.2. Assume that each wheel has a mass, m , and a rotational inertia of J . Assume there is no energy loss due to bearings and that the wheels do not slip.

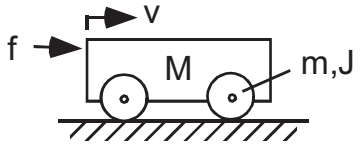


Figure 14.8.2: Cart of mass M and wheels each of inertia J.

(c) Derive an expression for the equivalent rotational inertia in Figure 14.8.3. Assume that the two pulleys have negligible inertia, there are no losses due to bearings and that the flywheel has inertia, J .

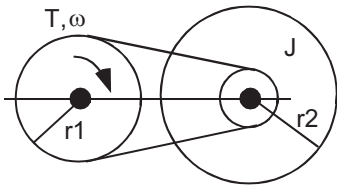


Figure 14.8.3: Pulley System

14.9 Inertia in Geartrain

This problem considers the system shown in Figure 14.9.1 in which two rotational inertias J_1 and J_2 are connected by a gear train.

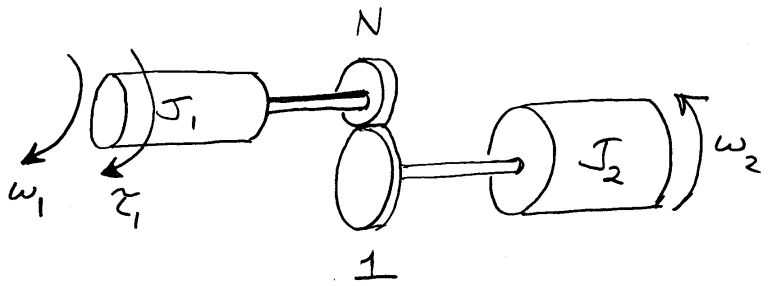


Figure 14.9.1: Rotational inertias connected through a gear train.

The rotational speed of J_1 is ω_1 and the rotational speed of J_2 is ω_2. An input source of torque τ_1 is applied to J_1 in the direction of ω_1. As shown in Figure 14.9.1, the gear train has a ratio of $N:1$; that is, $\omega_2 = N\omega_1$.

(a) Assume that the input torque source has a constant value of $\tau = \tau_0$. What value of N will maximize the acceleration of the load $\dot{\omega}_2$?

(b) For this acceleration-optimum gear ratio, what is the equivalent inertia of J_2 as seen by J_1 looking through the gear train? That is, what is the reflected inertia of J_2 on the J_1 side? How does this compare with J_1?

(c) The power input to the system is $P_{in} = \tau_1\omega_1$. For the optimum gear ratio calculated above, make a plot of $P_{in}(t)$ assuming that the load starts at rest at $t = 0$.

14.10 Motor Driving Inertia Through Gear Train

This problem focuses on the motor connected to a load as shown in Figure 14.10.1. The motor is driven by an input voltage V in series with a coil resistance R. The motor is assumed ideal, with no energy storage or losses inside the motor. The motor is connected to a load inertia $J > 0$ and rotational damper $b \geq 0$. The motor shaft and load rotate at angular velocity ω. The motor applies a torque to the load $\tau = Ki$ in the direction of ω; correspondingly, the back emf is $e = K\omega$.

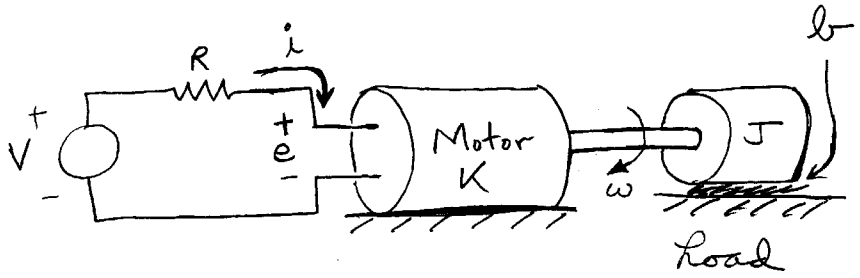


Figure 14.10.1: Motor driving a rotational inertia and damper.

(a) Write a differential equation describing the system in terms of the input voltage $V(t)$ and the output speed $\omega(t)$. Write an equivalent differential equation with input $V(t)$ and output $i(t)$.

(b) Assume that the system is initially at rest, and that at $t = 0$ the input voltage takes a step $V(t) = V_0 u_s(t)$. Solve for the resulting transient in $i(t)$ and $\omega(t)$, and make a plot of these two quantities as a function of time.

(c) What are the steady-state values of i and ω? In steady-state, write an expression for the power being dissipated on the mechanical side in the load damper b, and on the electrical side in the resistor R. How much power is being supplied in steady-state by the voltage source? Is this in balance with the dissipation? In this steady-state, how much kinetic energy is stored in the load inertia J?

(d) Make a plot of steady-state load power dissipation as a function of load damping b for $b \geq 0$. What value of b results in maximum power dissipation in the load? How does this compare with the electrical equivalent damping term K^2/R? For this maximum power value of b, how much power is being dissipated in the resistor R?

(e) Finally, suppose we allow negative values of the load damping. Note that a negative damper will *supply* power to the load. For what range of $b < 0$ will the system be stable? For what range of $b < 0$ will the system be unstable?

15 MATLAB and Simulink

15.1 Matlab Spirograph

Your instructor has written a small Matlab m-file called `spirograph.m` which he claims makes spirograph plots. This file is listed below, and is available for download from the course web site. Most of you have probably played with a simple spirograph made from a sheet of plastic. In such a device, an inner gear of some radius rolls inside an internal gear of larger radius. A pencil is inserted in the inner gear in one of a number of possible locations. As the inner gear rolls, the pencil thus traces out a trajectory on a piece of paper lying under the spirograph. The spirograph has a number of choices for the sizes of the inner and outer gears, which result in interesting variations in the generated patterns.

Unfortunately, the instructor was too lazy to add much in the way of comments when he wrote the file. Your job is to look through this file to understand the underlying geometry and computations. Specifically, we ask you to:

(a) Develop a diagram which explains the geometry which is being computed. Be sure this diagram includes all three of the radii r_0, r_2, and r_3, as well as the angles θ_1 and θ_3.

(b) Draw a diagram of the gear system, showing how this relates to the geometry above.

(c) Explain the operation of the program and the function of each line of code, by annotating a listing of the m-file.

(d) Run the m-file in Matlab by choosing some values for the parameters in the edit window, saving the file, then typing the name `spirograph` at the Matlab command line. Find some interesting values for the choice of radii and include the resulting plots. Can you explain the essential features of the plots on the basis of the chosen radii?

(e) Rewrite the file as a Matlab function which can be called with the three radii as arguments. Include a listing of your function. To find out more about Matlab m-files and functions, review the material on Matlab in Ch. 1 of the text. You can also use the on-line Matlab help system.

15.2 MATLAB Plotting

Use MATLAB to plot the following signals. Choose the appropriate time range and sampling points.

(a) $10 \sin 100t$

(b) $e^{-t} \cos 100t$

(c) $Re\{e^{st}\};\ s = -1 + 10i$

(d) $5e^{-10t}$

(e) $5e^{10t}$

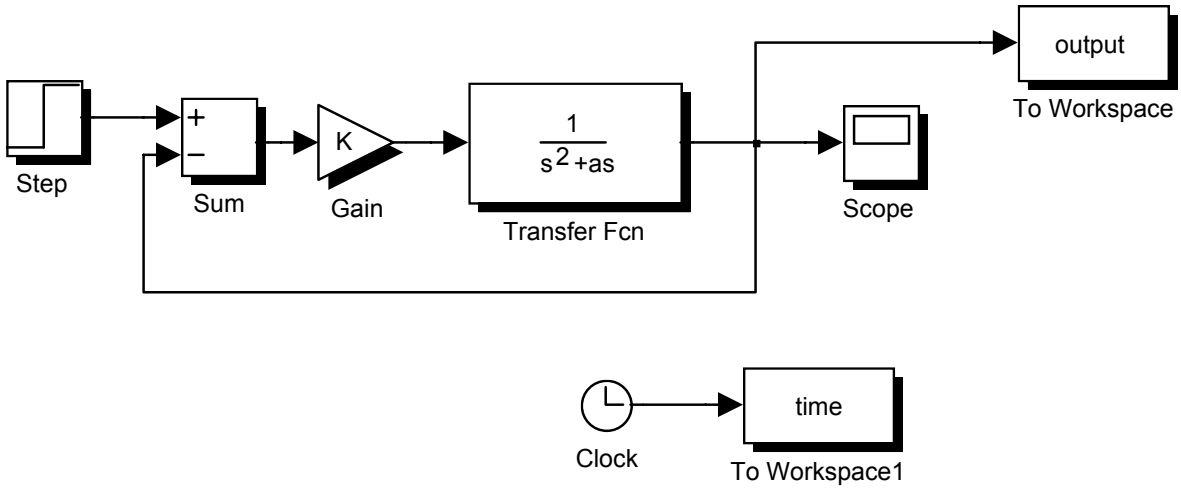
(f) $8e^{-100t} - 5e^{-1000t}$

15.3 Simulink Introduction

Fun and games with SIMULINK

The purpose of this short but *sweeeeet* assignment is to learn about the block diagram language within MATLAB know as SIMULINK. SIMULINK lets you create block diagrams of dynamic systems, invoke feedback control designs and then "run" the system to see the predicted performance.

To get started, simply type "Simulink" in the MATLAB command window. The GUI will then include a menu of various block options, along with a model window in which you build the system. For example, the simple velocity servo we have discussed in class would look like this in SIMULINK:



And if I set K=5 and a = 1 in the command window, the simulation (using default simulation parameters) becomes:

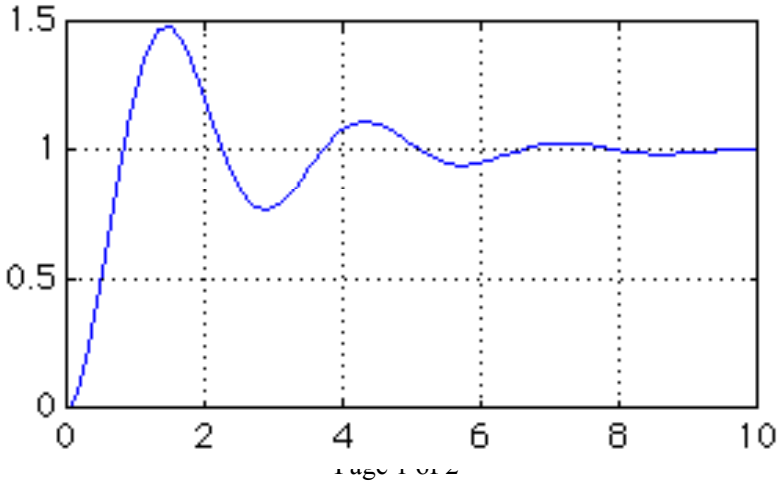


Figure 15.3.1:

For this assignment I encourage you to look at the various blocks in SIMULNIK and to run some of the DEMOS.

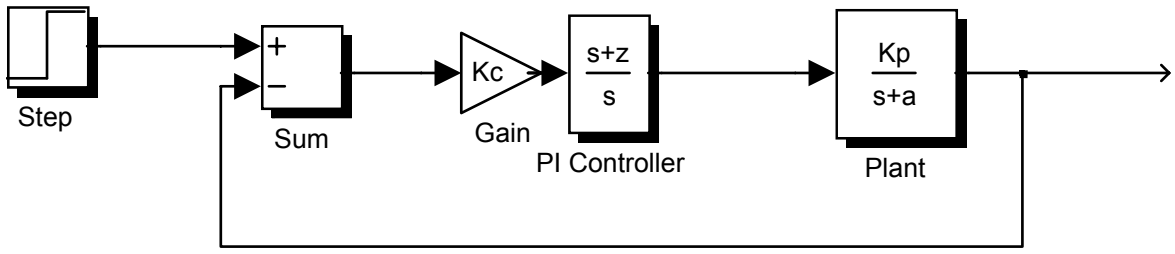
In addition, please look at the following:

Consider the problem we have done in class: The PI control of the velocity servo. However, here we address the problem of noise in the velocity measurement. This is typically done with a tachometer (really a generator) and leads to high frequency noise from the brushes in the generator. As a result, we often place a "low pass filter" in the feedback loop. Thus:

$$H(s) = \frac{K_f}{\tau_f s + 1}$$

Let's look at this effect by using both analysis and SIMULINK simulation:

a) First create the model of the PI controller for the velocity servo:



b) Calculate analytically the values for K_c and z (for the PI controller) that will give a critically damped response with a settling time of 0.5 second. (Assume a=Kp=1)

c) Now implement those gains in SIMULINK and see if it agrees with expectations. If there is any discrepancy, please explain it.

d) Now add the feedback filter H(s) to the SIMULINK diagram. Assume that K_f= 1 and that the filter time constant (τ_f = 0.5 sec). Without changing K_c *or z*, run the simulation again, and plot the result. What happened and why?

e) Now go ahead and vary the gain K_c until you get critically damped response again as seen by the simulation response. (Try to keep track of how many simulations you have to do until you get the "right" answer.) What is the settling time for this case?

Figure 15.3.2:

15.4 Wackygraph

Your instructor has written a small Matlab m-file called `wackygraph.m` which he claims makes interesting plots. This file is listed below, and is available for download from the course web site.

```
% wackygraph
% DL Trumper 9/9/02


r0 = 10;
r2 = 1;
N = 10;

g1 = 40;
g2 = 0.2*r0;


theta1 = [0:pi/1000:100*pi];
theta3 = -g1*theta1;

r1 = r0 + g2*sin(N*theta1);

position = (r0+r1).*exp(j*theta1) + r2*exp(j*theta3);
figure(1)
plot(position)
axis equal
```

Unfortunately, the instructor was too lazy to add much in the way of comments when he wrote the file. Your job is to look through this file to understand the underlying geometry and computations. Specifically, we ask you to:

(a) Develop a diagram which explains the geometry which is being computed. Be sure this diagram includes all three of the radii r_0, r_1, and r_2, as well as the angles θ_1 and θ_3. What are the effects of the parameters g_1, g_2, and N?

(b) Write expressions for the real part of the `position` variable, and the imaginary part of the `position` variable, *i.e.*, Re{`position`} and Im{`position`}, respectively.

(c) Explain the operation of the program and the function of each line of code, by annotating a listing of the m-file. Note the use of the `.*` operator which does element by element multiplication. Why did the program use this. What happens if the regular `*` operator is used?

(d) Run the m-file in Matlab by chosing some values for the parameters in the edit window, saving the file, then typing the name `wackygraph` at the Matlab command line. Find some interesting values for the choice of parameters and include the resulting plots. Can you explain the essential features of the plots on the basis of the chosen parameters?

(e) Rewrite the file as a Matlab function which can be called with the three radii and three parameters as arguments (6 total arguments). Include a listing of your function. To find out more about Matlab m-files and functions you can use the on-line Matlab help system.

16 Case Studies

16.1 CD Player 1

In a typical CD player the disc is mounted directly on the shaft of an electric motor. The rotor of the motor has a moment of inertia I_r and experiences a frictional retarding torque which can be modeled by a linear damper with damping coefficient B_m.

(a) The mass of a compact disc is 0.028 kg, and its diameter is 0.120meters. The diameter of the hole in the center is 0.015 meters. Calculate the moment of inertia I_l of the CD.

(b) Assume that the electric motor, when it is powered, produces a constant torque T_m independently of the rotational speed ω_m. It is observed that,after an initial starting transient, the steady-state rotor speed ω_{ss} is the *same*, whether a CD is mounted or not. The start-up transients are, however, different. With no disc mounted, the rotor spindle-speed reaches 95% ofω_{ss} in 0.30 seconds, but when a disc is mounted it takes 2.0 seconds toreach that same speed. Assuming that this behavior can described by a simple linear model, estimate the value of the damping coefficient B_m,

   (i) when there is no disc mounted, and

   (ii) when a CD is mounted.

(c) Estimate the moment of inertia of the motor rotor I_r.

(d) Construct a state-determined representation of your linear model for the system with the CD mounted. Take T_m to be the input and ω_m to be the output.

16.2 CD Player 2

A common design for a speed controller uses simple proportional velocity feedback. The actual speed of the disc is sensed from information written onto the disc. The motor torque T_m is then generated according to the following control algorithm

$$T_m = G(r - \omega_{sensed})$$

where r is a constant reference speed, and G is a constant gain.

(a) What are the dimensions of the gain parameter G? What would be the units for G in the SI system of units?

(b) Consider that this controller is applied to the CD player modeled in Problem 16.1 (d). Construct a state-determined representation of a linear model for the controlled system with r as the input and ω_m as the output.

(c) Compare the inputs required to achieve a steady-state disc rotation speed of 4.0 revolutions per second for the uncontrolled open-loop system of Problem1 and for the controlled closed-loop system of the present Problem:

  (i) What is the magnitude of the constant, suddenly applied, motor torqueT_m required to reach the desired steady-state speed in the uncontrolled system.

  (ii) Obtain a formula (involving the gain parameter G) for the magnitude of the constant, suddenly applied, reference speed r required to reach the desired steady-state speed in the controlled system.

(d) Design the controller; *i.e.,* choose the value of G, such that the time to reach 95% of the steady-state speed for the controlled system is 1/5 of thecorresponding time for the uncontrolled system.

(e) Write a MATLAB script which can integrate the equation (or equations) ofPart (b) and plot the time history of the motor torque T_m during the startingtransient of the controlled system. Compare the magnitude of the *maximum* motortorque in the controlled case with the magnitude of the constant torquefound in (c{i}) for the uncontrolled case.

16.3 CD Player 3

In a CD player the disc with moment of inertia I_d is mounted directly on the shaft of an electric motor, whose rotor has a moment of inertia I_r. The unit experiences a frictional retarding torque that can be modeled by a linear damper with damping coefficient B_m. The parameter values for the system are estimated to have the following values:

$$\begin{aligned} I_d &= \text{5e-5 kg-m}^2 \\ I_r &= \text{9e-6 kg-m}^2 \\ B_m &= \text{9e-5 N-m/rad/sec} \end{aligned}$$

Formulate the state-determined equation of motion for rotation with the angular velocityω_m as the output and the motor torque T_m as the input.

(a) Consider the steady-state sinusoidal oscillation of ω_m in response to an input torque of the form $T_m = T_a \sin \Omega t$. Calculate the response amplitude and phase angle for three frequencies:

$$\begin{aligned} \Omega &= 1 rad/sec \\ \Omega &= 5 rad/sec \\ \Omega &= 15 rad/sec \end{aligned}$$

for each of the following cases:

(i) No disc mounted on the motor spindle

(ii) A disc is mounted on the spindle

(b) Evaluate the *break frequency* for each of the following cases:

(i) No disc mounted on the motor spindle

(ii) A disc is mounted on the spindle

(c) Make an accurate, labeled, sketch of the Bode plots for amplitude ratio and phase angle for each of the following cases:

(i) No disc mounted on the motor spindle

(ii) A disc is mounted on the spindle

(d) If the input torques are applied to the rotor initially at rest, estimate how long it takes to reach the steady state (for most practical purposes, steady state may be assumed to be reached at the end of 4 decay time-constants, when 98% of the transienthas decayed). Do this for each of the three values of Ω, for each of the followingcases:

(i) No disc mounted on the motor spindle

(ii) A disc is mounted on the spindle

16.4 Pinewood Derby 1

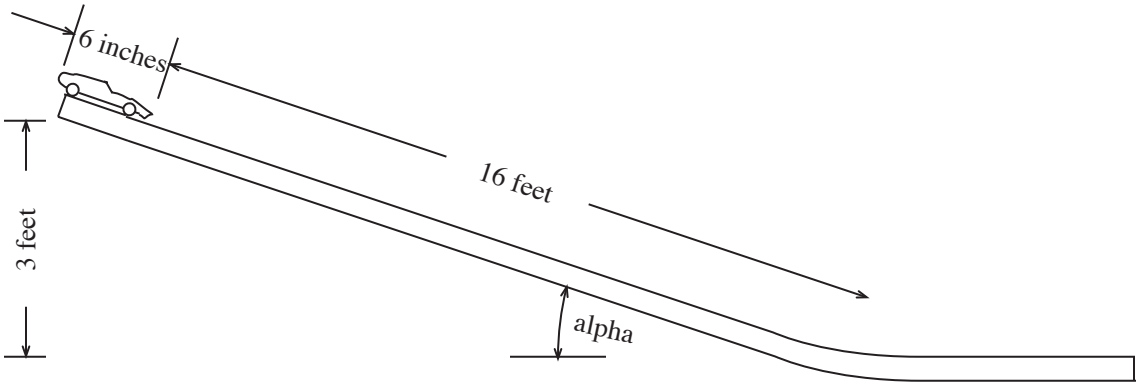


Figure 16.4.1: Pinewood Derby

Referring to the Pinewood Derby (Fig. 16.4.1), answer the following questions:

(a) In SI units, how much energy is available to move the race car?

(b) A typical Derby car is 6 inches long. If there were no friction of any kind,what would be the maximum speed (in SI units) a race car could attain if all its weight were concentrated at the front end of the car?

(c) What would be the maximum speed if all its weight were concentrated at the rear end of the car?

(d) In case (c) how long does it take the car to reach the bottom of the inclined section of track?

16.5 Pinewood Derby 2

Consider adding a viscous friction retarding force 'bv' to the previous model. With such a retarding force which increases with velocity, the speed of the car, even on an inclined track of infinite length, is limited by the speed at which the retarding force just balances the accelerating force. If this limiting speed is twice the velocity obtained in part (c) of Problem 16.4, what is the value (in SI units) of the viscous friction parameter 'b'?

16.6 Pinewood Derby 3

The two MATLAB scripts which follow permit you to integrate the equation of motion for a racecar on a long incline with angle 'alpha', subjected to a viscous friction force. Study the programs and experiment with them , using the Pinewood Derby parameters and the friction parameter 'b' obtained in Problem 16.5.

(a) Plot a graph of the racecar speed vs. time for a car starting at rest, over the time period obtained in Problem 16.5, part (d).

(b) With a long incline the speed of the car is asymptotic to the limiting speed described in Problem 16.5. Use the scripts to determine how long it takes for the speed to reach 99.9% of the limiting speed.

16.7 Pinewood Derby 4

In Problem 16.4 a Pinewood Derby race car was analyzed under the assumption that its motion was opposed by linear viscous friction with a constitutive relation

$$f_{viscous} = bv$$

(a) Formulate a revised model in which the viscous friction element is replaced by a model of air resistance (sometimes called *form drag*) with a constitutive equation

$$f_{drag} = AC_d v|v| = AC_d v^2 \text{sgn}(v)$$

where A is the frontal area of the car and C_d is a drag coefficient that depends on the car's geometry; *i.e.,* on its *form.*

(b) What are the dimensions, in SI units, of the drag coefficient C_d in the above constitutive relation?

(c) Derive a mathematical expression for the time-history of the speed of the car, on a horizontal track, if it starts at $t = 0$ with initial velocity v_o.

16.8 Pinewood Derby 5

Formulate a differential equation for the velocity of a race car in a model in which the car of mass m descends a very long inclined track with angle (alpha), under the influence of gravity and a formdrag, which has the constitutive equation described in Problem 12.8.

(a) Show that this model predicts a terminal velocity v_{ss} on an infinitely long track.Derive a formula for v_{ss} in terms of the parameters m, (alpha), g, A, and C_d.

(b) If, for the same parameter values considered in Assignment 1 (m = 0.1418 kg, (alpha) = 10.81degrees), the terminal velocity is the same as it was for viscous friction (v_{ss} = 8.60 m/s), what is (are) the parameter value(s) in the form-drag model?

(c) Study the MATLAB scripts 'car2.m' and 'car_form.m' which can be used to explore the model described in this Problem. Determine the time T it takes for the car starting at rest to accelerate to 99.9% of the terminal velocity.

(d) Compare the result of 2(c) with the corresponding result in 4(b) of Problem 16.7. Which friction model has a quicker approach to the terminal velocity? Give a physical and/or graphical explanation for this result.

16.9 Engine Block Vibration 1

Consider uniaxial vibration of an engine block that weighs 200 pounds and is supported on mounts that have an effective stiffness of 18,000 pounds per inch. and an effective linear damping coefficient of 2 pounds per inch per second, and is subjected to an oscillating force of the form

$$f(t) = f_a \sin \Omega t$$

where the amplitude f_a equals 2 pounds. Derive an equation of motion for the displacement $y(t)$, of the engine block from its equilibrium position.

(a) Write an expression for the complex amplitude A of the steady-state displacement response.

(b) Obtain:

(i) the low-frequency asymptote for the complex amplitude (a);

(ii) the high-frequency asymptote for the complex amplitude (a);

(iii) the break frequency Ω_{break} where the magnitudes of (i)and (ii) are equal.

(c) Evaluate the input frequency Ω_{peak} at which the engine displacement amplitude has the greatest magnitude.

(d) Evaluate the ratio $|A(\Omega_{peak})/A(0)|$ where $A(0)$ is the limit of $A(\Omega)$ as $\Omega \to 0$.

(e) Use MATLAB to make Bode plots for:

(i) the magnitude of the ratio $A(\Omega)/A(0)$;

(ii) the phase of the complex amplitude $A(\Omega)$.

Use a logarithmic scale for frequency Ω which extends at least one decade below, and one decade above, the break frequency. Express the magnitude scale in decibels; *i.e.,* plot $10 \log_{10}(\text{magnitude})^2$.

16.10 Engine Block Vibration 2

Reconsider Problem 16.9 for the same engine block and mounts, but instead of the input force with constant amplitude $f_a = 2$ pounds, consider the amplitude of the input force $f(t) = f_a sin\Omega t$ to increase with the square of the frequency, as it does when the force is a reaction to the force required to accelerate a mass $m = w/g$ which oscillates at frequency $\Omega$with displacement amplitude r. Take $f_a = (w/g)r\Omega^2$, where $(w/g)r = (1.93/386)(1) = 0.005$ pound-sec^2.

(a) Write an expression for the complex amplitude A of the steady-state displacement response.

(b) Obtain:

(i) the low-frequency asymptote for the complex amplitude (a);

(ii) the high-frequency asymptote for the complex amplitude (a);

(iii) the break frequency Ω_{break} where the magnitudes of (i) and (ii) are equal.

(c) Evaluate the input frequency Ω_{peak} at which the engine displacement amplitude has the greatest magnitude.

(d) Evaluate the ratio $|A(\Omega_{peak})/A(\infty)|$ where $A(\infty)$is the limit of $A(\Omega)$ as $\Omega \rightarrow \infty$.

(e) Use MATLAB to make Bode plots for:

(i) the magnitude of the ratio $A(\Omega)/A(\infty)$;

(ii) the phase of the complex amplitude $A(\Omega)$.

Use a logarithmic scale for frequency Ω which extends at least one decade below, and one decade above, the break frequency. Express the magnitude scale in decibels; *i.e.,* plot 10 $\log_{10}$(magnitude)^2.

16.11 Engine Block Vibration 3

Reconsider Problem 16.9, but now with emphasis on the start-up transient. Cast the equations of motion in the standard form for state-determined systems:

$$\begin{aligned} \tfrac{d}{dt}\mathbf{x} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned}$$

with the force $f(t)$ as input and the displacement $y(t)$ as the output. Consider that the engine is at rest in its equilibrium position until$t = 0$. Take the force $f(t)$ to be an abruptly initiated sinusoidal force of the form

$$f(t) = \left\{ \begin{array}{ll} 0, & t < 0 \\ f_a \sin \Omega t, & t > 0 \end{array} \right\}$$

with $f_a = 2$ pounds.

(a) Write a MATLAB script to integrate the state equations from the given initial condition. The script should have the capability of accepting an arbitrary value of the forcing frequency Ω.

(b) The MATLAB solution for $\mathbf{x}$ is a two-column matrix consisting of the functions $y(t)$ and $v(t) = dy/dt(t)$ evaluated at the many intermediate t-values listed in the one-column $\mathbf{t}$-matrix. Write some additional script to make MATLAB construct three one-column matrices of the following power quantities evaluated at the same set of intermediate$t$-values:

(i) $P_{in}(t) = f(t) * v(t)$, the power transmitted to the vibratory system by the input force $f(t)$.

(ii) $P_{diss}(t) = f_{fric}(t) * v(t) = bv^2(t)$, the power dissipated by the damping element.

(iii) $P_{stored}(t) = P_{in}(t) - P_{diss}(t)$, the power transmitted to the storage elements where it is continually shifted back and forth from kinetic energy of the engine block to potential energy of the suspension springs.

(c) Have MATLAB make the following four plots:

Engine block displacement $y(t)$ vs. t;

Power input $P_{in}(t)$ vs. t;

Power dissipated $P_{diss}(t)$ vs. t;

Power stored $P_{stored}(t)$ vs. t;

for each of three input frequencies:

(i) $\Omega = 1/3\Omega_{break}$;

(ii) $\Omega = \Omega_{break}$;

(iii) $\Omega = 3\Omega_{break}$.

In order to see both the transient response and the steady state response in the plots, the time span in the plots should be at least ten times longer than the system decay time-constant.

16.12 Plate On Springs 1

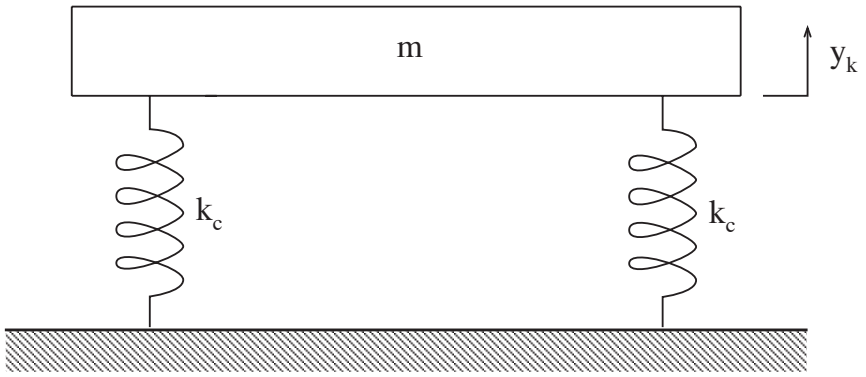


Figure 16.12.1: Steel Plate on Springs

In class a model of the vertical motion of a steel plate mounted on four springs (Fig. 16.12.1) analyzed. Consider the limiting case where all friction is absent. Derive an equation for the displacement history of the plate when it starts at $t = 0$ from its equilibrium position with the initial velocity v_o.

16.13 Plate On Springs 2

Consider the experiment demonstrated in class, in which a textbook is dropped on the spring-supported plate. Again, for simplicity, consider the limiting case where all friction is absent. Take the mass of the textbook to be one half the mass of the plate, and assume that during the motion the textbook remains in contact with the plate. There are two cases to be considered. In Case I there is a "zero height drop"in which the book is released from rest when just in contact with the plate. In Case II the drop is from a non-zero height such that the book strikes the motionless plate with a downward velocity of v_o. In both cases the motion begins at $t = 0$, at the instant when the book contacts the plate.

(a) What is the ratio of the frequency of oscillation in Case I to the frequency of free oscillation of the unloaded plate?

(b) What is the ratio of the frequency of oscillation in Case II to the frequency of free oscillation of the unloaded plate?

(c) Locate the equilibrium position of the (book + plate) system with respect to the equilibrium position of the unloaded plate .

(d) Determine the initial conditions for y_k and v_k for Case I.

(e) Determine the initial conditions for y_k and v_k for Case II.

(f) For each case write a mathematical expression for the vertical motion of the book and plate for $t > 0$.

16.14 Mousetrap Dynamics 1

This problem makes use of the mouse traps which were handed out in class. The problem is intentionally open-ended; we haven't told you what methods to use to find the requested information. Think physically about the problem, and develop the simplest methods which will allow you to answer the questions which we pose with reasonable accuracy. In your answers, we expect you to provide a good description of the experiments you conducted and the assumptions made in developing answers. Relevant and clear sketches, figures, and experimental data clearly presented will greatly help in telling us what you did. We will come back to this problem over the course of the next few weeks, so please retain copies of your data and calculations for later reference. We will not provide solutions for this problem, so it is up to you to develop a solid understanding now so that you will be able to effectively answer later modeling questions.

You are welcome to work in groups of your own creation for solving the experimental parts of this problem. However, it is up to you to be sure whatever collaboration you undertake is effective for increasing your own learning. The mousetraps are yours to keep, so feel free to drill holes, tie on strings, etc., if that will facilitate your experiments. Just be sure that your trap can remain functional. If you are working in a group, you can take one of your traps completely apart if that helps. We recommend putting some super glue on the two staples in the base where these enter the base in order to glue them into the base. Otherwise the staples have a tendency to pull out in about 20-30 triggers of the trap. **Safety note: Parts of the trap attain significant velocities when the trap is triggered.Thus, you must wear the provided safety glasses at all times when you are conducting experiments on the trap! Your vision is a precious commodity! Please return the safety glasses to us when requested at the close of this experiment.**

To have a common vocabulary, we will use the following terminology: The trap is built on a wood piece we will call the base. We will refer to the main pivoting wire portion of the trap as the bail. The bail is driven by the main spring wrapped around its pivot point. When cocked, the bail is retained by a wire that we will call the trigger bar.The bait is placed on a piece of metal we will call the trigger plate. When cocked, the trigger bar is captured by the trigger plate, and then released when the trigger plate is touched with sufficient force.Please answer the following questions. Think carefully about how to develop these answers in a accurate,efficient manner. Use fundamentals to guide your thinking!

(a) Define the cross-bar as the portion of the wire bail at the largest radius from the pivot, i.e., the portion of the wire that hits a pencil when the trap is triggered. What is the force acting on the crossbar when the trap is cocked, i.e., how much force is required to keep the trap in the cocked position? What element supplies this force to the cross-bar? How is this force transferred to the base?About what fraction of the force is carried by the trigger plate? Include a diagram to explain your thinking.Devise a means to measure this *hold-open* force to an accuracy on the order of a fraction of a Newton. Tell us how you made your measurements, and why they should be valid. What provides the weight calibration? Include relevant diagrams and experimental data.

(b) When the trap is in the closed position (i.e., in its resting state), there is still a torque acting on the bail. For example, when the cross-bar is resting on a pencil, a force is exerted on the pencil,forcing it against the base. We will call this the *resting-state* force. Devise a means to measure this resting-state force to an accuracy on the order of a fraction of a Newton. Tell us

how you made your measurements, and why they should be valid. What provides the weight calibration? Include relevant diagrams and experimental data.

(c) Use the force measurements you made above to develop a model for the spring torque as a function of opening angle, under the assumption that the force, and thus the torque, varies linearly with opening angle. As a convention, assume that the resting-state angle is0 radians, and thus that the hold-open angle (i.e., when the trap is cocked) is π radians. Express your torque model in units of Nm/rad. What is the angle at which zero torque is exerted (it's not 0 rad,since there is still a significant torque at this angle)? Make a sketch of the torque as a function of angle.

(d) On the course web site there are several high-speed videos which show the trap being triggered. The video labeled `mtrap1` shows a pencil triggering the trap. The frame rate of this video is 1000frames/sec. On the basis of your observation of the video, about how long does it take for the trap to close after it is triggered? What is approximately the highest velocity in m/sec attained by any part of the trap? About how fast is the cross-bar going in m/sec when it hits the base after being triggered?Explain how you developed these estimates. To give a sense of scale, an arrow shot from a modern bow goes about 100 m/sec, and a bullet from a rifle goes about 1000 m/sec.

16.15 Mousetrap Dynamics 2

This problem further considers the mousetrap system. Now we develop a model which will predict the closing time of the trap.

(a) From the geometry of the bail, and using the density of steel of about $8 \times 10^3/m^3$, calculate the moment of inertia of the bail about the pivot point, in units of kg-m^2.

(b) Last week you measured the force of the bail in the open and closed positions, and thereby developed a model of the torque acting on the bail as a function of angle. For the rest of this week's problem, we will assume that the torque does not vary as a function of angle, but rather has a constant value equal to the open position torque. This will allow us to simplify the analytical calculation of the trap closing time. What is the value of this constant torque in N-m?

(c) Use the simplified constant spring torque model developed above as part of an analysis which will predict the time required for the trap to close when triggered. Clearly describe your thinking in developing this model. Please provide plots of the angular velocity and position as a function of time as well as the magnitude of the velocity of the cross-bar as a function of time. What is the velocity of the cross-bar when it reaches the closed position ($\theta = 0$)? How long does it take the trap to close from when it is triggered?

(d) Compare your predicted transition time and velocity with data that you extract from the high-speed video which we have placed on the web. Tell us how you made this comparison. How might your model need to be modified in order to match the measured data on the video?

16.16 Mousetrap Dynamics 3

This problem revisits the continuing saga of mousetrap dynamics. In our last exciting installment, you calculated the time to close under the assumption that the torque is constant with angle. This allowed a fairly simple analysis to predict the closing time with reasonable accuracy. This week, we will allow the torque on the bail to vary linearly with angle. With this linear variation, and assuming no damping, the mousetrap is really a second-order oscillator, with a pair of poles on the imaginary axis during the time that the bail is closing. Nonlinearity manifests itself in a very strong fashion when the bail hits the wood part of the trap. However, the closing motion can be well-predicted by this linear model. Specifically, we ask you to:

(a) Recall your earlier results to write a model for the spring torque as a function of angle in the form

$$\tau = K(\theta + \theta_0).$$

Here τ is the torque in Nm applied to the bail from an external source, K is the spring constant in Nm/rad, θ is the rotational angle of the mousetrap, and $\theta = -\theta_0$ is the angle at which the torque applied to the bail is zero.

(b) Use this spring model along with the value of moment of inertia that you computed last week to write an undamped second-order differential equation describing the mousetrap motion in the form

$$J\ddot{\theta} = K(\theta + \theta_0).$$

In the open position, we have defined $\theta = \pi$. From this initial condition, solve for the motion assuming that the bail is released at $t = 0$. Be clear as to how you handle the θ_0 term.

How long does it take for the bail to reach the closed position $\theta = 0$? You may assume for the purposes of this analysis that the motion comes to an instantaneous stop when the angle reaches 0. Solve for and write an expression for the force required to arrest this motion in zero time.

(c) Use Matlab to make a plot of $\theta(t)$ during the closing event based upon your solution above. Also plot on the same axes the function $\theta(t)$ based upon last week's analysis in which the torque was constant as a function of angle. How do these two solutions differ? Why?

(d) Use the high-speed video of the mousetrap to develop a measure of $\theta(t)$ for all the frames during the closing event. Overlay this experimental data on your plot from part c). How does this data differ from the analytical results? Why? How might you refine your analysis to better model the event? You do not need to carry out this analysis, just tell us what additional elements or analysis might be added.

16.17 Mousetrap Dynamics 4

Trigger your mousetrap with a wood pencil, and notice the dent that appears in the pencil. In this last installment of the mousetrap problem, we ask you to develop a calculation which will predict the force experienced by the pencil, and the approximate time-scale in which this force is applied.

(a) Take a look at the high-speed video on the web of the mousetrap being triggered with a pencil. On the basis of the 1000 frame per second rate of the video, what can you say about an upper bound on the time in which the pencil is dented?

(b) Observe and estimate the depth of the dent in the pencil. In the simplest of models, we might assume that this dent is made at constant force. Under this assumption, you can readily calculate the force and duration of the denting process. You will need to rely on the values of angular velocity and moment of inertia that you developed earlier. Show your assumptions and calculations, and make a graph of your estimated force as a function of time. How does this estimate compare with the upper bound that you established in part a)?

(c) During the denting process, what is the instantaneous power flow as a function of time? That is, what is the time rate of energy transfer from the kinetic energy of the bail to the mechanical deformation of the pencil? Express this result in watts. Also, convert this power to units of horsepower. How does this power flow compare to typical devices such as light bulbs, lawn mower engines, etc.?

16.18 Hydraulic Elevator Design

For buildings of moderate height, passenger elevators are frequently driven by hydraulic actuators. Such elevators also include some form of spring/damper assembly in the bottom of the elevator shaft to catch the elevator in the event of a failure in the hydraulic drive or in the elevator control system. This problem studies an idealized model of the design of this crash stop. Specifically, suppose we model the elevator as a lumped mass $m = 2000$ kg. This mass (the car) rides in a vertical shaft, driven from the bottom by a hydraulic cylinder. At the bottom of the shaft are a spring and damper which form the crash stop. Define motion in the vertical direction as x. Assume that at $t = 0$, with the bottom of the elevator at a height of 10 m above the crash stop, and with the elevator at zero velocity, a valve fails in the hydraulic cylinder. The elevator begins falling under the influence of gravity. During the fall, the oil remaining in the cylinder sprays out of the failed valve, and acts as a damper with a damping coefficient $c = 2000$ Ns/m.

(a) Develop a model to predict the motion of the elevator during the fall. Make plots of the elevator position and velocity as a function of time during the fall. What is the total force exerted on a passenger during the fall? For this purpose, we model the passenger as a rigid mass of 100 kg (already included in the total $m = 2000$ kg).

(b) After falling 10 m, the bottom of the elevator contacts the crash stop. Your job is to design the crash stop so that the elevator comes to rest in a manner which is safe for the passengers and which also doesn't use up too much space in the building.Develop a model for the behavior of the elevator system after it hits the crash stop. In your model, assume that the damping c from the oil cylinder is present throughout the motion. You may also assume that the crash stop mass is insignificant.Use this model to choose the values of the crash spring k_r and damper c_r to meet the following specifications:

(i) The maximum acceleration d^2x/dt^2 on the car must be less than 20 m/s^2 at all times.

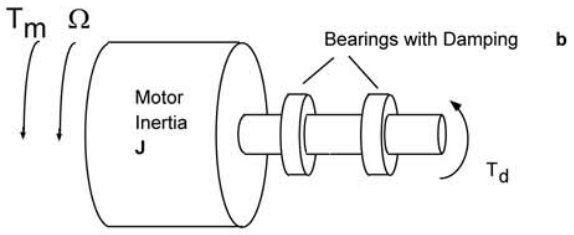
(ii) The maximum deflection of the crash stop is to be minimized.

(iii) The settling time will be defined as the time from initial contact with the crash stop until the time that the elevator velocity remains below 0.1 m/sec. The settling time should be made small.

Develop a model and analyses which allow you to meet the specifications above. Clearly show the key steps, assumptions, and results. Note that this is a typical design problem in that there is no one *right* answer, only degrees of goodness in satisfying the specifications. Tell us how you attempted to solve this problem. Be sure to include plots of the car position and velocity as a function of time, the deflection of the crash stop, and the force exerted on a 100 kg passenger. Also show on your plots the peak acceleration of the car and the time at which it occurs, the maximum deflection of the crash stop and the associated time, and the time at which the car velocity meets the settling specification.

16.19 Servo Position Control

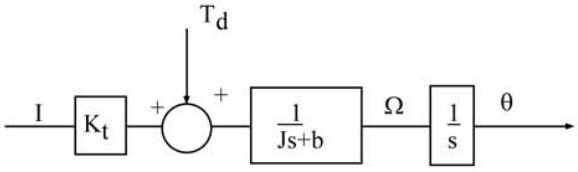
The model shown below and developed in the class notes is typical of an idealized position servo that might be used in a machine tool, robot, or VLSI mask positioning system. The basic system is a simple inertia with bearing friction and a disturbance torque T_d applied to the shaft:



The inertia would model the robot arm mass, the table mass for a mask positioning system or the motor / leadscrew / carriage mass for the machine tool. The disturbance torque models an external force such as the machining force induced torque, interference with the robot path, or encountering a compliant surface of some type.
For this system we can derive a model of the form:

$$\Sigma T_J = K_t I - b\Omega + T_d = J\frac{d\Omega}{dt}$$

This motor model, and the fact that the shaft position θ is the integral of Ω (i.e. $\theta = \frac{1}{s}\Omega$) leads to the following process block diagram similar to that developed in class:



Assuming the values J = 10 Nm-sec^2, b= 8 Nm-sec, K_t= 1 Nm/amp; and for now T_d = 0:

a) Draw a complete control system block diagram to regulate θ to a reference value θ_r.

b) Design a proportional controller ($G_c = K_p$) such that the closed-loop overshoot is zero and settling time is a minimum. This can be done using algebra only by looking at the closed-loop transfer function and characteristic equation.

c) What is the expected 5% settling time to a step change in the command position θ_r?

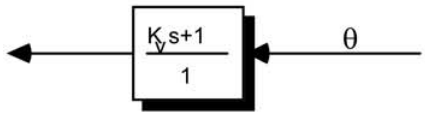
d) Using MATLAB, plot the step response of your system and compare it to what was expected. Please comment on similarities and any differences.

Figure 16.19.1:

16.20 Servo Frequency Compensation

This problem is a continuation of Problem 16.19.

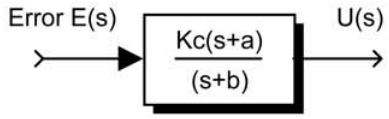
To improve the response it is common to feedback the velocity Ω as well as the positior θ. This is equivalent to adding the block



in the feedback path of the controller

However, to obviate measuring velocity, we often use a so-called "lead-lag" controller of the form:

$$G_c(s) = \frac{K_c(s+a)}{(s+b)} = \frac{U(s)}{E(s)} \quad \text{or:}$$



Lead-lag Controller

which is easily implemented with an op amp circuit with appropriate feedback and forward path impedances.

a) With this lead-lag block substituted for the controller block, derive the new closed-loop transfer function, but don't do any further algebra

b) By starting with a bode diagram of the open – loop system, design the compensator to achieve the following specification for the system:

- settling time = 0.8 sec.
- minimum damping ratio = 0.5

c) For your specific design, plot the resulting closed – loop step response using MATLAB and comment on how well it conforms to your predictions. Please explain any divergence from what you expect.

d) How does the response of part c) compare to that you would expect from using the velocity feedback block show above? (You should not need to do a complete design but rather just look at the possible closed-loop transfer function for this case.)

Figure 16.20.1:

16.21 Servo Torque Disturbance

This problem is a continuation of Problems 16.19 and 16.20.

Now we encounter a true disturbance: the application of some "unknown" and undesirable torque on the motor shaft. Our design goal will be to completely reject this disturbance, and to do so we must make use of more complex controller forms.

For the moment keep K_p at the value used in Problem 1b), but now assume $\theta_r = 0$ and T_d is a unit step.

a) What is the disturbance transfer function $\frac{\theta}{T_d}$ for this system?

b) Sketch the expected response to the step in T_d based on this transfer function.

c) What will be the steady-state error to this disturbance?

Now, to completely eliminate this error, we'll try an integral controller of the form

$$G_c = K_p/s.$$

With this new controller:

d) Explain why this controller form will not yield an acceptable control system by reference to the bode plot for this new system.

As a remedy to this problem, we could replace the integral controller with a proportional integral (PI) controller of the form:

$$G_c(s) = K_p + K_I s$$

that can be re-written as:

$$G_c(s) = \frac{K_c(s+\alpha)}{s}$$

Figure 16.21.1:

16.22 Plate On Springs Damped 1

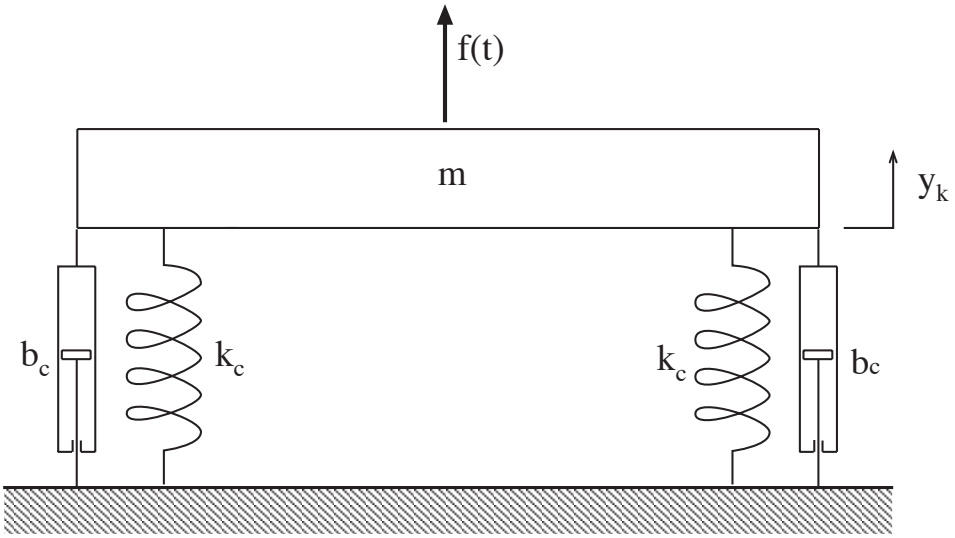


Figure 16.22.1: Plate On Springs With Damper

Consider the vertical oscillations of a 5-pound steel plate supported on springs, as demonstrated in class (Fig. 16.22.1). Begin by formulating a simple linear model for the motion of the plate in response to an applied force $f(t)$. Your job is to use the MATLAB scriptsMassSprgDmpr1, 2 & 3 based on closed form analytical solutions, of the type developed in class, to obtain twelve plots of the time history of response for twelve different sets of model parameters. In every case the plate is at rest in its equilibrium position, when, at$t = 0$ a constant force of 5 pounds is suddenly applied in the direction of positive displacement. In each case you must select the model stiffness and damping parameters to achieve the specified behavioral parameter values.

(a) In each of the first four plots the undamped natural frequency is 5 Hz(cycles/second), and the damping ratios have the following values: (i) $\zeta = 0.1$; (ii)$\zeta = 0.5$; (iii) $\zeta = 1.0$; (iv) $\zeta = 1.5$. You must choose the model parameters to insert in the MATLAB scripts to get the time histories of response corresponding to these combinations of ω_o and ζ. The script 'MassSprgDmpr1.m' must be used for Cases(i) and (ii). The script 'MassSprgDmpr2.m' must be used for Case (iii), and the script 'MassSprgDmpr3.m' must be used for Case (iv). For Cases (i), (ii), and (iii) take the total duration of the time history to be equal to 5 times the decay time constant. In Case(iv) take the total duration of the time history to be 5 times the *longest* time constant.

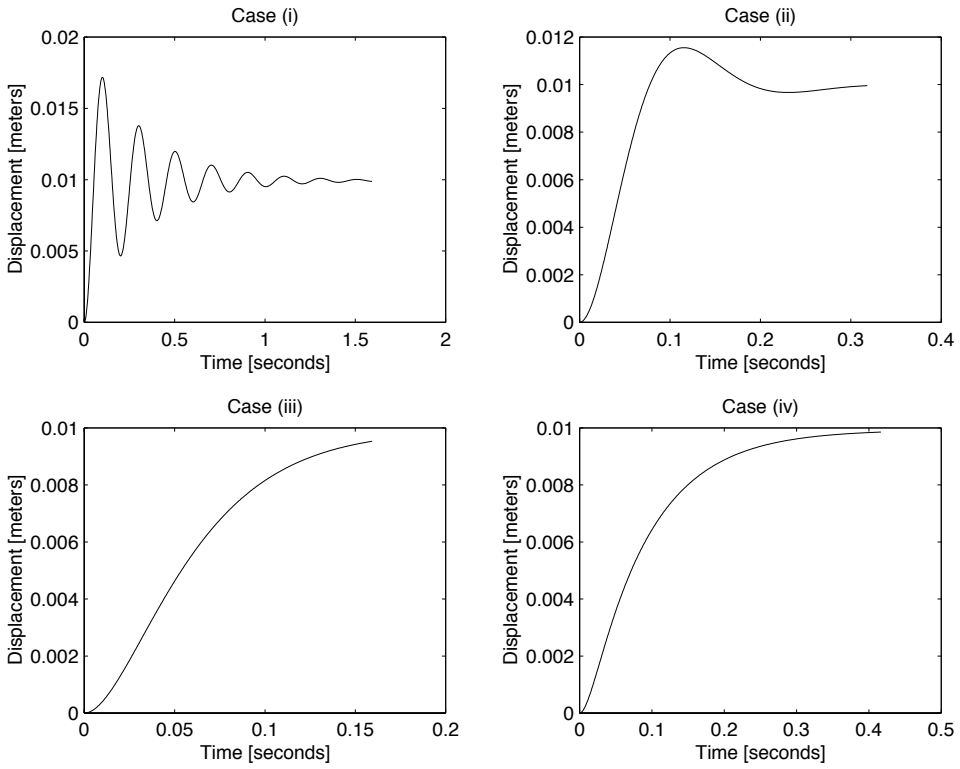


Figure 16.22.2: Figure 1

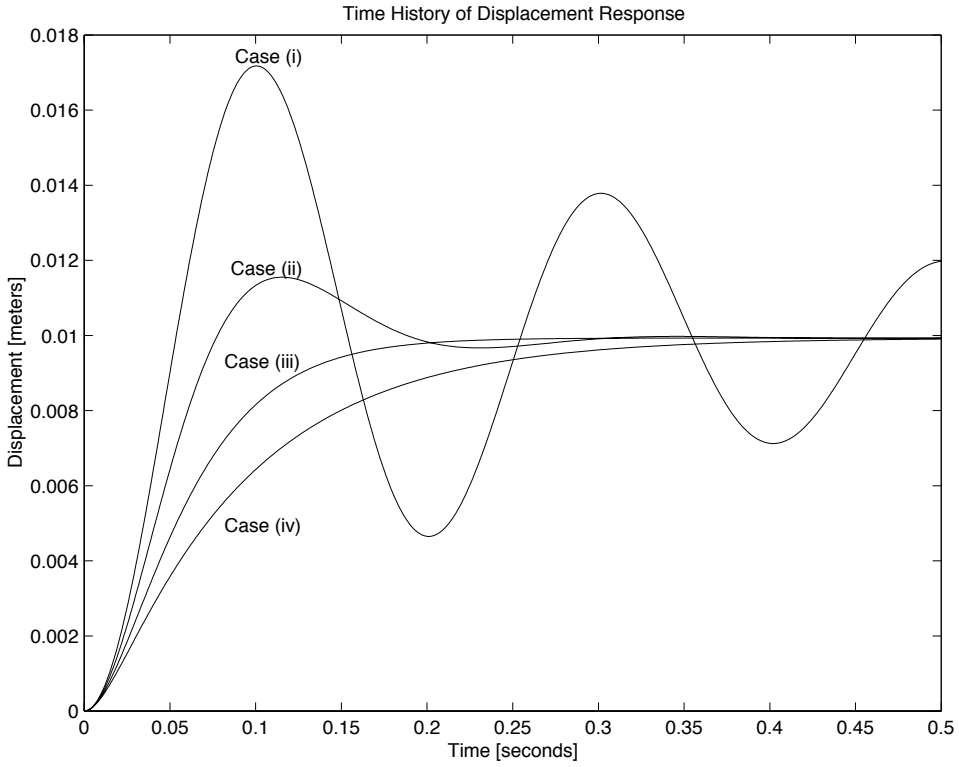


Figure 16.22.3: Figure 2

(b) In each of the next four plots, the *damped* natural frequency is 5 Hz, and the damping ratios have the following values: (i) $\zeta = 0.1$; (ii)$\zeta = 0.3$; (iii) $\zeta = 0.5$; (iv) $\zeta = 0.7$. Again choose model parameters to get the time histories of response corresponding to these combinations of ω_d and ζ.Here all cases can be run with 'MassSprgDmpr1.m'. In each case take the total duration of the time history to be equal to 5 times the decay time constant.

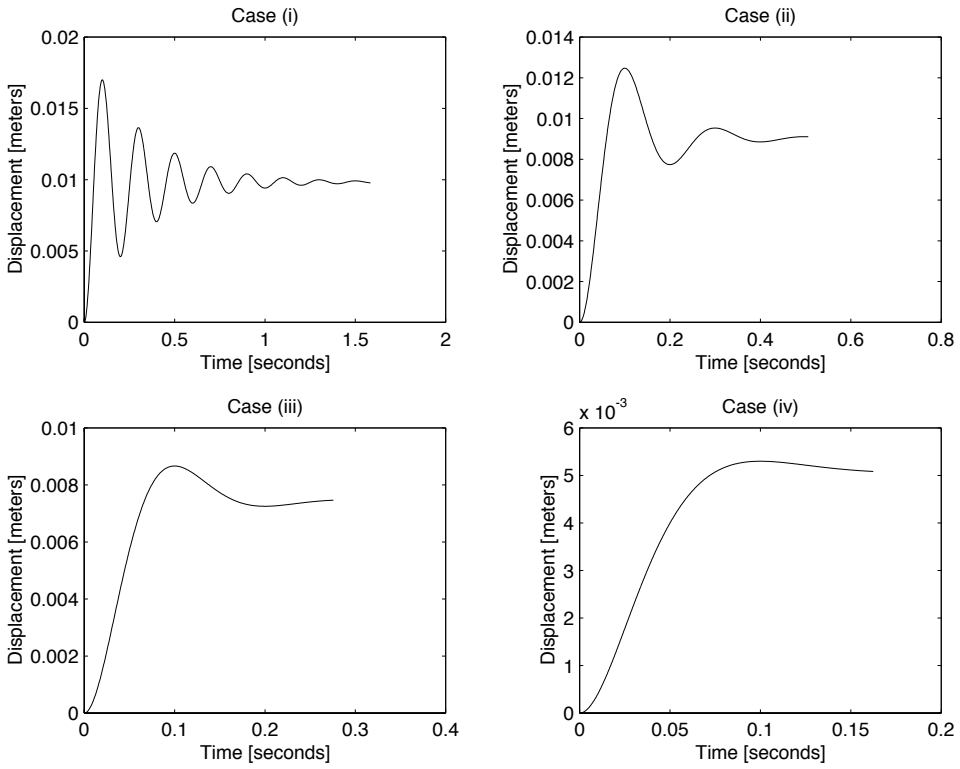


Figure 16.22.4: Figure 3

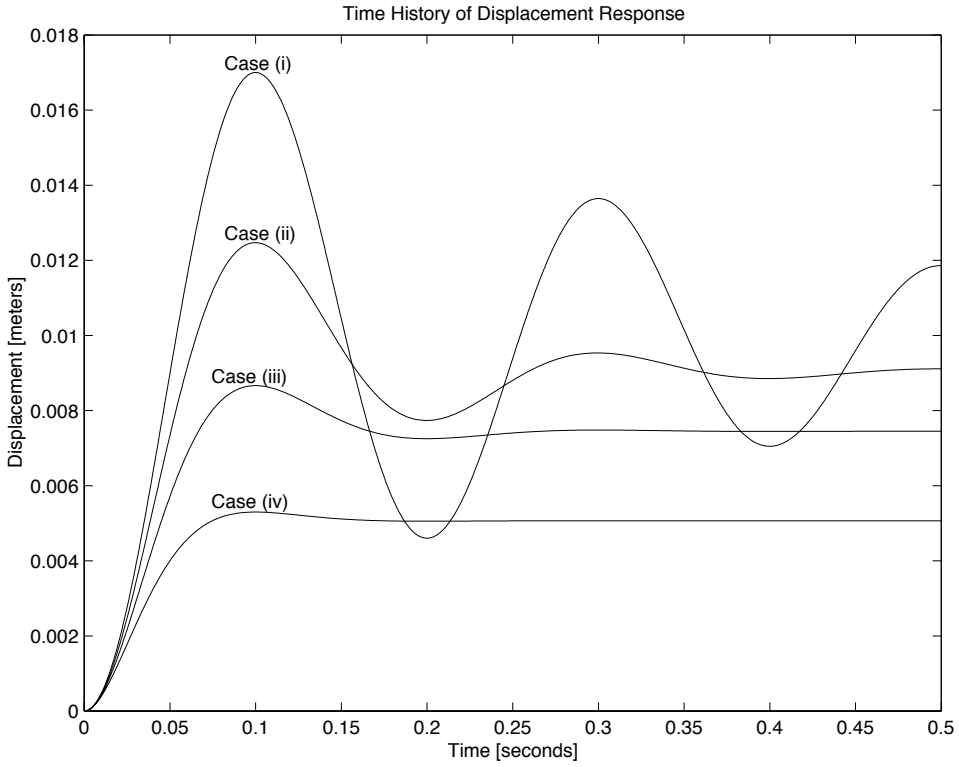


Figure 16.22.5: Figure 4

(c) In the final four plots the decay time constant is fixed at $\tau = 0.10$seconds, and the damping ratios have the following values: (i) $\zeta = 0.3$; (ii)$\zeta = 0.5$; (iii) $\zeta = 0.7$; (iv) $\zeta = 0.9$. Choose model parameters to get the time histories of response corresponding to these combinations of decay time constant and ζ. In each case take the total duration of the time history to be equal to 5 times the decay time constant.

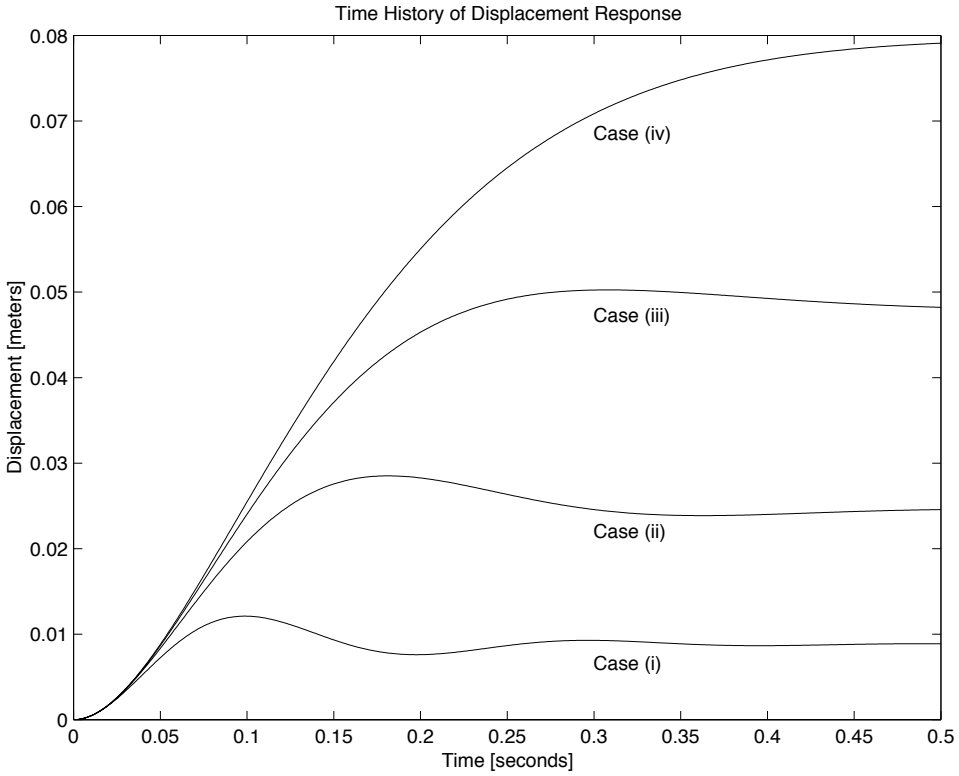


Figure 16.22.6: Figure 5

16.23 Plate On Springs Damped 2

Make a careful sketch of the complex plane showing the location of the eigenvalues, λ corresponding to the twelve parameter sets used to produce the time histories in Problem 16.22.

16.24 Plate On Springs Damped 3

Reconsider the spring-supported plate of Problem 16.22, but now take the motion to be caused by a 2-pound rubber ball which strikes the plate and bounces away. Assume that the plate is at rest in its equilibrium position, when it is struck by the ball at $t = 0$. Assume the ball is falling vertically with a velocity of 10 feet/second and rebounds with an upward velocity of 3 feet/second, and that it does not impact the plate again.

(a) Determine the initial velocity of the plate, immediately after the impact.

(b) Select the model stiffness and damping parameters so that the damped natural frequency is 5 Hz, and the damping ratio ζ is 0.3.

(c) Insert the model parameters in 'MassSprgDmpr1.m' and plot the displacement response of the plate for an interval equal to 5 times the decay time constant.

(d) What is the relationship between the time history in Problem 16.23 Part (c) and the time history in the plot for b(ii) in Problem 16.22?

16.25 Plate On Springs Damped 4

A rigid plate supported on four identical springs, like the system in Problem 16.22, has unknown model parameters, m, k, and b. A dynamic test is performed in which a constant force of 1.0 Newton is suddenly applied at $t = 0$. The measured displacement response is displayed in Fig. 16.25.1

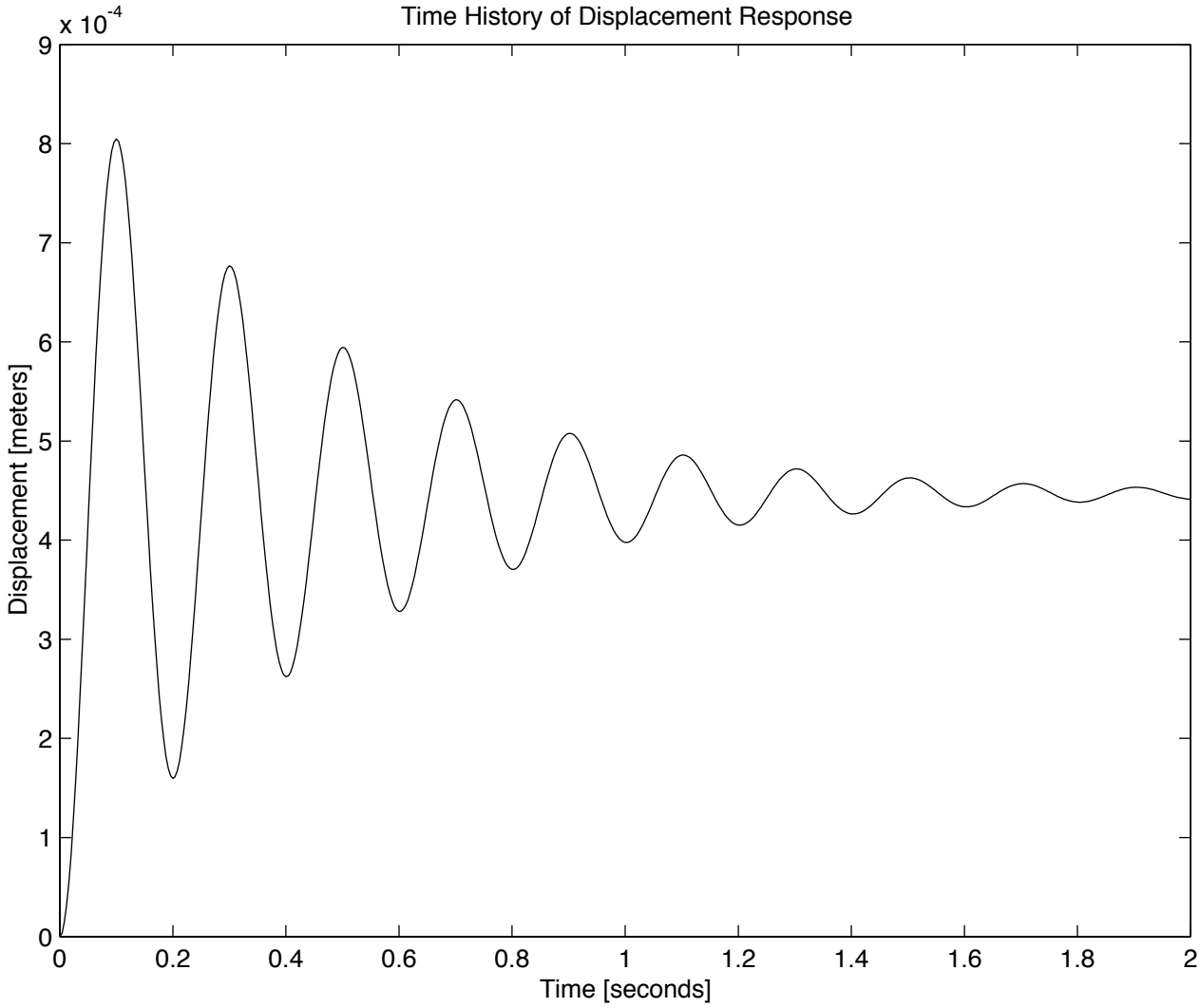


Figure 16.25.1: Response to 1.0 Newton Step-Force

Your job is to use the data revealed in this Figure to estimate:

(a) The effective stiffness k of the four springs;

(b) The effective mass m of the plate;

(c) The effective damping coefficient b of the system.

16.26 Plate On Springs Damped 5

Each, of two identical plate-on-springs assemblies, like those of problem 16.22,is accurately represented by a model with $m = 2.0$ kg, $k = 2000$ N/m, and $b = 20$N/m/s. Consider the system obtained by placing the two individual units face-to-face, with the first plate resting on a firm table with its springs extending upward, and the second plate reversed, so that its springs extend downwards and are connected to the springs of the first plate. Your job is to derive a differential equation for free vertical motion of the second plate, and:

(a) Evaluate the undamped natural frequency ω_o for the model of the combined system;

(b) Evaluate the damping ratio ζ for the model of the combined system;

(c) Evaluate the decay time constant τ for the model of the combined system.

16.27 Plate On Springs Damped 6

The steel plate on springs shown in Problem 16.22 has been modeled by the differential equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t)$$

Remodel this as a state-determined system with the force $f(t)$ as the input and the vertical position and velocity of the plate as the outputs.

(a) The effective stiffness of the springs is known to be 3000 N/m. Choose the mass parameter m and the damping parameter b such that the steady state deflection of the plate under its own weight is $\Delta = 7$ millimeters and the damping ratio is$\zeta = 0.5$.

(b) Write a MATLAB script to integrate the state equations for the case where,at $t = 0$, an abruptly applied force equal to the weight of the plate is applied in the *upward* direction.

(c) Write a MATLAB script to produce plots of the first two seconds of transient response in (b). Produce plots of the following forms:

(i) position vs. time

(ii) velocity vs. time

(iii) velocity vs. position

16.28 Shipping Crate 1

Sensitive machinery and instruments are typically packed with shock-absorbing material for shipping. Fig. 16.28.1 shows a plan view sketch of a 500-pound engine in a shipping crate with packing material at each end. In reality packing material is provided on all sides, but to keep things simple we only consider the end-to-end horizontal translation of the engine with respect to the crate. Assume that the packing material at each end can be adequately modeled as a linear elastic element with stiffness k combined with a linear friction element with damping parameter b.

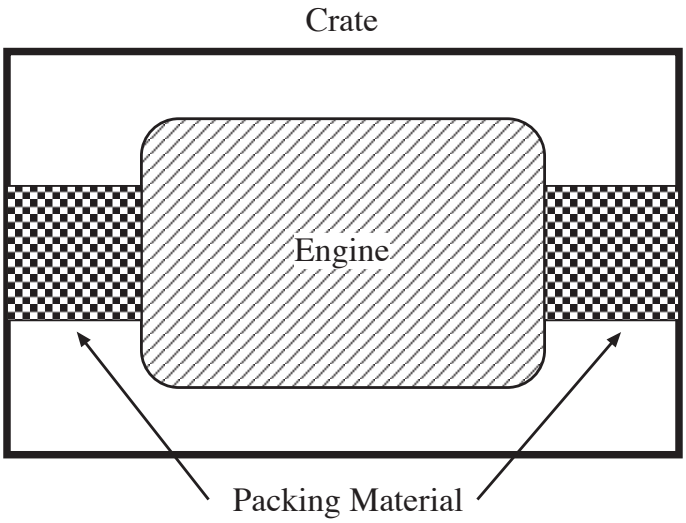


Figure 16.28.1: Engine in Shipping Crate

(a) Derive a differential equation for the end-to-end motion of the engine with respect to a stationary crate.

(b) Specify the values of k and b required to provide a damped natural frequency of 1 Hz and a damping ratio of 0.707.

16.29 Shipping Crate 2

Reconsider the 500-pound engine with shock-absorbing packaging in a shipping crate of Problem 16.28. Take the packing material at each end to have an effective stiffness k of 100 pounds/inch, and an effective damping parameter b of 150 pounds/inch/second. Now suppose the crate is placed on a flat-bed truck so that the end-to-end orientation of the crate is parallel to the direction of motion of the truck. The crate is secured to the truck so that there is no relative motion of the crate with respect to the truck. However, because of the shock-absorbing packaging, the engine may move with respect to the crate when the truck accelerates and decelerates.

(a) Formulate a mathematical model to describe the longitudinal motion of the engine *with respect to the crate* in response to longitudinal motion of truck.

(b) Derive a single ordinary differential equation relating the (input) absolute velocity of the truck $v_{truck}(t)$ to the the (output) relative velocity of the engine to the crate $v_{engine}(t)$.

(c) Derive a state-determined system in which $v_{truck}(t)$ is the input and$v_{engine}(t)$ is the output.

17 Quiz Problems

17.1 Fun with Block Diagrams

(a) Reduce the block diagram in Figure 17.1.1 to derive the transfer function for the system. Find the value of K that will result in a critically damped response.

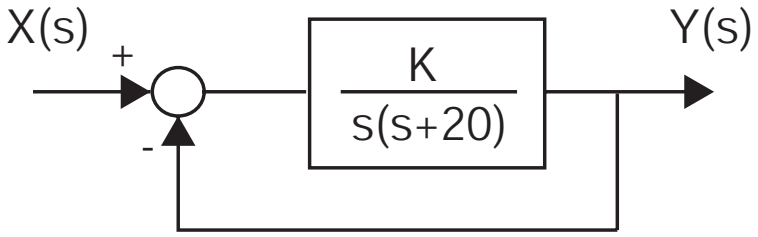


Figure 17.1.1: Block diagram for part (a)

(b) Derive the transfer functions $\frac{V_{out}(s)}{V_1(s)}$ and $\frac{V_{out}(s)}{V_2(s)}$ for the block diagram shown in Figure 17.1.2. (Hint: you can always label individual positions along the block diagram with variable names and write out and solve the relevant algebraic equations.)

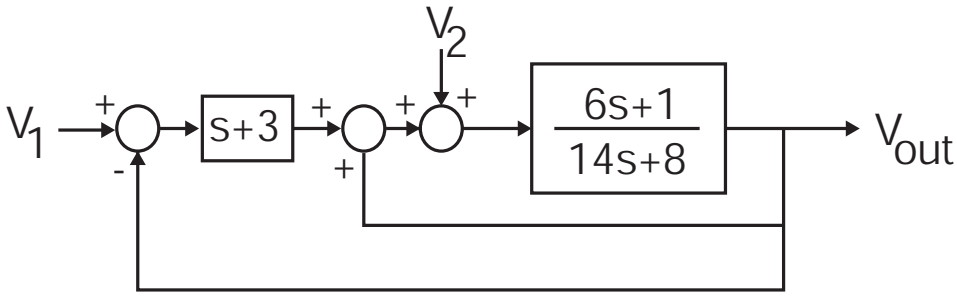


Figure 17.1.2: Block diagram for part (b)

(c) Write the complete differential equation for $V_{out}(t)$ in terms of $V_1(t)$ and $V_2(t)$.

17.2 Complex Translation

This problem concerns the translational mechanical system shown in Figure 17.2.1

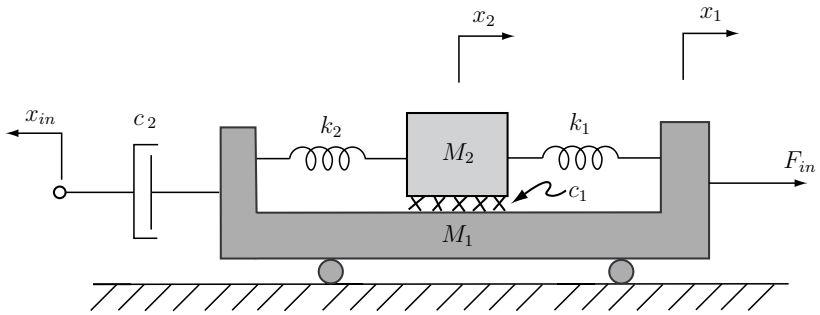


Figure 17.2.1: Translational Mechanical System

The system is driven by two inputs, a force input F_{in}, and a position input x_{in}. The damper c_1 shown acts between the masses M_1 and M_2. The mass M_1 rolls on the ground surface without friction.

(a) Draw free body diagrams for the two masses, clearly labeling the forces and associated reference directions which act on the mass. The functional dependence of these forces on the system variables should be indicated.

(b) Use the free body diagrams developed above to write the governing system differential equations in the form:

$$M_1\ddot{x}_1 =$$
$$M_2\ddot{x}_2 =$$

where you supply the right hand side of the equations. You do not need to further reduce the form of the system equations.

17.3 Cylinder Step Response

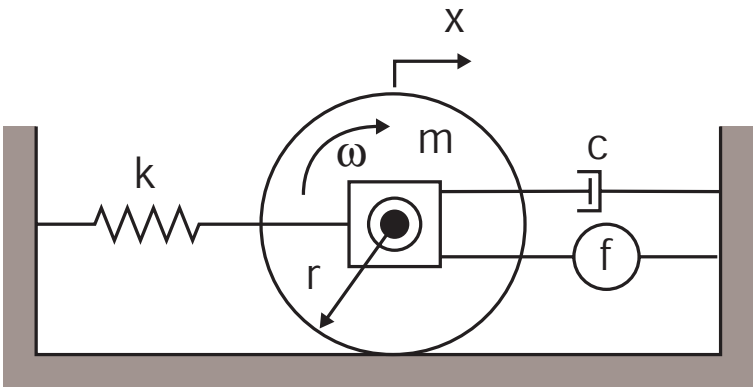


Figure 17.3.1: System Model

You have been given the system illustrated in Figure 17.3.1. The system consist of a cylinder with a mass m with a radius $r = 0.5$ m which spins about an axel. The cylinder rolls without slip on the ground. Attached to the axel housing are a damper c, a spring $k = 200$ N/m, and a force source f. You measure the following response $x(t)$ to a step input of the force source.

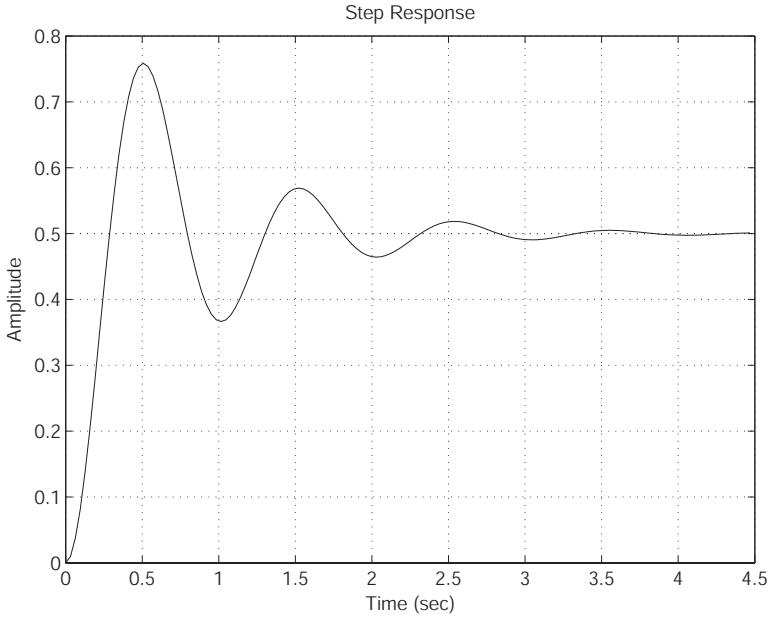


Figure 17.3.2: Step Response

(a) Using the provided parameters and the step response, determine the damping constant c and the equivalent mass m_{eq}, where m_{eq} is mass equivalent of the combined inertia and mass.

(b) If the cylinder has a $m = 3$ kg, determine the inertia of the cylinder.

17.4 JKC Frequency Response 1

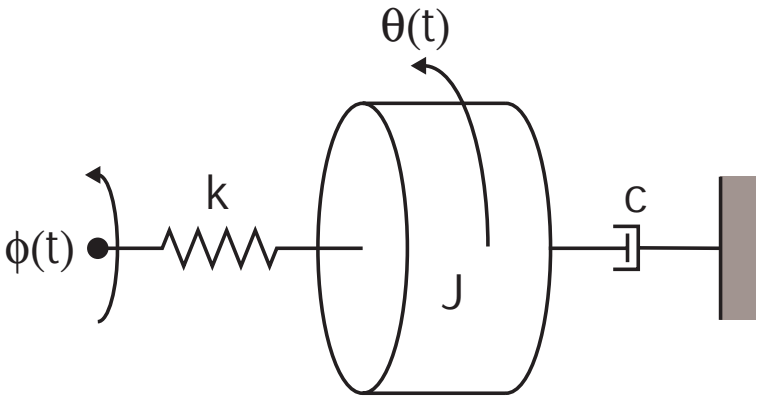


Figure 17.4.1: Rotational System Model

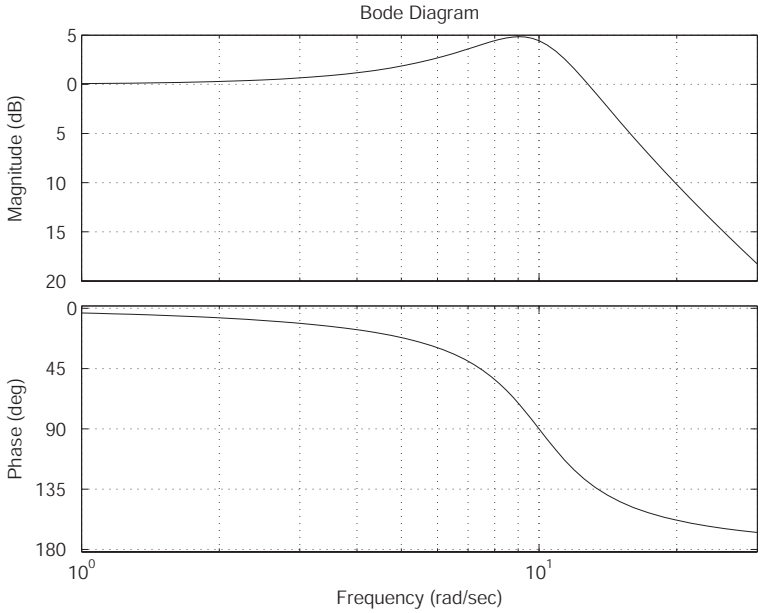


Figure 17.4.2: Bode Plot

You perform a frequency analysis of the system shown in Figure 17.4.1 and obtain the bode plot shown in Figure 17.4.2. The inertia $J = 15$ N/m^2.

(a) Using the data in the bode plot determine, ζ and ω_n for this system.

(b) Using J and the values determined in part (a), determine c and k for the system.

(c) Determine an expression for the system output $\theta(t)$ when $\phi(t) = \sin(\omega t)$ and $\omega = 1.1$, 10, and 20 rad/s.

17.5 Mass Spring Damper Dynamics

A second-order system is driven with a position source x_{in}.

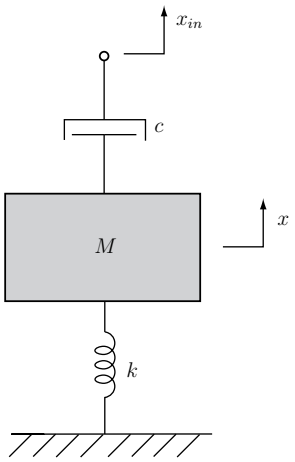


Figure 17.5.1: Mass Spring Damper System

The mass position is $x(t)$. There is no gravity acting in this problem.

(a) Write the governing differential equation for this system with an input x_{in}, and an output x. Clearly show the steps in your solution.

(b) Now let the parameters take the values $M = 0.1$ kg, $k = 10$ N/m, and $c = 0.2$ Ns/m. What are the values of ζ and ω_n?

(c) Assume the system is in a condition of initial rest. The input position is given to be a unit step, $x_{in}(t) = u_s(t)$ m. Solve for the mass position $x(t)$ as a function of time. Clearly show the steps in your solution.

(d) Make a carefully labeled and dimensioned sketch of $x(t)$.

17.6 Op-Amp Analysis

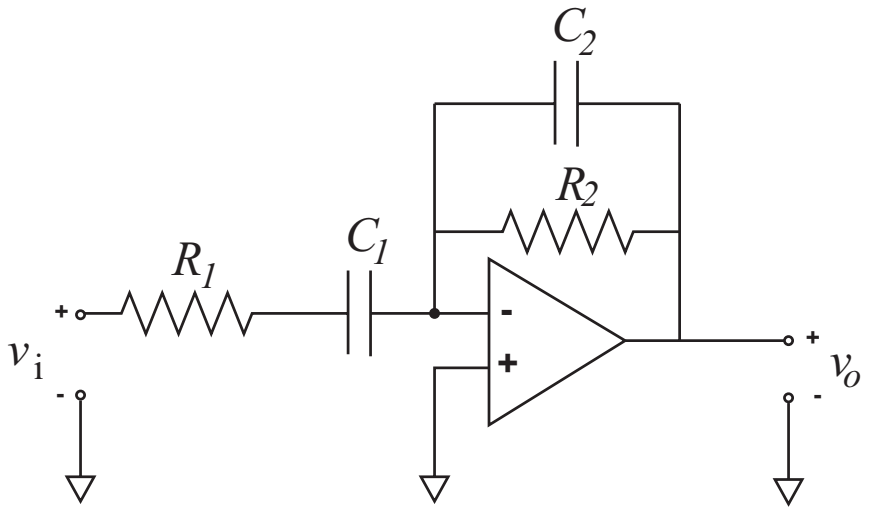


Figure 17.6.1: Op-Amp Circuit, $R_1 = 1M\Omega$, $R_f = 100k\Omega$, $C_1 = 1\mu F$, $C_2 = 1nF$

(a) Derive the Transfer function H(s) relating V_o to V_i (assume the Op-Amp acts as an infinite gain).

(b) Derive the expressions for the magnitude and phase as a function of frequency.

(c) Make a bode plot of the system indicating the major points.

(d) Determine the pole and zero locations(s) and plot them on the s-plane.

17.7 RLC Circuit Analysis

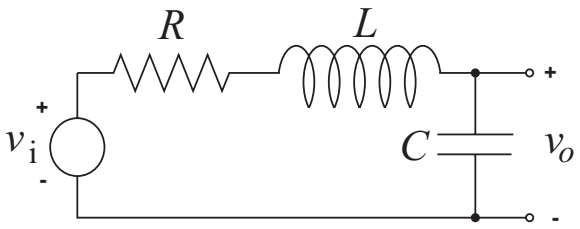


Figure 17.7.1: RLC Circuit

(a) Write the transfer function, $\frac{V_{out}(s)}{V_{in}(s)}$, for the circuit shown above.

(b) Given $C = 1 \times 10^{-6}\, F$, find the values of R and L such that $\zeta = 0.707$ and the undamped natural frequency is 5 kHz. (*Don't forget to convert to rad/sec!!!*)

(c) Using the same values of L and C from part(b), find the locations of any system pole(s) and zero(s) given $R = 1000\Omega$. Sketch the unit step response, clearly indicating the time and magnitude scaling. (Hint: use a dominant pole approximation.) Use the IVT and FVT to show that your response starts and ends at the appropriate values.

(d) Sketch the log Magnitude vs log frequency, and linear phase vs log frequency (Bode plot) for this system based on your calculated poles from part 3. Show that your plot approaches the correct values of magnitude both as $\omega \to 0$ and $\omega \to \infty$.

(e) Again using $C = 1 \times 10^{-6}\, F$ and $R = 1000\Omega$, now let $L = 0\, H$. (i.e. Remove the inductor from the circuit) Calculate the location of the pole and compare this to the dominant pole found in part 2.

(f) Write the transfer function, $\frac{V_{out}(s)}{V_{in}(s)}$, for the two circuits shown below.

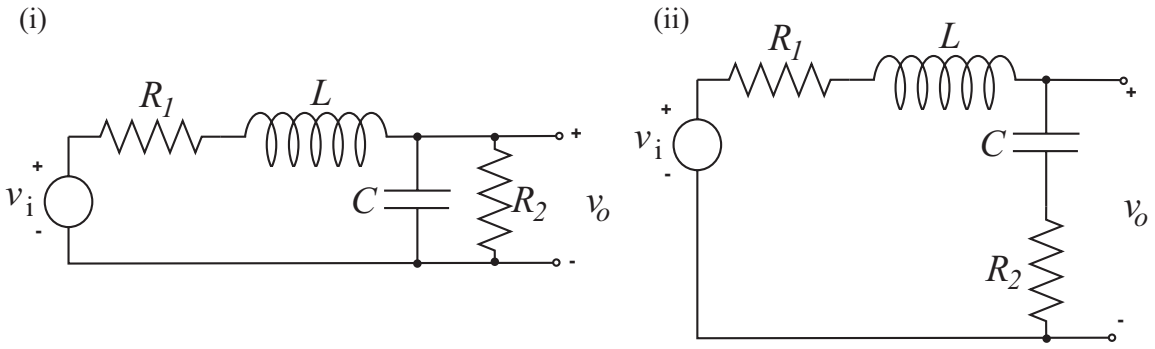


Figure 17.7.2: RLC Circuit

17.8 Sailing for Engineers

A sailing ship of mass m is initially at rest, i.e. $v(0) = 0$. At time $t = 0$, a strong wind arises of magnitude $V_o = 10$ m/s.

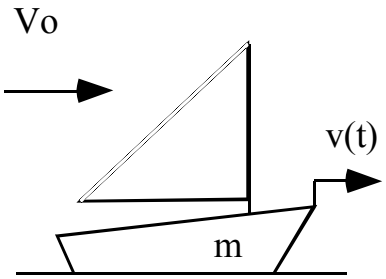


Figure 17.8.1: Sailboat Diagram

Assume that the force of the wind on the sails in the direction of travel is given by $F_w(t) = B_w[V_o - v(t)]$. Assume that the viscous drag of the water on the ship is given by $F_b(t) = B_d v(t)$

(a) Formulate a differential equation that describes the ship's velocity $v(t)$.

(b) Solve the differential equation from (a) and write an expression for the ship's velocity $v(t)$.

(c) Sketch the response $v(t)$.

(d) Write an expression for the steady-state velocity v_{ss} in terms of system parameters.

17.9 Second Order Step Response

The following plot represents the step response of a second order system, starting from zero initial conditions.

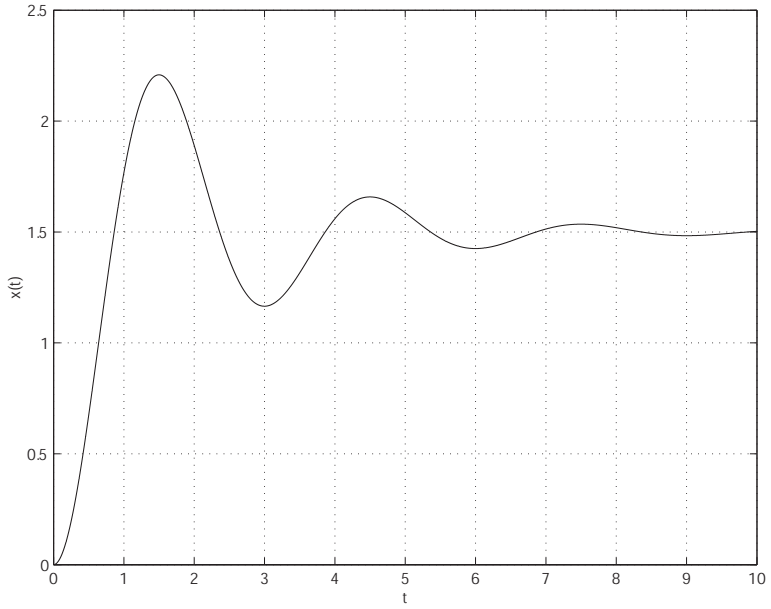


Figure 17.9.1: Step Response Plot

(a) From the plot, find the 10%-90% rising time t_r, the peak time t_p, the 1% settling time t_s and the overshoot M_p.

(b) Estimate the natural frequency ω_n and the damping coefficient ζ.

(c) Write down the ODE corresponding to the plot.

17.10 Spring Damper Dynamics

This problem studies the spring-damper arrangement shown in Figure 17.10.1. The spring and damper are joined by a massless link, which has a vertical position x. A force F acts on this link.

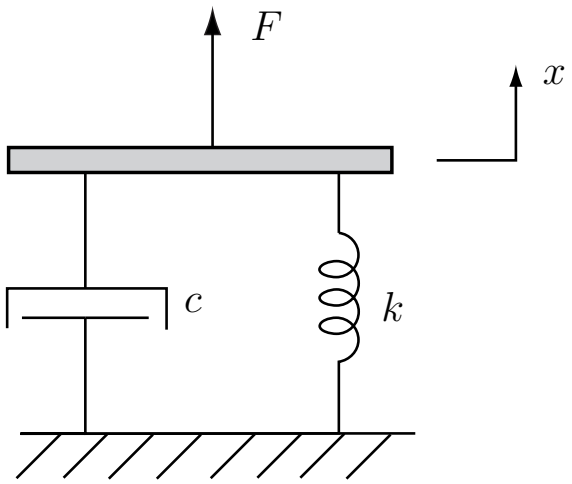


Figure 17.10.1: Spring Damper System

(a) Make a sketch of the pole location in the s-plane.

(b) The input force has the following form, valid for *all time*

$$F(t) = 10 + 3\delta(t-1),$$

where $\delta(t)$ is the unit impulse. Note that the driving impulse $\delta(t-1)$ is delayed by 1 second. Be sure you see that this input is for all time, not just after $t = 0$.

Calculate the link position x as a function of time, and give an expression for this position valid for all time. Clearly show your thinking and the steps in your calculation. Your result should be in terms of the parameters of the system, as we have not given numerical values for these parameters.

(c) Make a carefully labeled and dimensioned sketch of $x(t)$. Again, this sketch should be dimensioned in terms of the system parameters.

17.11 Derive Blocks for Op-amp Circuit

Consider the Op-amp circuit shown below.

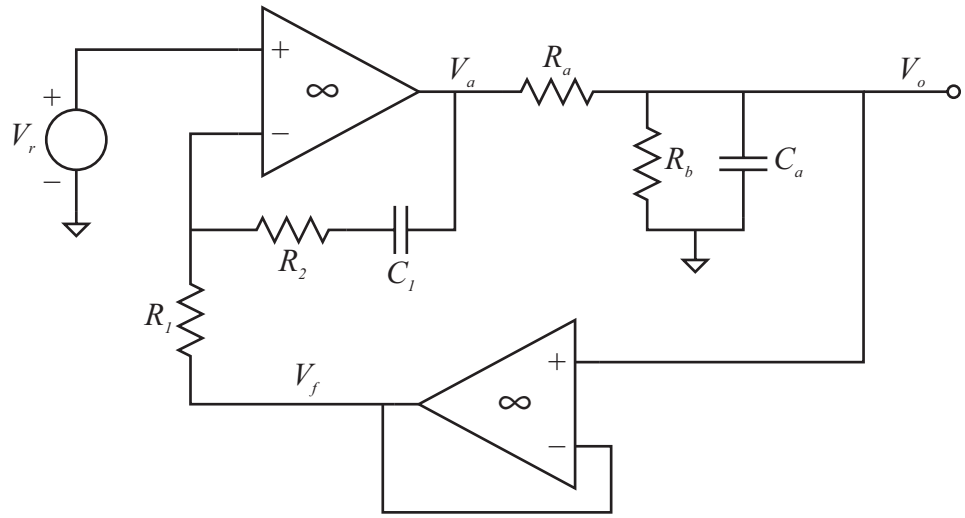


Figure 17.11.1: Op-Amp Circuit.

(a) An incomplete block diagram for this circuit is shown on the next page. Develop expressions for each of the transfer functions in the blocks of this block diagram, and fill them in on the block diagram. Carefully note the system variables already shown on the block diagram, and fill in the transfer functions appropriately.

(b) Set V_r to 1 [V]. What are the steady state values of V_o and V_a?

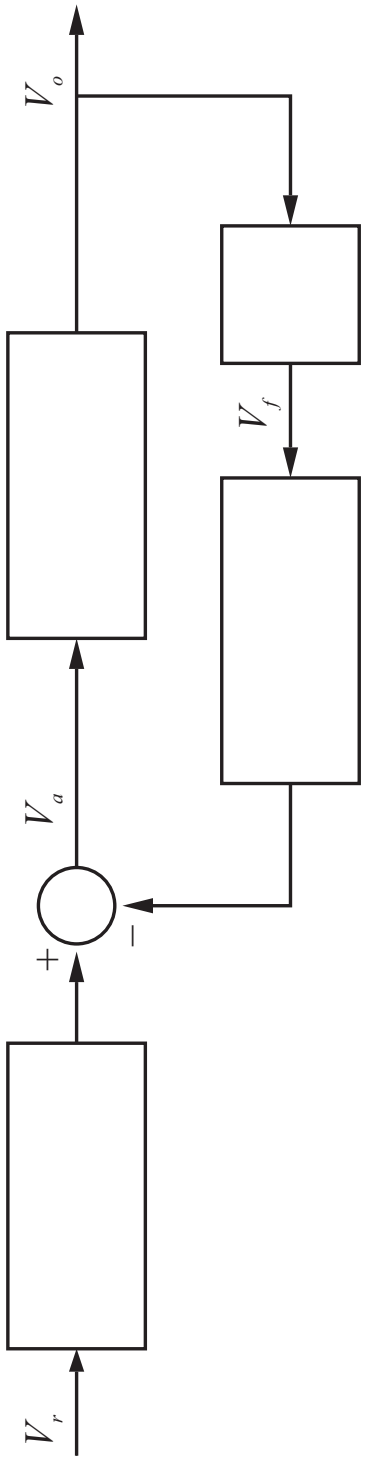


Figure 17.11.2:

17.12 Automobile Fender Spring/Damper System

A fender is mounted on an automobile though dampers (to absorb collision energy) and springs (so that the fender can recover after low-speed collisions). During a crash-test, the automobile is moving at 2 m/s when its fender strikes a concrete barrier. The vehicle mass, m, is 1,000 kg. (In comparison, the fender itself is essentially massless.) The springs that mount the fender have a stiffness, k, of 1,000,000 N/m.

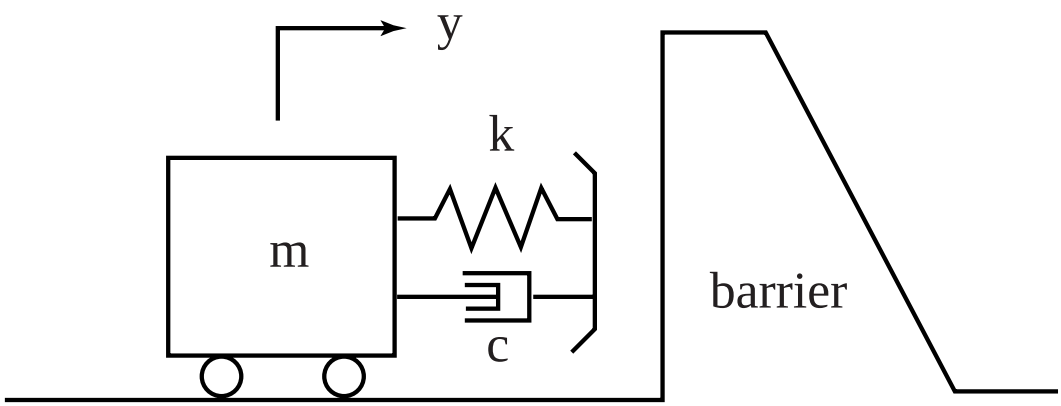


Figure 17.12.1:

(a) Write a differential equation for the deflection of the springs when the fender is in contact with the barrier. (We define y = 0 at the point of first contact.)

(b) If the damping coefficient, c, is 30,000 N-s/m, what is the damping ratio of the mass-spring-damper system when the fender is in contact with the barrier?

(c) Is there any value of the damping coefficient, c, that would yield no rebound of the vehicle from the barrier? If so, what it is? If not, why not?

17.13 Cu Flywheel with Eddy Current Damper

The figure below shows a stylized picture of the copper flywheel, eddy current damper, and mass used in Laboratory 2.

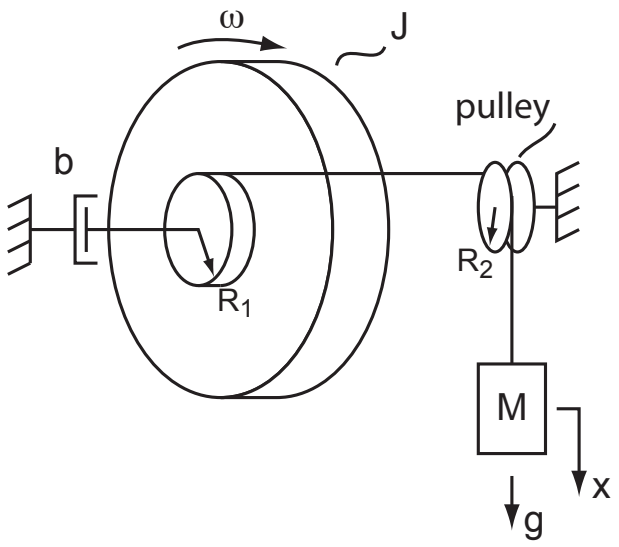


Figure 17.13.1: Stylized version of Lab 2 setup.

(a) Sketch two free body diagrams for the flywheel and hanging weight respectively, detailing all the forces or torques acting on these elements. In developing these free body diagrams, express the tension in the string as T. Note that $T \neq mg$ in general.

(b) Write the equations of motion for each of your free body diagrams. Eliminate T and x to combine these into one first-order differential equation in terms of the flywheel angular velocity ω. To eliminate x, note that $\dot{x} = \omega R_1$.

(c) Write the characteristic equation for the system in terms of the given system parameters. What is an expression for the time constant?

(d) At $t = 0$, with the system initially at rest, the mass is released and begins to accelerate downward. Make a plot of the resulting flywheel angular velocity $\omega(t)$. What is the final angular velocity as $t \to \infty$?

17.14 Current Driven RC Circuit

This problem considers the circuit shown below

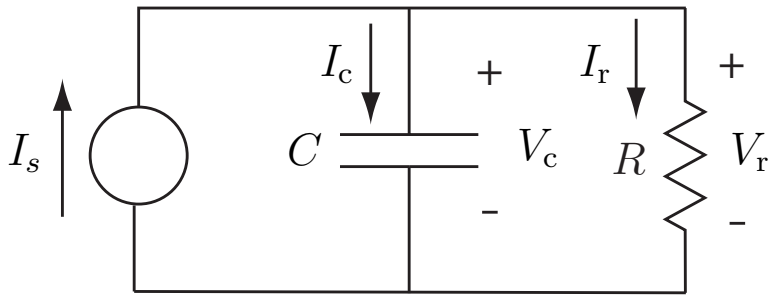


Figure 17.14.1: RC circuit driven by current source.

(a) Note that the circuit is driven with a current source $I_s(t)$. Write a differential equation for this circuit in terms of the capacitor voltage $V_c(t)$, and with input $I_s(t)$. Clearly show the steps in your development.

(b) Show that with the values $R = 1$ kΩ, and $C = 1$ μF, the differential equation becomes

$$10^{-3}\dot{V}_c(t) + V_c(t) = 10^{-3}I_s(t)$$

We assume that the input current takes the form of a 0.1 A step $I_s(t) = 0.1u_s(t)$ [A], and that the system is initially at rest. Write an expression for the resulting output $V_c(t)$ for $t > 0$, and make a properly-dimensioned plot of this response.

17.15 Driven Mass Spring System

Consider the mechanical system shown below.

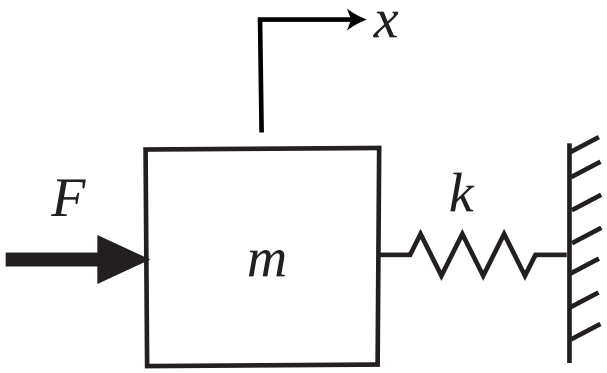


Figure 17.15.1: 3.0in

(a) Write the governing differential equation in terms of F and x.

(b) The input is a unit step $F(t) = u_s(t)$. Calculate $x(t)$ for $t > 0$, assuming initial rest. Write an expression for $x(t)$, and make a properly dimensioned sketch of $x(t)$.

17.16 Flywheel Driven by Hanging Mass

Consider the mechanical system shown below.

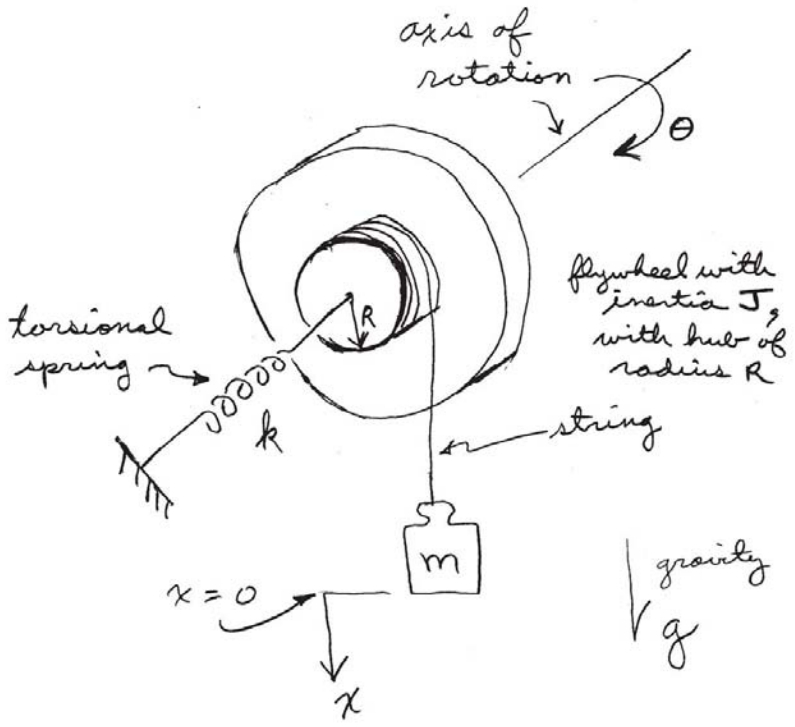


Figure 17.16.1:

In this system, a flywheel of inertia J [kgm^2] is constrained to rotate about the indicated axis. A torsional spring k [Nm/rad] is attached to the flywheel. A weight of mass m hangs on a massless string which is wrapped around a hub of radius R on the flywheel. The force of gravity mg acts on the weight in the downward direction. In the absence of any torque on the spring, the weight sits at the position $x = 0$, and x increases as the weight moves down from this position. There is no damping in the system.

(a) Write a differential equation describing the dynamics of this system in terms of the motion of the weight x, with input force mg. The variable θ should be eliminated from this equation.

(b) Assume that the weight is initially supported at the position $x = 0$ by a shelf. At $t = 0$ this shelf is removed. Write an expression for the resulting motion $x(t)$ for $t > 0$, and make a dimensioned graph of this motion as a function of time.

17.17 Homogeneous Second Order DE

Consider the homogeneous differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + (1 + 100\pi^2)y = 0.$$

This has poles at $s_1 = -1 + j10\pi$ and $s_2 = -1 - j10\pi$.

(a) What are the values of the damped natural frequency w_d, the natural frequency w_n, the attenuation σ, and the damping ratio ζ?

(b) The homogeneous solution is of the form

$$y(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}.$$

For the initial values of $y(0) = \dot{y}(0) = 1$, what are the numerical values of c_1 and c_2?

(c) The solution can be written in the form

$$y(t) = Me^{-\sigma t}\cos(\omega_d t + \phi).$$

What are the numerical values of M and ϕ?

(d) Make a reasonably accurate plot of $y(t)$, indicating key values on the time and magnitude axes.

17.18 Match Pole/Zero Plots with Step Response

Six pole/zero plots labeled A-F are provided below. The following page has eight step response plots labeled 1-8. Match each of the six pole/zero to its corresponding step response. Your solution should include a list of A through F with the corresponding number directly next to it. Note the time and amplitude scales are not necessarily the same in each figure. No partial credit will be given.

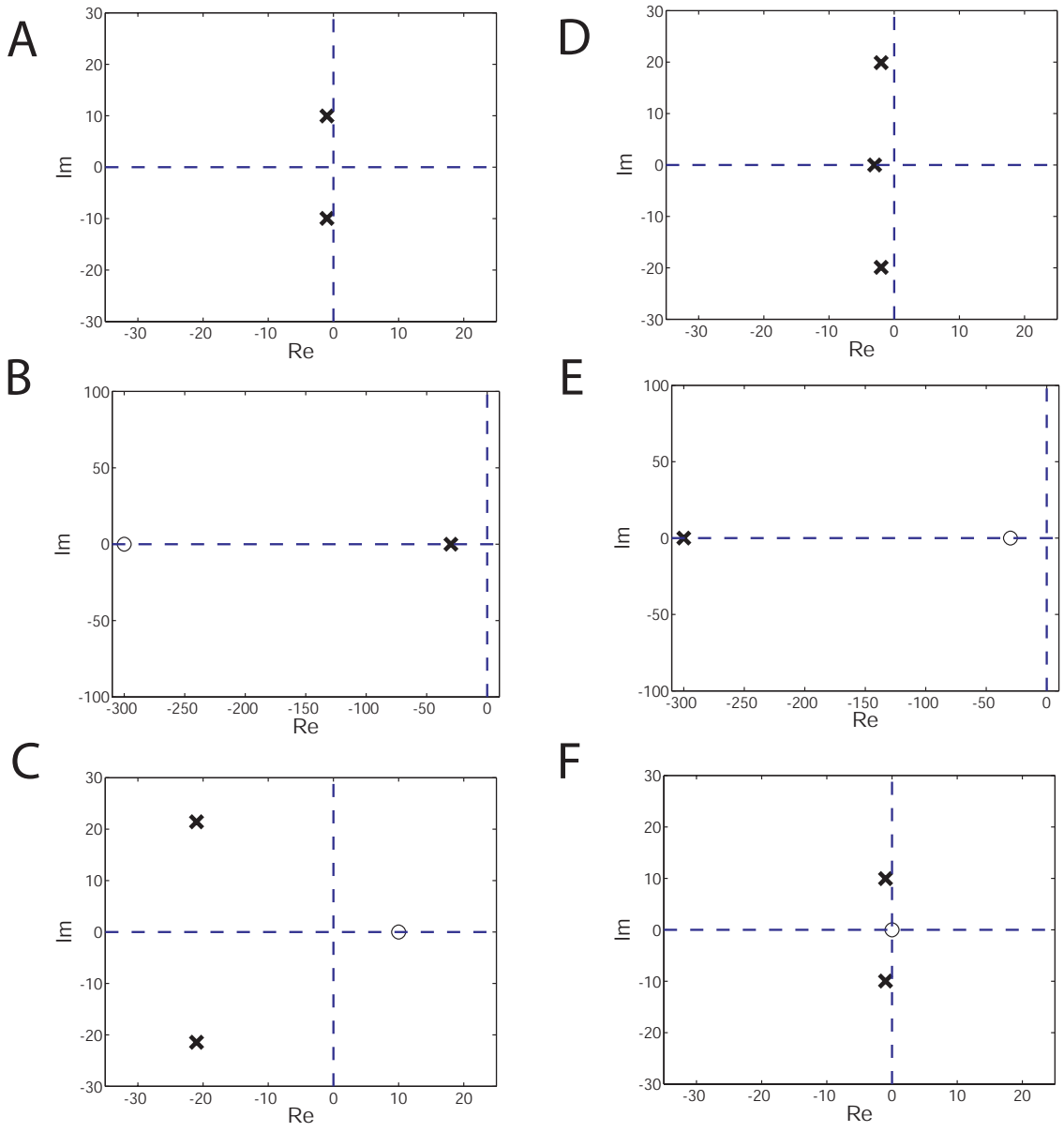


Figure 17.18.1:

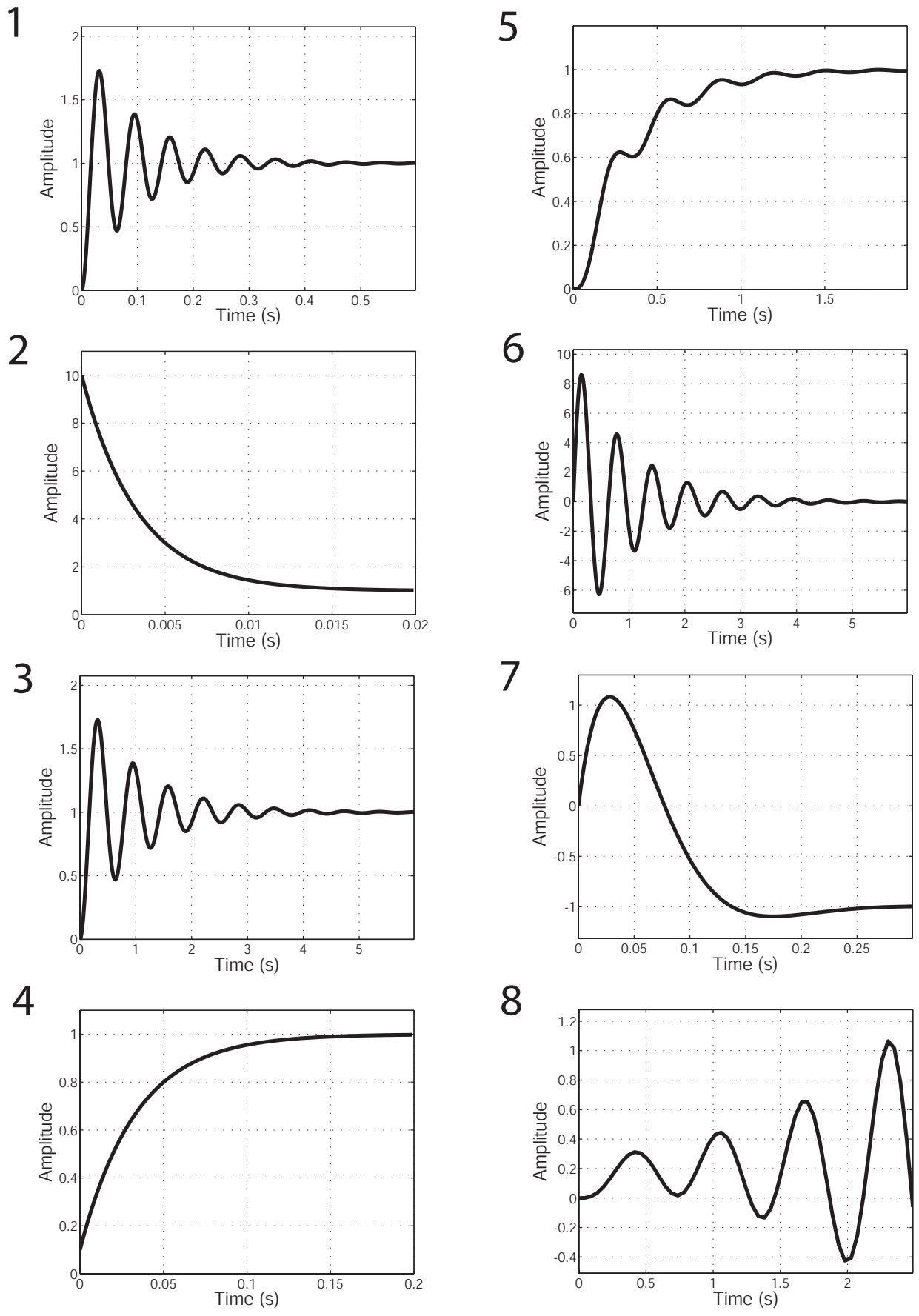


Figure 17.18.2:

17.19 Opamp Block Transfer Functions

This problem considers the op amp circuit shown below

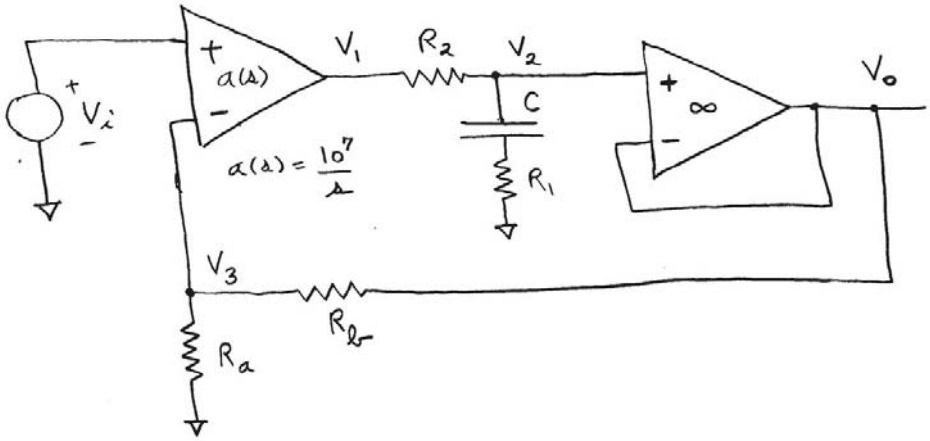


Figure 17.19.1: Op amp circuit.

(a) The circuit can be represented by the block diagram given on the next page. You are to fill in each of the blocks with the correct transfer function on that page, and return the page still attached to the quiz. Show the relevant calculations in your quiz book.

(b) Suppose that the input is $V_i = 1$ V, and that this has been applied for a long time. What is the value of V_o? Explain your reasoning.

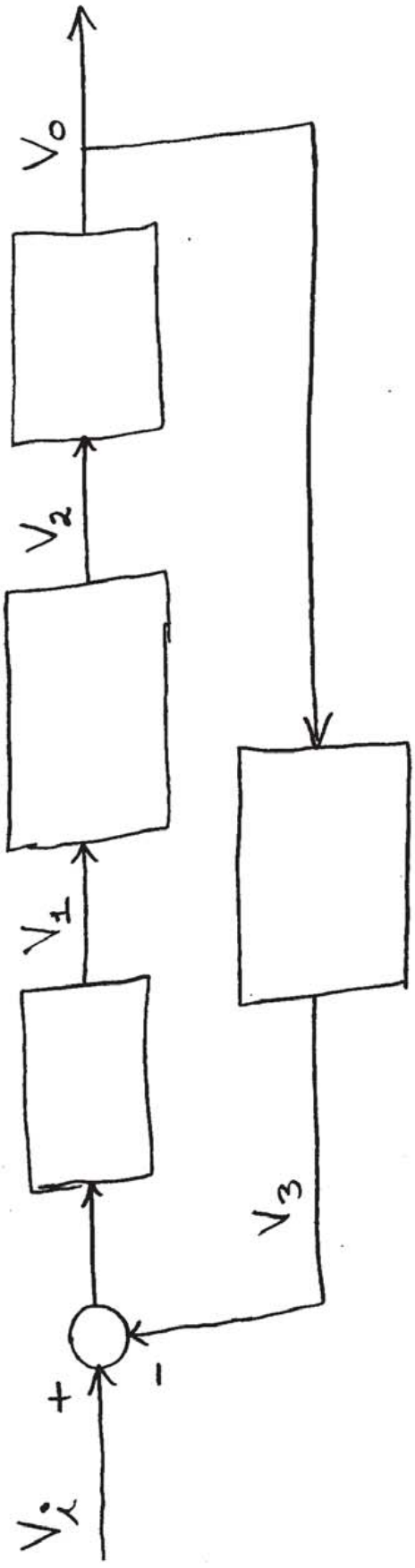


Figure 17.19.2:

17.20 Piston with 2nd Order Translation

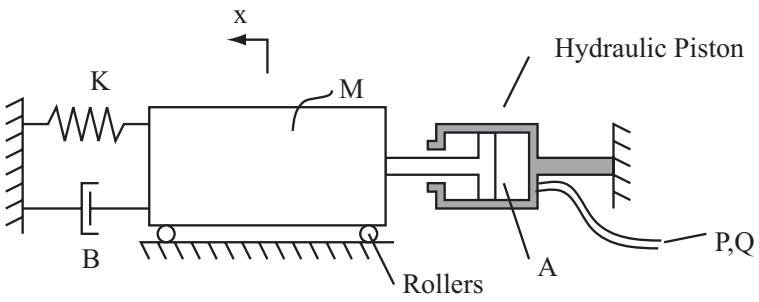


Figure 17.20.1: Mass Spring Damper System with Hydraulic Piston

The figure above shows a large stage with mass M (kg) driven by a hydraulic piston. The hydraulic piston is driven by a pressure source P(t) (N/m^2) with a volume flow $Q(t)$ (m^3/s) (the flow resistance of the piping is ignored). The piston has a force/pressure relationship of F=P*A, and thus $Q = A\dot{x}$.

(a) Find the differential equation for this system in terms of an input P(t) and an output x. Note: Ignore the roller dynamics.

(b) Given M=5000 kg, K=250,000 N/m , B=75,000 N*s/m , and A=0.1 m^2 show that the transfer function from the pressure input P to flow Q (the fluidic admittance) is given by

$$\frac{Q}{P}(s) = \frac{0.01s}{5000s^2 + 75,000s + 250,000}$$

(c) The system has been held in equilibrium with P = 500 kPa = 500,000 N/m^2 when the pressure is removed suddenly at t=0. That is, $P(t)=P_0(1\text{-}u_s(t))$. Find and make a dimensioned plot of Q(t). Use the transfer function from part b) for this calculation even if you were not able to derive it.

17.21 Pole Zero Bode Matching

Six pole/zero plots labeled A-F are shown below. The following page has eight Bode plots labeled 1-8. Match each of the six pole/zero plots to its corresponding Bode plot. Your solution should consist of a list of letters A through F with the corresponding Bode plot number directly next to it.
No partial credit will be given.

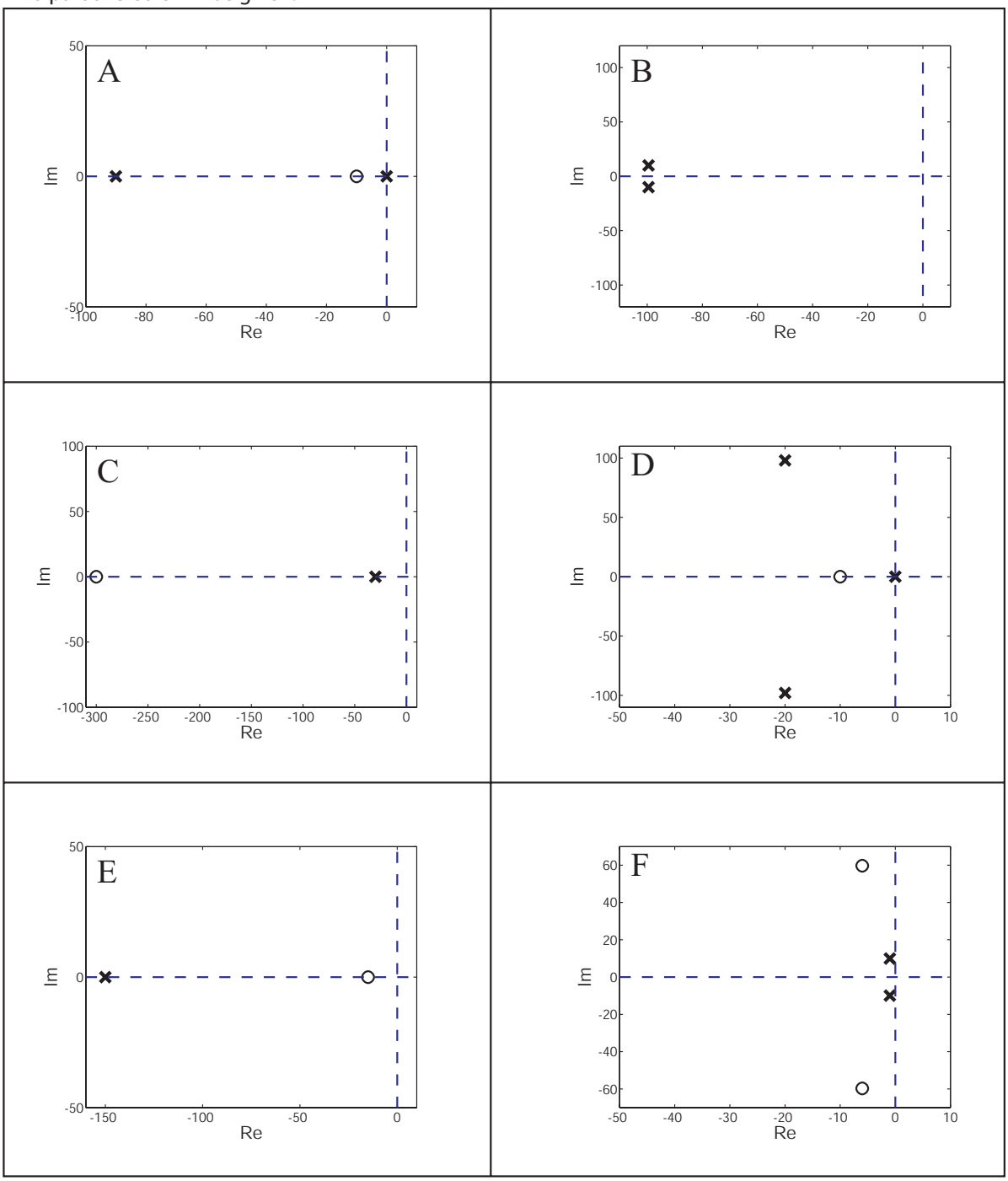


Figure 17.21.1:

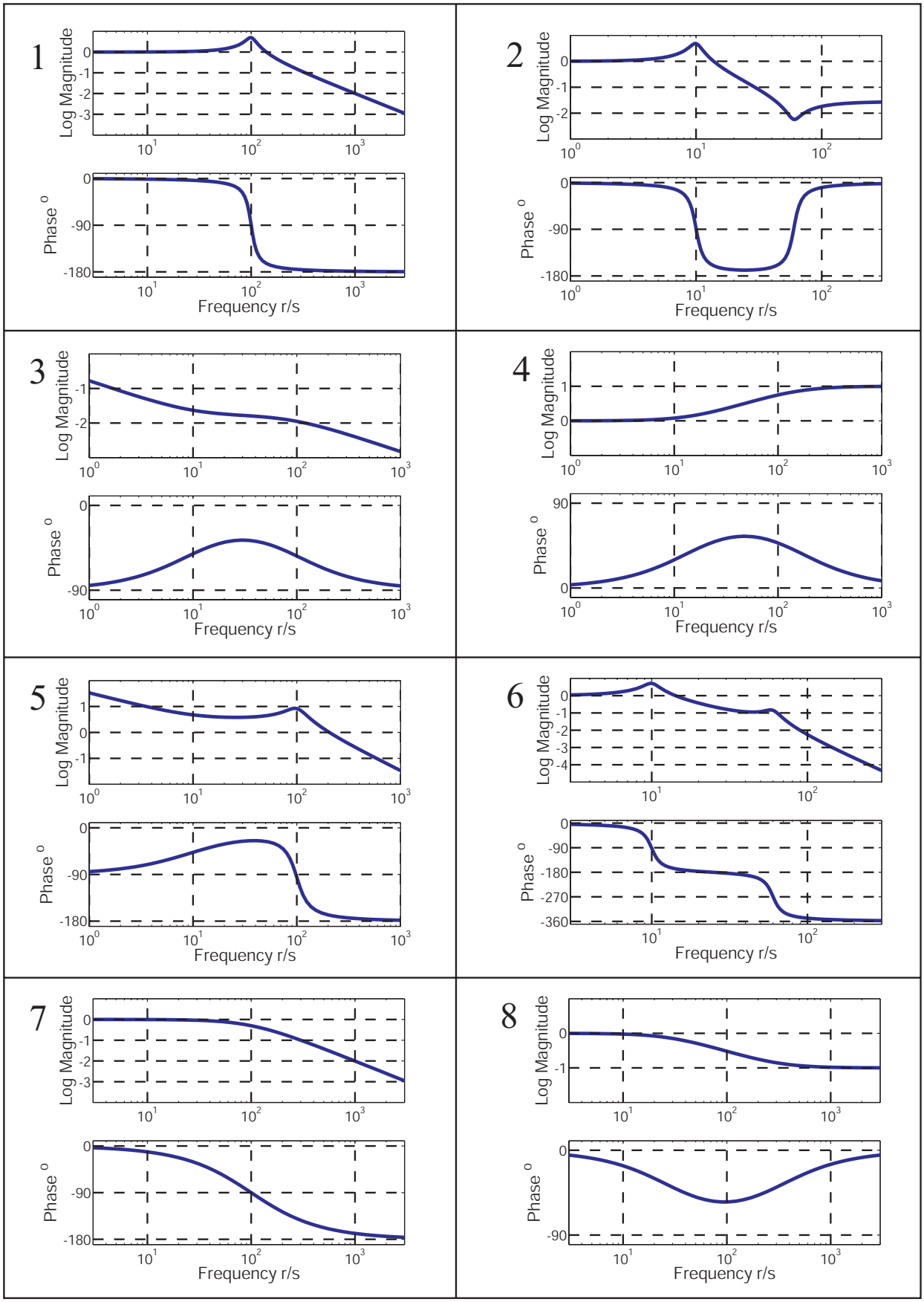


Figure 17.21.2:

17.22 Power Semiconductor Thermal Problem

The thermal system shown below represents large power semiconductor device which is captured between two plates. The device has a thermal capacitance $C_1[J/K]$, and its temperature is $T_1[K]$. Thermal resistances $R_1[K/W]$ and $R_2[K/W]$ connect the device to the upper and lower plates respectively. These plates are maintained at ambient temperature $T_A[K]$. Power dissipation in the device is modeled as heat flow $q_{in}[W]$ into the device. Assume no heat flow occurs through the side walls of the device.

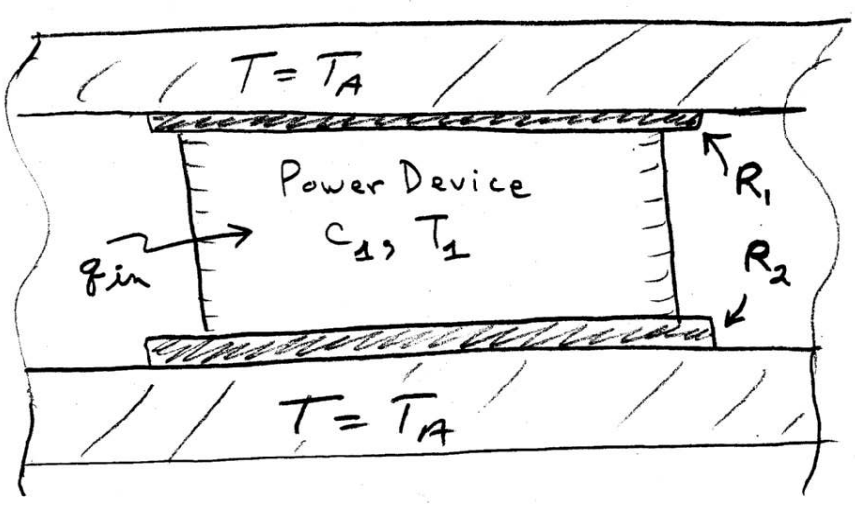


Figure 17.22.1: Schematic of thermal configuration.

(a) Write the governing differential equation in terms of T_1 and q_{in}.

(b) Assume that in steady state, $q_{in} = q_o$, a constant. What is the steady state temperature T_{1ss}?

17.23 Submersible Capsule Hoist System

A submersible capsule requires in case of malfunction a rescue hoist system, as shown in Figure 17.23.1. The capsule mass is M, and the hoist cable stiffness is assumed constant and of value K. The drum has inertia J, and radius r. The drum is driven by a velocity source $w(t)$. The capsule is subject to a drag force equal to $F_b = bv(t)$ and to a flotation force F_w. **Note**: The system is designed in a way that the hoist cable will be always in tension, under normal operation.

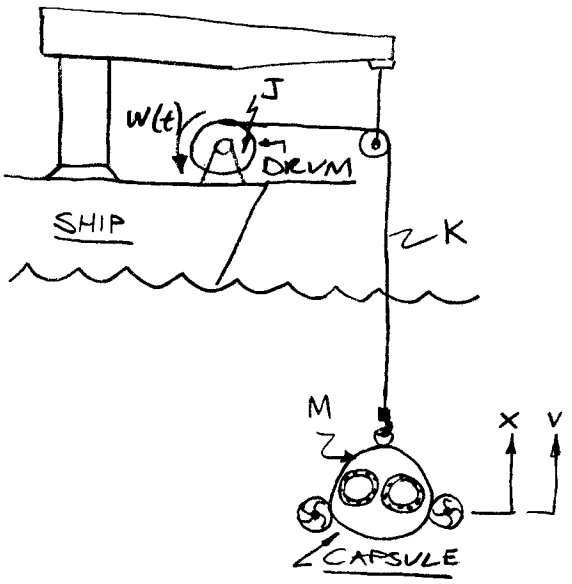


Figure 17.23.1: Capsule hoist system

(a) When the capsule is at rest the cable is extended by an amount e. Find e as a function of M and F_w.

(b) Find the transfer function V(s)/W(s).

(c) Given $M = 10000$ [kg] and the bode plot shown on Figure 17.23.2, try to find the values of J, r, K, and B. Use the correct units for all values, and indicate what value cannot be determined.

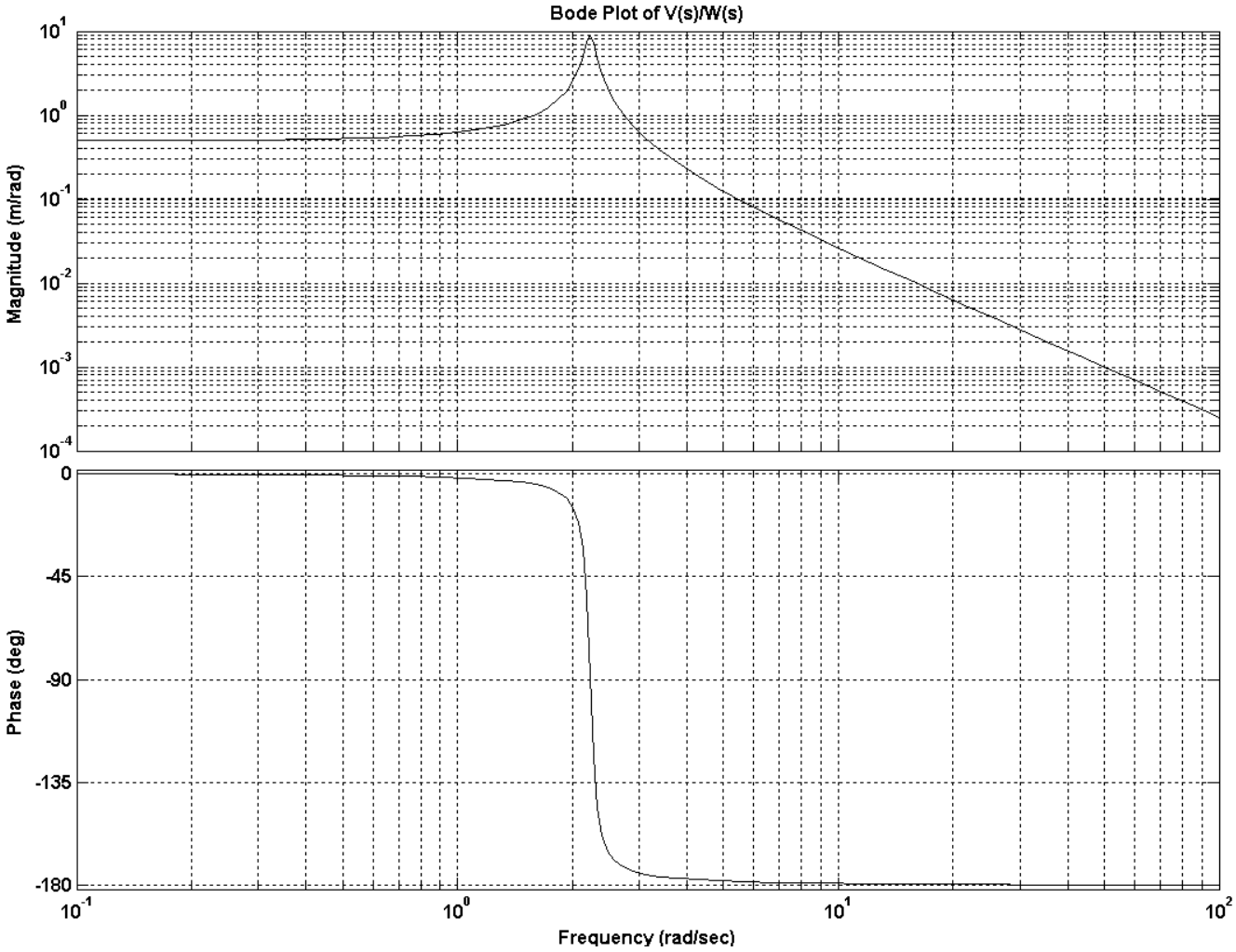


Figure 17.23.2: Bode Plot of $V(s)/W(s)$

17.24 Transfer Function from Pole Zero Plot

Consider the following pole-zero plot:

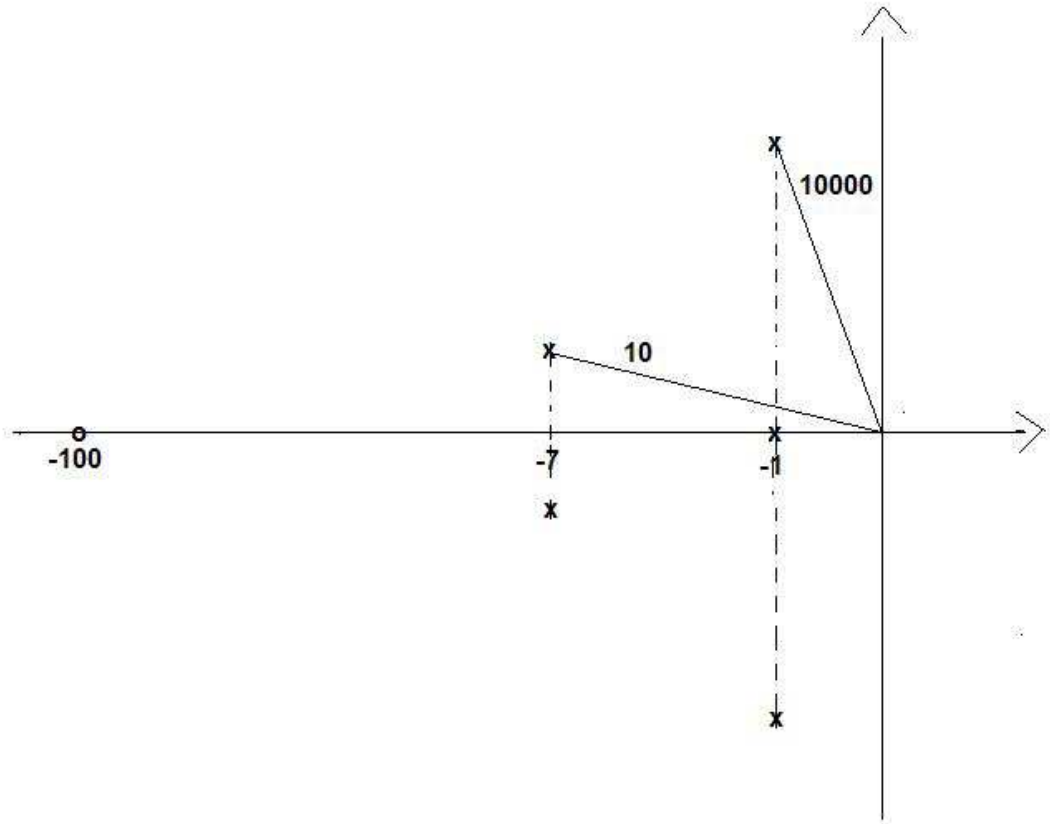


Figure 17.24.1: System Bode Plot.

The DC gain for the system is 10.

(a) Derive the system's transfer function.

(b) Sketch the system's Bode plots.

17.25 Vaccine Cooler

You're a 2.009 student trying to verify the feasibility of a portable vaccine cooler. The vaccine chamber of the cooler is in the shape of a cube and is shown below in Figure 17.25.1. The vaccines within the chamber have mass m and specific heat c. The combined effective resistance of all six walls is given by R. T_{vac} is the temperature of the vaccines within the cooler, and T_{amb} is the temperature outside the cooler. The heat flow into the vaccine chamber due to the cooling system is given by q_{in}. The width of each surrounding wall is L.

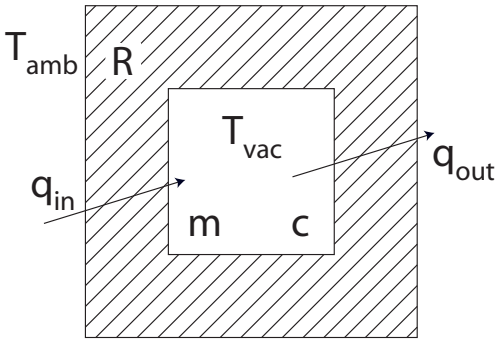


Figure 17.25.1: Cross-section of a cubical portable vaccine cooler

(a) Derive the ordinary differential equation that describes this system.

(b) You are told that the mass of the vaccines is 1 kg, their specific heat is 4.2 kJ/(kgK), and the total effective thermal resistance resistance around the vaccine chamber is given by the following equation:

$$R = 40 \left[\frac{\text{K}}{\text{Wm}} \right] L$$

If the width of the surrounding walls is 2.5 cm, the ambient temperature is 25 °C, the vaccine chamber is initially 2 °C, and the cooling system is off, how long will it take for the vaccine chamber to reach a temperature of 10 °C?

(c) You've been swamped with work for 2.003, and your 2.009 team makes an incredible amount of progress without you. You come back 2 weeks later and find an almost working prototype cooler. The cooler uses a non-linear on/off controller with a dead band. On a workbench you see the following plots. What wall thickness did your teammates choose for your cooler? When the cooling system is turned on, what is the value for q_{in}? Please use appropriate units.

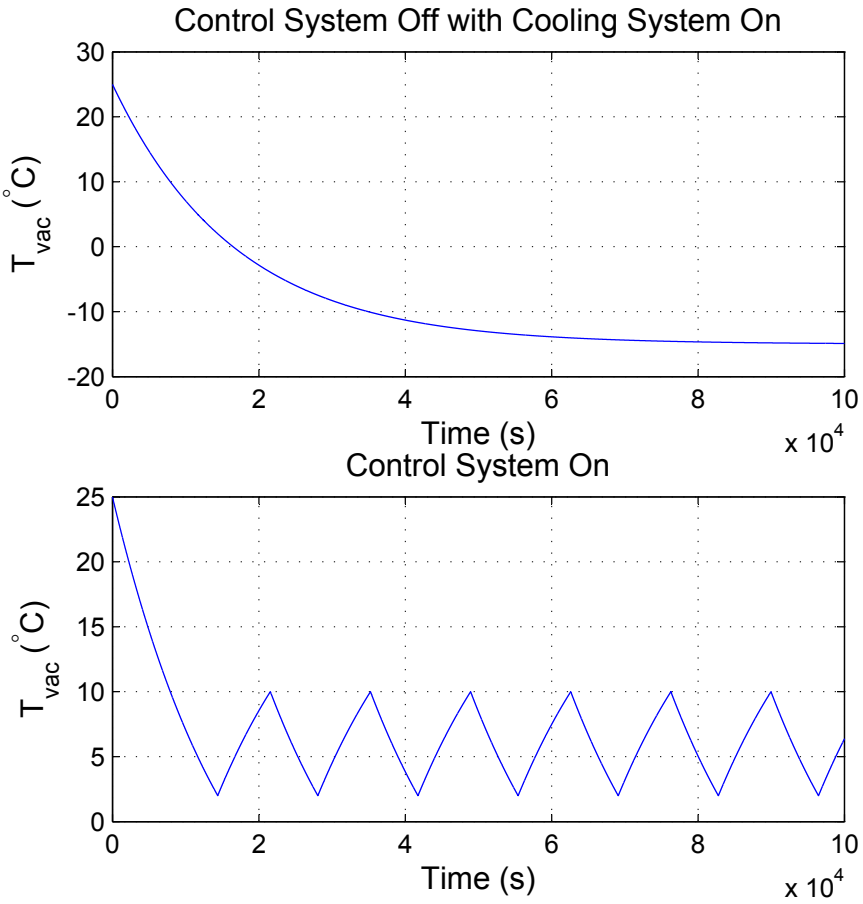


Figure 17.25.2: Prototype Cooler Tests. The top plot shows the cooling system being run continuously. The bottom plot shows the cooling system being run with the non-linear controller.

17.26 Voltage Driven RRC Circuit

Consider the circuit shown below.

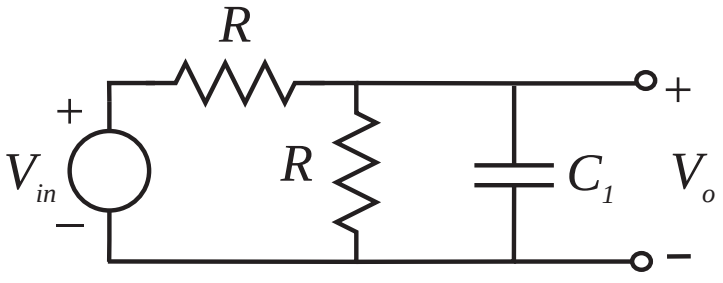


Figure 17.26.1: Voltage divider with capacitor in parallel

(a) Derive the differential equation describing $V_o(t)$ in terms of $V_i(t)$.

(b) The output $V_o(t)$ is given by

$$V_o(t) = 1; t <= 0$$

$$V_o(t) = 2e^{-10t} - 1; t > 0$$

This waveform is sketched below

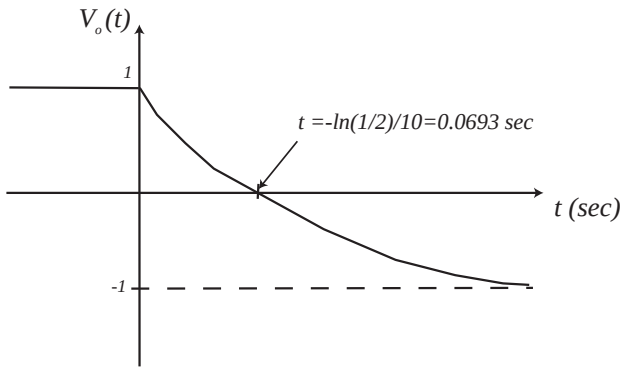


Figure 17.26.2: Sketch of V_o

What is the input $v_i(t)$ for all t? Explain your reasoning, and make a sketch of $v_i(t)$.

17.27 Linear Mechanical System with Position Input

Consider the mechanical system shown in Figure 17.27.1, where $w(t)$ is an input position source, and $x(t)$ is the mass position.

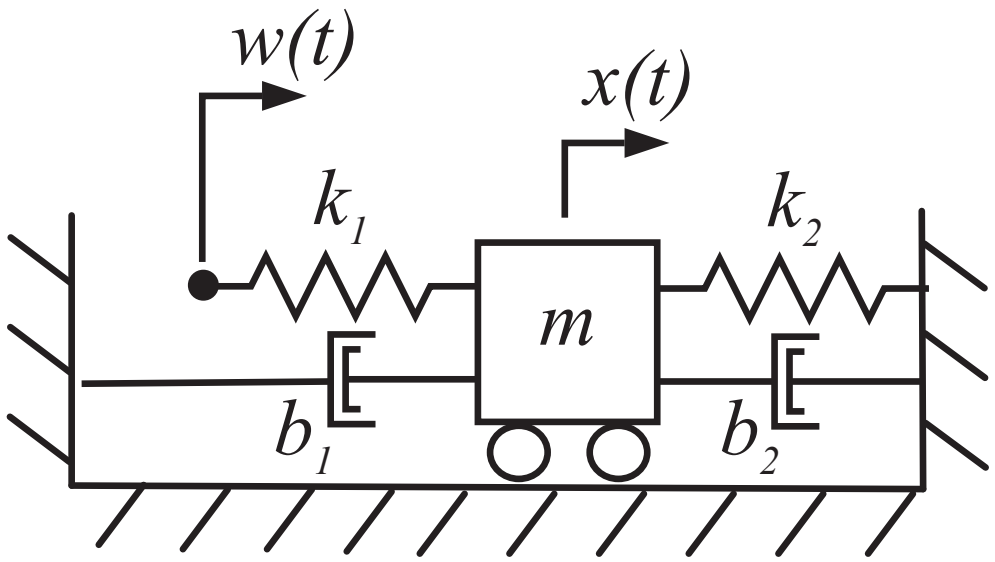


Figure 17.27.1: Linear mechanical system with two springs, two dampers, and a position source input, $w(t)$.

(a) Draw a free body diagram for the mass m, showing forces acting on the mass, as a function of the displacements $w(t)$ and $x(t)$ and in terms of the system parameters.

(b) Write a differential equation for the system, in terms of input $w(t)$ and output $x(t)$.

(c) Let $m = 10\,\text{kg}$, $k_1 = 30\,\text{N/m}$, $k_2 = 10\,\text{N/m}$, $b_1 = 2\,\text{Ns/m}$, $b_2 = 6\,\text{Ns/m}$, with $w(t) = 1u_s(t)$ [m] (a unit step). Plot the system poles in the s-plane. Solve for $x(t)$, $t > 0$, from rest initial conditions.

(d) Plot $x(t)$, using accurate dimensions, and units. Also indicate the overshoot value, with its associated peak time, as well as the 10–90 % rise time.

17.28 Tank with Pump Inlet Lower than Outlet

This problem considers the system shown in Figure 17.28.1.

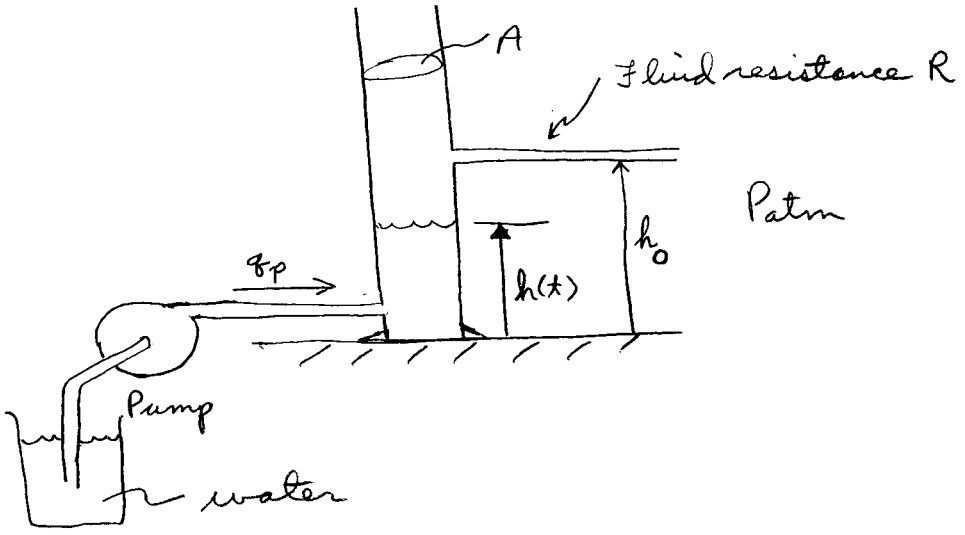


Figure 17.28.1: Diagram of tank configuration.

As shown in the figure, a pump acts as a source of flow $q_p(t)$ into the cylinder; that is the pump flow is a specified constant independent of load pressure. The cylinder is of cross-sectional area A [m^2]. At a height h_o [m] from the bottom of the cylinder, we connect a fluid resistance R [Pa $\cdot$ sec/m^3]. The entire system is exposed to atmospheric pressure P_{atm}.

At $t = 0$, the cylinder is empty ($h = 0$), and the pump is turned on at constant flow rate $q_o = 10^{-6}$ m^3/sec, and so the cylinder begins to fill up. The fluid height above the bottom of the cylinder is defined as $h(t)$ [m]. Note that no flow enters the resistance R until $h(t) \geq h_o$. The liquid in the system is water with $\rho = 10^3$ kg/m^3. The height of the fluid is recorded as a function of time, and the data is plotted as shown below.

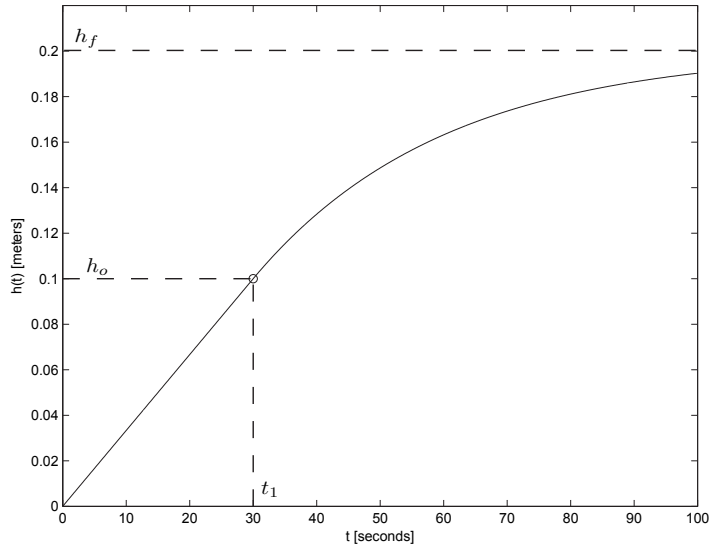


Figure 17.28.2: Fluid height as a function of time.

Note that the fluid height reaches the height of the resistance at $t = t_1 = 30$ sec; that is $h(t_1) = h_o = 0.1$ m. For $t > t_1$, the height exponentially approaches a final value of $h_f = 0.2$ m.

(a) Using the data for $0 \leq t \leq t_1$, what is the value of A in $[\mathrm{m}^2]$.

(b) Using the data for $t > t_1$, what is the value of R in $[\mathrm{Pa} \cdot \mathrm{sec}/\mathrm{m}^3]$? (This can be answered relatively easily if you think about it correctly.) What is the time constant τ [sec] associated with the response for $t > t_1$? (It is acceptable to determine this either graphically or via an appropriate calculation.)

(c) A long time later at $t = t_2 >> t_1$, the pump flow is set to zero, $q_p = 0$. Make a dimensioned sketch of $h(t)$, $t > t_2$.

17.29 Thermal Power Chip Analysis

A power electronic chip is attached to a substrate and also exposed to air as shown in Figure 17.29.1. Heat is lost to the air through a thermal resistance R_1 $[°K/W]$ and to the board through a thermal resistance R_2 $[°K/W]$. The air and circuit board are at ambient temperature T_A $[°K]$. The chip has a thermal capacitance of C $[J/°K]$ and is at temperature T_c $[°K]$. Power dissipation in the chip is modeled as heat flow in, q_{in} $[W]$.

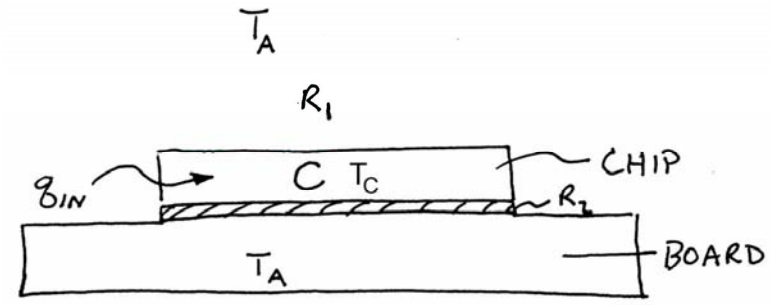


Figure 17.29.1:

(a) Write the governing differential equation for this system in terms of T_c and q_{in}.

(b) For $R_1 = 90$ $°K/W$, $R_2 = 10$ $°K/W$, and $C = 1$ $J/°K$, solve for $T_c(t)$ and plot T_c as a function of time for $q_{in} = 10$ W from rest initial conditions where $T_c(0) = T_A$. Be sure to label and dimension your time and temperature scales.

(c) If the maximum allowable steady-state temperature of the chip is 100 $°K$ above ambient, what is the maximum allowable power the chip can dissipate?

17.30 Current Divider in Simple Network

For the resistive circuit:

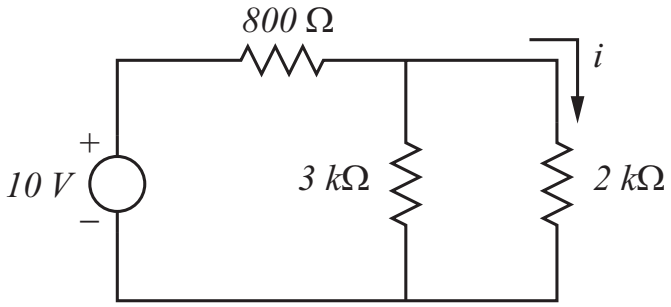


Figure 17.30.1: Simple circuit

What is the value of i? Show the steps in your derivation.

17.31 Voltage and Current Source Driving network

The following circuit has two inputs, a voltage source v_{in} and a current source i_{in}. We have also defined the common terminal as shown, and the node voltage v.

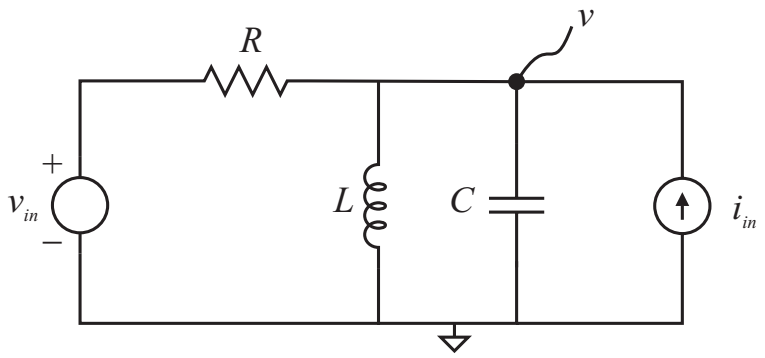


Figure 17.31.1: Network driven by a current source and a voltage source.

Write a differential equation in v for the circuit with inputs v_{in} and i_{in}. Show the steps in your derivation.

17.32 Rotary Motor with Damper

A rotary motor has an internal coil resistance R [Ω] and a torque constant K [Nm/A]. The manufacturer's data sheet for the motor provides a steady-state torque-speed curve for the motor under the condition that the applied voltage is 10 V. Here τ [Nm] is defined as the motor output torque applied to a load, and ω [rad/s] is the test load angular velocity.

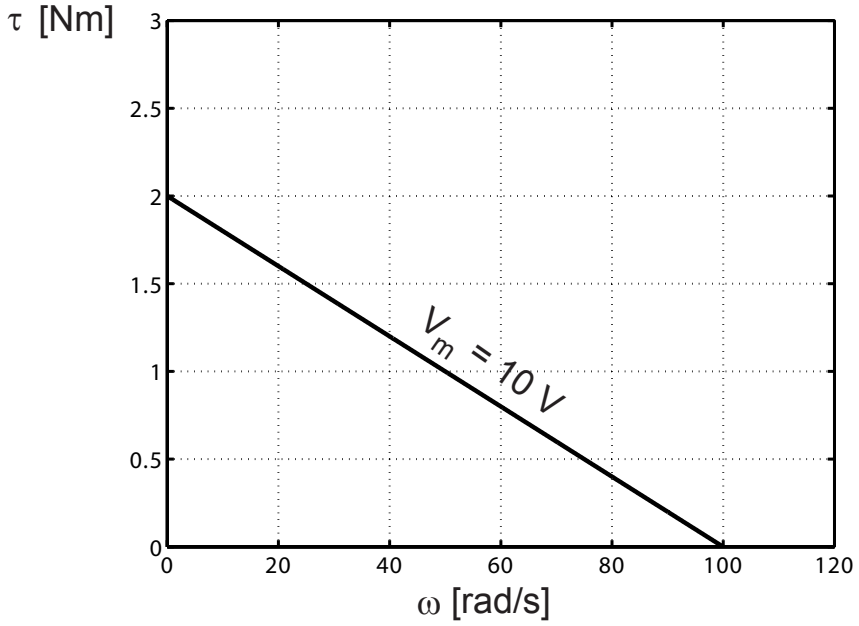


Figure 17.32.1: Motor torque speed curve.

(a) What are the numerical values of K and R? Be sure to show the steps in your derivation.

Now the rotary motor output shaft is connected to a load rotational damper $b = 0.03$ Nms/rad.

(b) Draw the rotational damper torque-speed curve on the graph above, and hand in with your quiz paper.

(c) With this load, what will be the steady-state operating speed?

(d) In steady-state with the load attached, how much power is being provided by the voltage source driving the motor? How much power is being dissipated in the motor? How much power is being dissipated in the load?

17.33 Patient Rehabilitation

Robot arms can be used to provide a controlled motion for exercises, for example in rehabilitation of stroke victims. This problem considers a single-link arm as shown below.

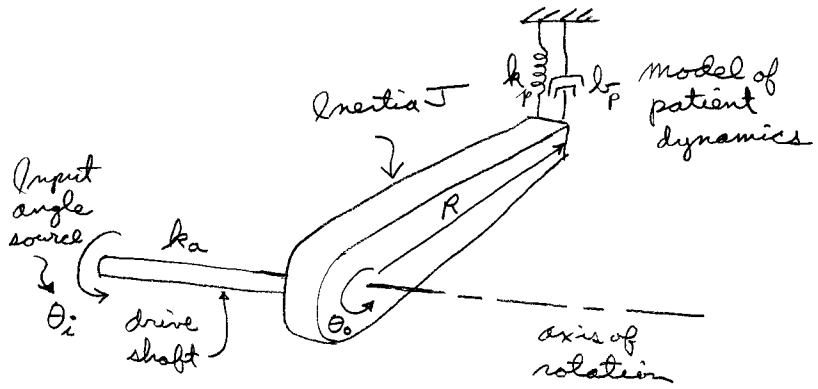


Figure 17.33.1: Schematic of single link arm.

The arm rotates about the indicated axis, with an arm angle of θ_0. The arm is driven by an input angle source θ_i through a thin drive shaft, which has a torsional spring constant k_a [Nm/rad]. The arm has a moment of inertia J [kgm^2] about the axis of rotation.

We model the patient, who is gripping the arm at a radius R, by the spring k_p [N/m] and a damper b_p [Ns/m]. For this problem, all rotations are small, and gravity is neglected.

a) Draw the free body diagram for the arm showing all applied torques and indicating their dependence on system motion.

b) Use this free body diagram to write a differential equation for the system in terms of $\theta_0(t)$, with input $\theta_i(t)$.

c) Now, let the numerical parameters take the values:

$$\begin{aligned} J &= 0.25 \text{ kg} - \text{m}^2 \\ k_a &= 72 \text{ Nm/rad} \\ R &= 0.5 \text{ m} \\ k_p &= 288 \text{ N/m} \\ b_p &= 0.12 \text{ Ns/m} \end{aligned}$$

Assume initial rest conditions. For an input step $\theta_i(t) = 0.1u_s(t)$ [rad], sketch the output $\theta_0(t)$ as a function of time. Be sure to show relevant parameters of the response.

17.34 Delay Transfer Function

The system shown below delays its input $u(t)$ by T seconds to produce the output $y(t)$.

Figure 17.34.1: Block diagram of delay.

a) Write an expression for $y(t)$ in terms of $u(t)$.

b) What is the transfer function $H(s) = \frac{Y(s)}{U(s)}$ for this system?

c) Make an accurately dimensioned sketch of the Bode plot magnitude and phase for this system on the standard loglog and semi-log axes, respectively.

17.35 Laplace and Complex Numbers

Consider the transform

$$X(s) = \frac{10}{(s+3-j4)(s+3+j4)} \tag{1}$$

a) Write $X(s)$ in a partial fraction expansion as

$$X(s) = \frac{A_1}{s+3-j4} + \frac{A_2}{s+3+j4} \tag{2}$$

What are the values of A_1 and A_2?

b) Take the inverse Laplace transform of the partial fraction expansion to find an expression for $x(t)$, $t \geq 0$; this must be simplified to purely real terms.

c) Make an accurately dimensioned sketch of $x(t)$ as a function of t.

17.36 Mechanical Stiffness

The stiffness of a mechanical system at a point is defined as

$$S(s) = \frac{F(s)}{X(s)}$$

where $x(t)$ and $f(t)$ are the motion of the point, and the force applied to the system at that point, respectively, as shown below:

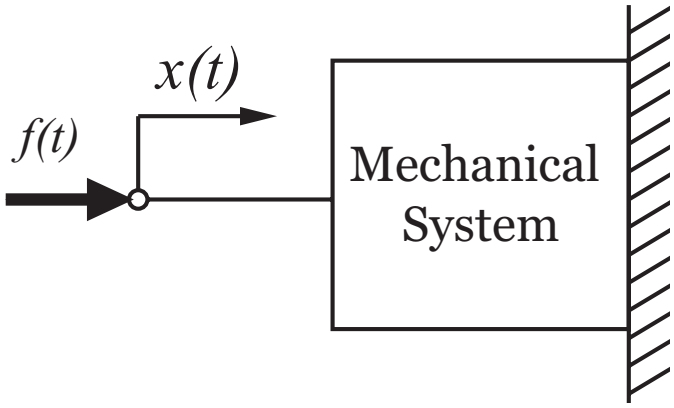


Figure 17.36.1: Illustration of stiffness definition.

To make things specific, consider the second-order mechanical system shown below:

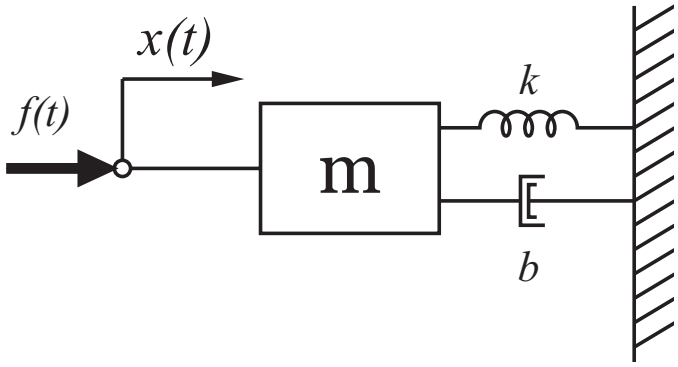


Figure 17.36.2: Mass, spring, and damper system.

(a) Write an expression for

$$S(s) = \frac{F(s)}{X(s)}$$

in terms of the system parameters. Show your reasoning.

(b) Assume the system has a damping ratio of $\zeta = 0.05$ and a natural frequency $\omega_n = 100$ [rad/sec]. Make an accurately-dimensioned sketch of $|S(j\omega)|$ and $\angle S(j\omega)$ in a Bode plot format. What is the DC gain of this system? What are the values of the magnitude and phase at $\omega = 500$ [rad/sec]?

18 Math Techniques

18.1 Complex Expression Reduction

(a) Reduce the following expressions to a single complex number in rectangular form:

1. $(2 + 3i) * (4 - 5i)$
2. $(2 + 3i)/(4 - 5i)$
3. $(-6 + 7i) * (3 + 7i)$
4. $(-6 + 7i)/(3 + 7i)$

(b) Express all above complex numbers in polar form, then calculate the final result in the polar formats.

18.2 Complex Expressions

(a) Reduce the following expressions to a single complex number in rectangular form:

(i) $(1 + 2i) * (3 - 4i)$

(ii) $(1 + 2i)/(3 - 4i)$

(iii) $(-6 + 4i) * (5 + 7i)$

(iv) $(-6 + 4i)/(5 + 7i)$

(b) Express all above complex numbers in polar form, then calculate the final result in the polar formats.

18.3 Matrix Operation Practice

Assume $X = [1\ 2\ 3\ ]'$, $Y = [9\ 8\ 0]'$ and $Z = [s^2\ s\ 1]'$, where $'$ denotes transpose operation. Calculate the following expressions:

(a) $-5X$;

(b) $X'Y$;

(c) XY';

(d) Find the roots of the equation $X'Z = 0$

19 Recitation Problems

19.1 Recitation 1 Problem

Just how effective are snow banks at stopping a sliding car? We can model the system and study its dynamic and static behavior.

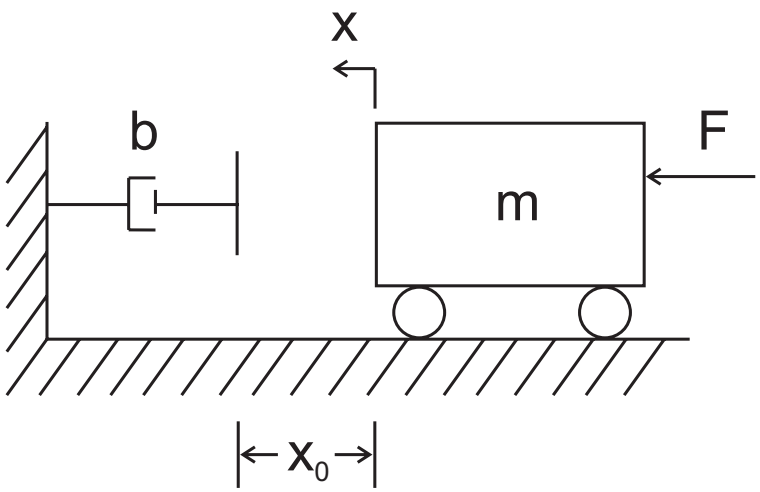


Figure 19.1.1: Sliding Car Model

Consider the model shown in Figure 19.1.1. A mass m, initially at rest, is subject to a constant applied force F. After the mass has travelled a distance x_o it impacts a damper with damping coefficient b. Assume the surface is frictionless.

(a) Formulate the first order differential equation describing the velocity of the car as a function of time, before the collision.

(b) What is the velocity at the collision?

(c) Formulate the first order differential equation describing the system after the collision.

(d) What is the time constant for the system? How long will it take to stop the car?

(e) What is the peak force? As a function of x_o? What if x_o is 0? What if b is ∞?

(f) Solve the problem again using energy methods.

19.2 Recitation 2 Problem

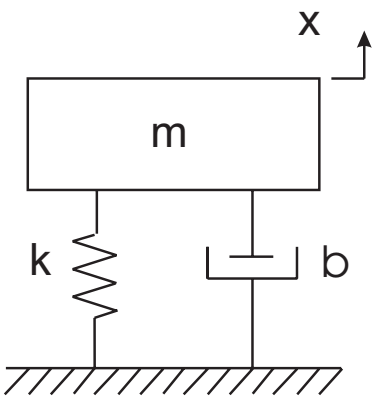


Figure 19.2.1: Automobile Suspension Model

The suspension for an automobile can be modeled by the spring-mass-damper system shown in the figure with mass $m = 500$[kg], spring constant $k = 4 \times 10^3$ [N/m], and damper $b = 2 \times 10^3$ [Ns/m].

(a) Determine the characteristic equation for the system. What is the form of the homogeneous response to an initial condition $x_0 = 0$ and $\dot{x}_0 = 1$? To an initial condition $x_0 = 1$ and $\dot{x}_0 = 0$?

(b) Find the damping ratio ζ, undamped natural frequency ω_n, damped natural frequency ω_d, attenuation σ, and the time constant of the exponential envelope τ.

Hint:

$$\zeta = \frac{b}{2\sqrt{km}} \qquad \text{and} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

(c) Is the system undamped, underdamped, critically-damped, overdamped (or just right)?

(d) From the homogeneous response determine if the system is stable, marginally stable, or unstable. Does this match your physical intuition?

(e) What if the shock absorber is leaky? Discuss what happens to the system as b changes.

(f) What happens when you drive down Massachusetts Avenue and hit a pothole? Think about the forced response.

(g) If you were designing the suspension for a car, which damping behavior (undamped, underdamped, critically-damped, overdamped) would you choose and why? For a sports car? For grandma's Cadillac? For a pickup truck?

19.3 Recitation 3 Problem

Consider a swinging door, such as you might find at the entrance to the kitchen in a restaurant or at the front door to a saloon in the Wild West. These types of doors don't slam; they can overshoot and oscillate. The schematic representation of the door is shown in the Figure 19.3.1.

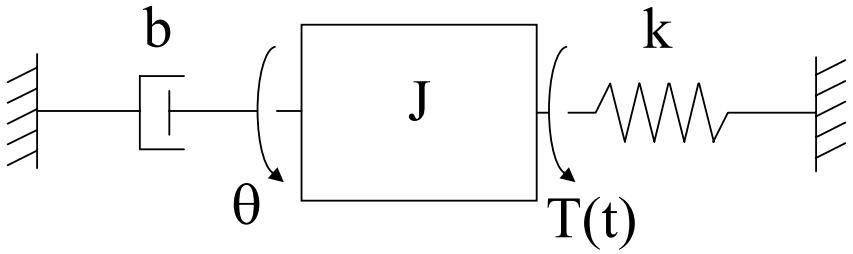


Figure 19.3.1: Swinging door model

The system rotates about a central shaft with angular displacement θ, moment of inertia J, spring constant k, and damping constant b. When someone pushes on the door, they apply a torque T.

(a) Determine the characteristic equation for the system.

(b) Find the damping ratio ζ, undamped natural frequency ω_n, and damped natural frequency ω_d. Remember the standard form:

$$f(t) = \frac{1}{\omega_n^2}\ddot{\theta} + \frac{2\zeta}{\omega_n}\dot{\theta} + \theta$$

(c) Sketch the response of the system to a step torque of the form

$$T(t) = T_0 u_s(t)$$

with initial conditions $\theta_0 = 0$, $\dot{\theta}_0 = 0$. Use $\omega_n = 1$ [rad/sec].

(d) Define the maximum value of the angular displacement θ. Add it to the sketch.

(e) Define rise time. Show rise time on the sketch.

(f) Define settling time. Show settling time on the sketch.

(g) How does the above forced response of a second-order system compare to the homogeneous response?

19.4 Recitation 4 Problem

Two identical tanks with cross sectional area A are partially filled with a volume V of incompressible fluid of density ρ. The fluid flows between the two tanks through a short tube with a small diameter which can be modeled as a lumped resistance, R. The top of tank 2 is open to the atmosphere (with pressure P_a).

Initially, the top of tank 1 is closed and the pressure at the top of tank 1 is set so that the level of the fluid in the first tank, h_1 is substantially higher than the level of the fluid in the second tank, h_2. At time $t = 0$, the top of tank 1 is opened to the atmosphere. 19.4.1 shows the system just after the top is opened.

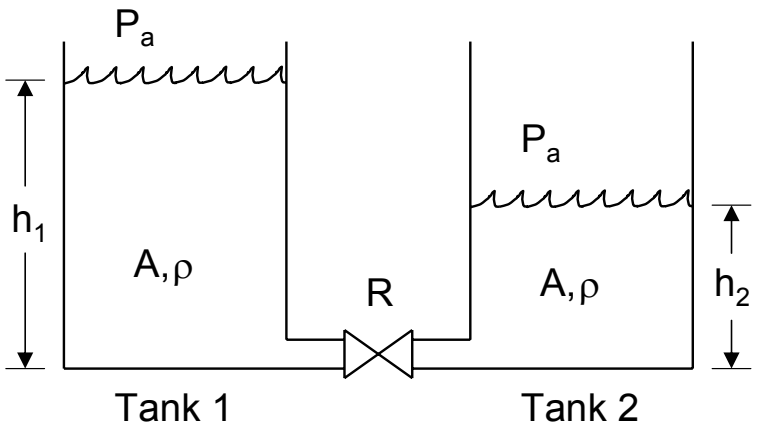


Figure 19.4.1: Tank Setup

(a) How many independent variables as a function of time do you need to completely describe the state of the system?

(b) What do you expect the system to do physically? Before making any calculations, sketch the height of the fluid in tank 1 and the height of the fluid in tank 2 as functions of time.

(c) Find a relationship between the height of the fluid in tank 1 to the height of the fluid in tank 2.

(d) What is the pressure P_1 at the bottom of tank 1? What is the pressure P_2 at the bottom of tank 2?

(e) Find a constitutive relationship for the lumped resistance in the tube.

(f) Find the differential equation in h_1 for the system.

(g) Sketch the height of the fluid in the tank 1 and tank 2 as a function of time, given initial conditions $h_1(0)$ and $h_2(0)$. Will this system oscillate?

(h) Does the time constant of the system τ depend on the amount of fluid? What could you do to make the system respond twice as quickly?

(i) This system is first-order, but the demonstration we saw in class on Wednesday of water flowing through a tube was a second-order system. Why are these systems different?

19.5 Recitation 5 Problem

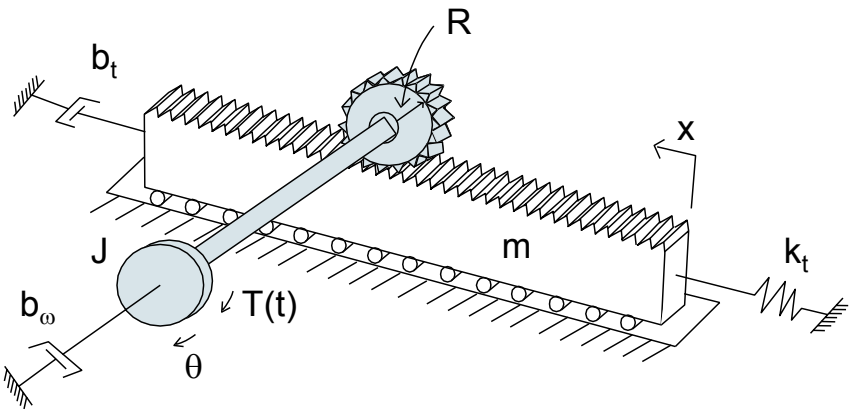


Figure 19.5.1: Rack and Pinion

A rack and pinion converts rotational motion to translational motion and vice-versa. The tracking mechanism in the CD player passed around in class contained a rack and pinion.

The figure shows a lumped model where the angular position of the pinion gear is given by θ. The combined shaft and gear inertia is given by J, the rotational damping on the shaft is given by b_ω and the applied torque $T(t)$ is a function of time. The rack position is x, the mass of the rack is m, the linear damping acting on the rack is given by b_t, and the linear spring constant is k_t. Assume that there is adequate room for the rack to roll back and forth.

(a) What is the relationship between x and θ?

(b) For each lumped element in the system find the energy stored or power dissipated. Compare it to the constitutive relationship. For example, the potential energy stored in the spring is: $PE = .5k_t x^2$ The constitutive relationship for the spring is: $F_k = k_t x$

(c) Determine the total kinetic energy, potential energy and power dissipated for the system. Express them as a function of θ (Use part (a) to eliminate x).

(d) Use the energy equations to find the second-order differential equation for the system as a function of θ.

(e) What is the equivalent inertia, equivalent damping coefficient, and equivalent spring constant for the system? Is the equivalent inertia for the system larger or smaller than J?

(f) Find the damping ratio ζ, undamped natural frequency ω_n as functions of the system parameters. Remember the standard form:

$$f(t) = \frac{1}{\omega_n^2}\ddot{\theta} + \frac{2\zeta}{\omega_n}\dot{\theta} + \theta$$

(g) If the radius of the gear R is increased, what happens to the damping ratio ζ and undamped natural frequency ω_n?

(h) Rewrite the equations as a function of x instead of θ. What happens to the damping ratio ζ and undamped natural frequency ω_n?

(i) Assume the system is underdamped and sketch the response of θ to a step torque of the form:

$$T(t) = T_0 u_s(t)$$

with zero initial conditions $\theta(0) = \dot{\theta}(0) = 0$.

19.6 Recitation 6 Problem

The circuit in the figure consists of a voltage source, a capacitor and three resistors.

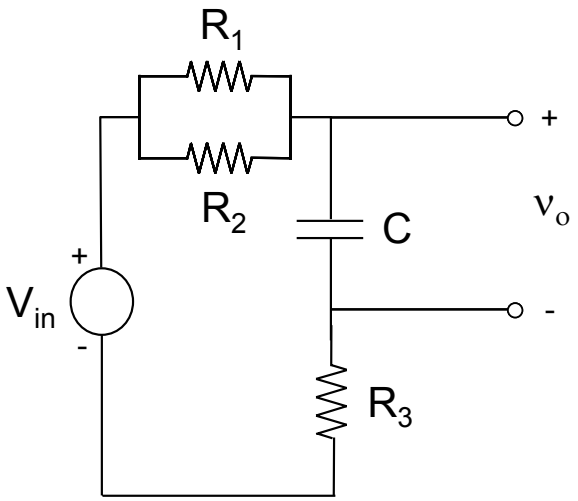


Figure 19.6.1: RC Circuit

(a) Find the differential equation for this circuit in terms of v_o.

(b) Find v_o as a function of time, if V_{in} is a step input and $v_o(0) = 0$.

(c) What is the equivalent resistance for the circuit?

19.7 Recitation 7 Problem

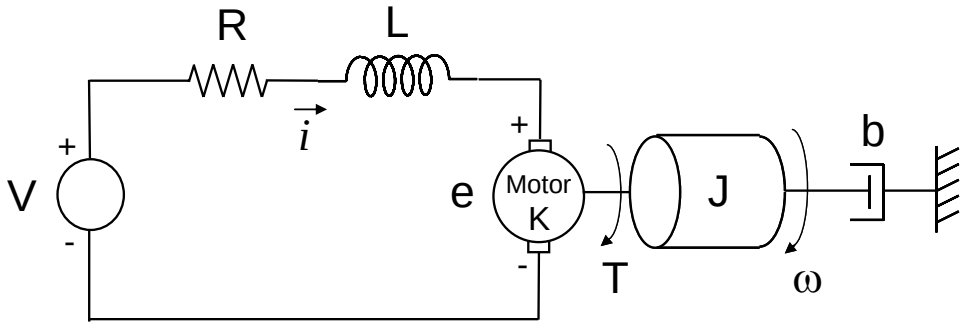


Figure 19.7.1: Motor Circuit

Figure 19.7.1 shows an ideal motor connected to a load rotating with angular velocity ω. The motor's resistance is lumped into the element R. This motor also has non-negligible inductance given by L. The motor is driven by an input voltage V. The motor applies a torque to the load $T = Ki$. The motor's back emf is $e = K\omega$. The combined motor and load inertia is given by J and the load is also subject to a rotational damper b.

(a) Find a differential equation for the rotational load in terms of T and ω.

(b) Find a differential equation for the circuit in terms of e, V and i. Remember that the elemental equation for an inductor is $V_L = L\frac{di}{dt}$.

(c) Combine parts (a) and (b) to find a differential equation for the system in terms of ω and V_{in} (Eliminate T and e). What is the steady state behavior of this system?

(d) When can the inductance L be neglected? What happens as L goes to zero? Does the steady state behavior of the system change?

(e) With $L = 0$, think about the torque input T to the rotational load. If the input voltage V is a step, would you also expect the torque T to be a step?

19.8 Recitation 8 Problem

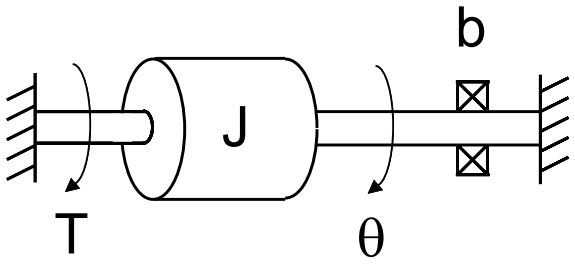


Figure 19.8.1: Rotational System

Figure 19.8.1 shows a rotational system whose angular position is θ and angular velocity is $\omega = \dot{\theta}$. The load has inertia J and rotates on bearings which are modeled as an ideal rotational damper with coefficient b. A torque T is applied to the load. The initial velocity is $\omega(0) = \omega_0$.

(a) Write the differential equation for the system in terms of T and ω.

(b) Find the Laplace transform for the differential equation.

(c) If the input torque is a unit step $T(t) = T_0 u_s(t)$ (with constant magnitude T_0), what is $\Omega(s)$?

(d) How would you solve for the angular velocity as a function of time $\omega(t)$?

(e) What is the transfer function $\frac{\Omega(s)}{T(s)}$?

19.9 Recitation 9 Problem

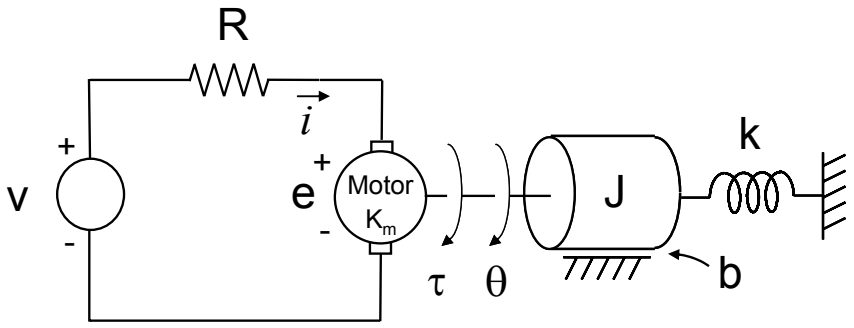


Figure 19.9.1: Motor with Rotational Load

Figure 19.9.1 shows a motor connected to a rotational load. The motor's resistance is given by R and the motor's inductance is negligible. The motor is driven by an input voltage v which generates a current i in the circuit. The combined motor and load inertia is given by J. The angular position of the load is θ. Flexibility in the load is modeled as rotational spring with constant k. Rotational damping from the motor and the load is lumped into the element b. The system is initially at rest.

The motor applies a torque to the load: $\tau = K_m i$

The motor's back emf is: $e = K_m \dot{\theta}$

(a) Find the ordinary differential equation (ODE) for the motor circuit in terms of v, τ and θ. Laplace transform this equation and put it in the form:

$$V(s) = H_1(s)T(s) + H_2(s)\Theta(s)$$

(b) Find the ODE for the load in terms of τ and θ. Find the transfer function:

$$H_3(s) = \frac{\Theta(s)}{T(s)}$$

(c) Combine the results of parts (a) and (b) to find the system transfer function from v to θ: $H_\theta(s) = \frac{\Theta(s)}{V(s)}$. How would you find the transfer function from v to τ? Determine $H_T(s) = \frac{T(s)}{V(s)}$

(d) If $v(t)$ is a unit step voltage with magnitude V_0, use the final value theorem to solve for the steady state value $\theta(\infty)$. What is the steady state angular velocity $\dot{\theta}(\infty)$? What is the steady state torque $\tau(\infty)$?

(e) Find the power dissipated by the rotational damper at steady state. Is this surprising? How would you determine when the power reached its maximum? Where is the power coming from?

(f) Repeat part (d) for a ramp input. What is the power dissipated by the rotational damper at steady state for this case?

(g) How would you redesign this system to get better behavior?

(h) If $H(s)$ can be written in the form

$$H_\theta(s) = \frac{0.7}{s^2 + 62 + 25}$$

sketch the poles and zeros of $H(s)$ in the s-plane. Based on your pole-zero sketch, how would you expect the system to respond to a step input?

holy crap you are a big pain in the ass what the hell is happening to the red sox they are totally sucking some major ass and i hate it.

19.10 Recitation 10 Problem

Construct Bode plots for the frequency response function

$$G(j\omega) = \frac{2}{j\omega(1 + j\omega/2)(1 + j\omega/5)}$$

19.11 Recitation 10A Problem

Consider the following feedback control system.

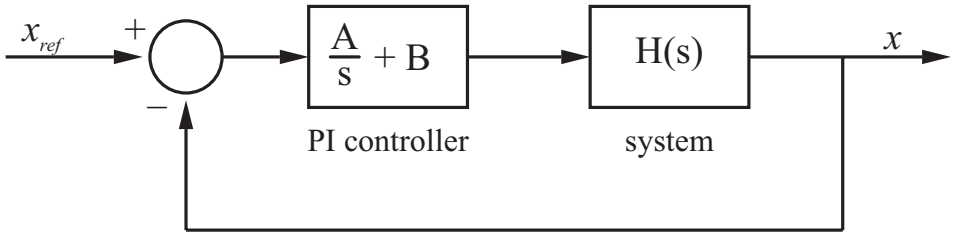


Figure 19.11.1: Feedback Control System Block Diagram

(a) Let $H(s) = \frac{1}{s+1}$. Choose A and B such that the closed-loop system has poles at $s = -10 \pm 100j$.

(b) Let $H(s) = \frac{1}{(10^{-4}s+1)(s+1)}$. Using the same values of A and B from part (a), compute the closed loop transfer function and find its poles.

(c) Sketch the bode plots and step response for the systems from part (a) and (b).

19.12 Recitation 11 Problem

Construct the asymptotic Bode plots for the frequency response function

$$G(s) = \frac{1 + s/2 + (s/2)^2}{s(1 + s/0.5)(1 + s/4)}$$

19.13 Recitation 12 Problem

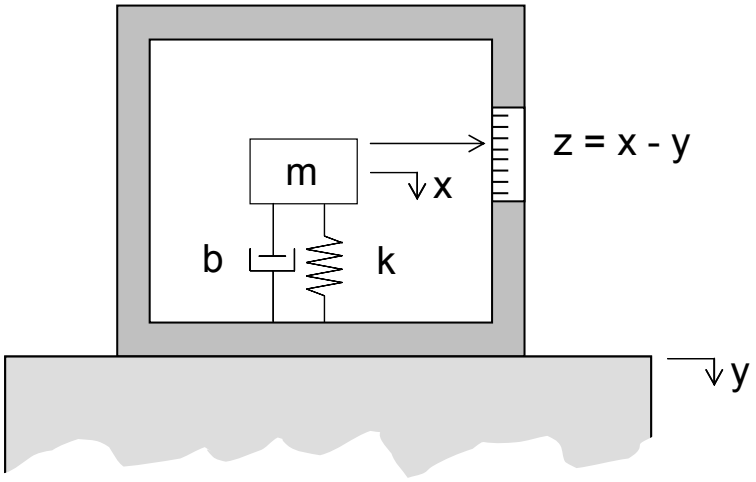


Figure 19.13.1: Seismograph Diagram

The figure shows a seismograph which measures ground displacement during earthquakes. The inertial position of the mass m is given by x. The mass is suspended by a damper b and spring k. The inertial displacement of the case (and ground) is given by the input y which is approximated as a sine wave $y(t) = A \sin \omega t$. The seismograph measure the relative displacement between x and y, so the output of the system is $z = x - y$.

(a) Find the transfer function between $Z(s)$ and $Y(s)$.

(b) Sketch the magnitude and phase Bode plots for the system.

(c) Show that the seismograph measures the displacement of the case y accurately if $\omega >> \omega_n$.

19.14 Recitation 13 Problem

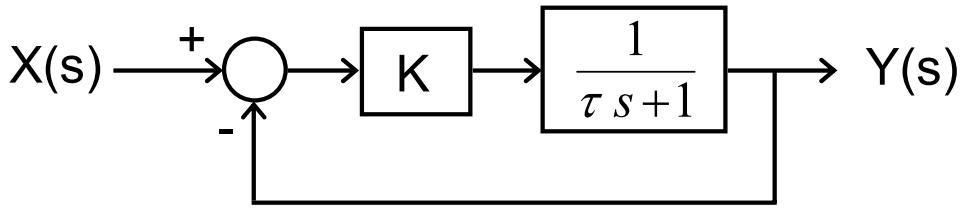


Figure 19.14.1: System Block Diagram

Sketch the open and closed-loop Bode plots for the system given above.

20 Recitation Quizzes

20.1 Recitation 1 Quiz

The snow from last week's storm on the flat roof of Building 3 has melted all at once, and now physical plant wants to know how long it will take for all that water to drain. We can model this as a first order system and calculate the time constant.

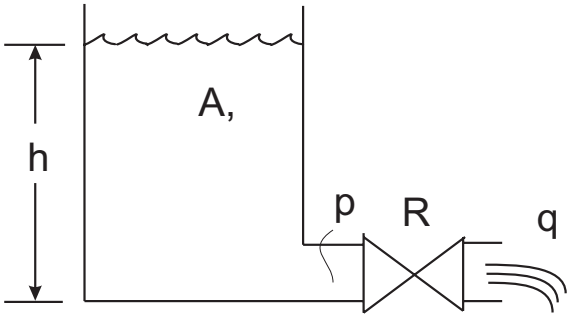


Figure 20.1.1: Fluid Model

Consider the system shown in Figure 20.1.1, which consists of a fluid tank filled to a height h. Water is draining out of the tank at a rate q through an opening with resistance R. The volume in the tank, V, is equal to the area A times the height $V = Ah$. The change in volume is thus:

$$\frac{dV}{dt} = A\frac{dh}{dt}$$

From conservation of mass, we know that the change in volume of liquid in the tank is equal to the outflow (note the sign is negative because the water is flowing out):

$$q = -\frac{dV}{dt}$$

The pressure p at the bottom of the tank is $p = \rho g h$ where ρ is the (incompressible) fluid mass density and g is acceleration due to gravity. The linear fluid resistance can be written as $p = qR$.

(a) Formulate the first order differential equation describing the height of the water as a function of time.

(b) What is the time constant for the system?

(c) If the initial height of the water is h_0, what will the height of the water be after one time constant? After 2.3 time constants?

20.2 Recitation 2 Quiz

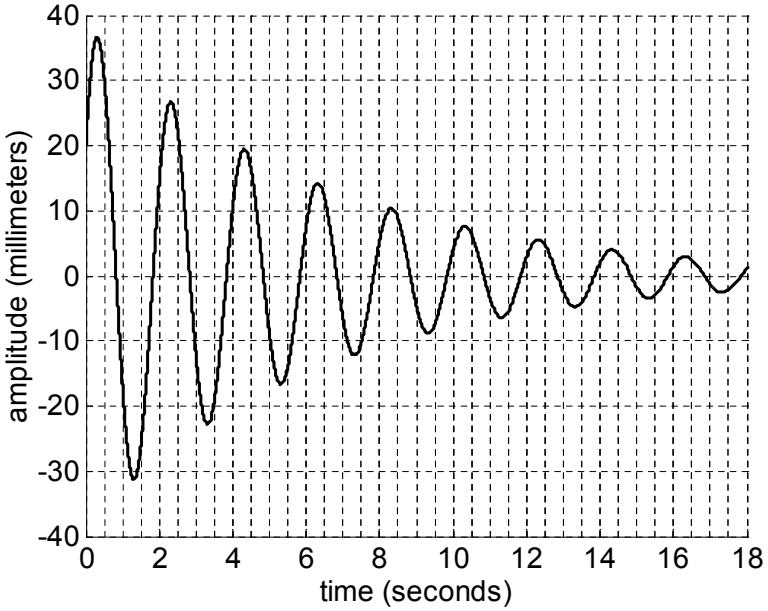


Figure 20.2.1: Second Order Response

Consider the plot shown above of the homogeneous response of a second-order system:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Estimate the undamped natural frequency ω_n, the damping ratio ζ, and the initial position $x(0)$. Is this system undamped, underdamped, critically-damped, or overdamped?

20.3 Recitation 3 Quiz

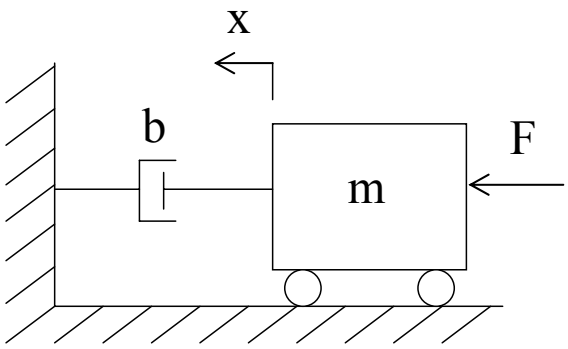


Figure 20.3.1: Cart System

The system shown in the figure has the first-order differential equation:

$$m\dot{v} + bv = F$$

where m is the mass, b is the damping coefficient, x is the displacement of the mass, $v = \dot{x}$ is the velocity of the mass, and the force F is given by a step function:

$$F(t) = F_0 u_s(t)$$

with magnitude F_0 constant. Assume zero initial position ($x_0 = 0$) and zero initial velocity ($v_0 = 0$).

(a) Sketch the velocity of the mass as a function of time.

(b) What is the velocity of the mass at steady state?

20.4 Recitation 4 Quiz

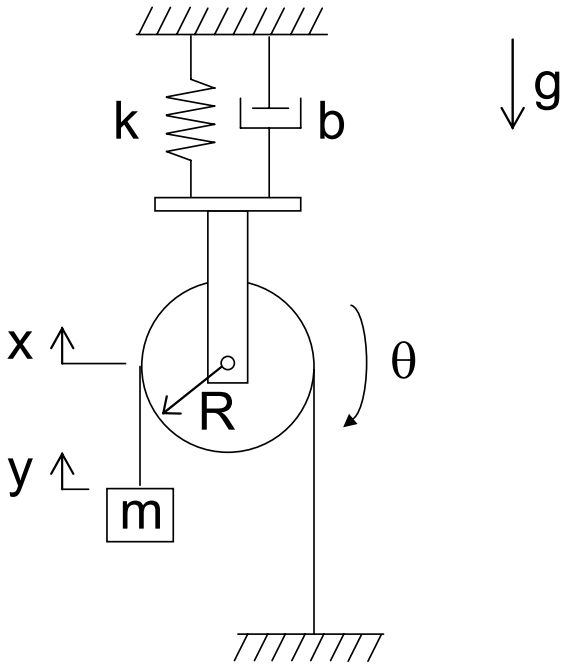


Figure 20.4.1: Pulley System

The mechanical system shown in Figure 20.4.1 consists of a pulley of radius R suspended from the ceiling by a spring and a damper. A cable runs over the pulley and is attached to ground on the right and a hanging mass m on the left. The vertical position of the center of the pulley is x, the vertical position of the mass is y, and the rotational angle of the pulley is θ. Note also that gravity acts in a downward direction.

Assume that the mass of the pulley, cable and support is negligible, the pulley rotates without friction, and that there is no slip between the pulley and the cable.

(a) Draw a free body diagram for the pulley.

(b) What is the relationship between x and y?

20.5 Recitation 5 Quiz

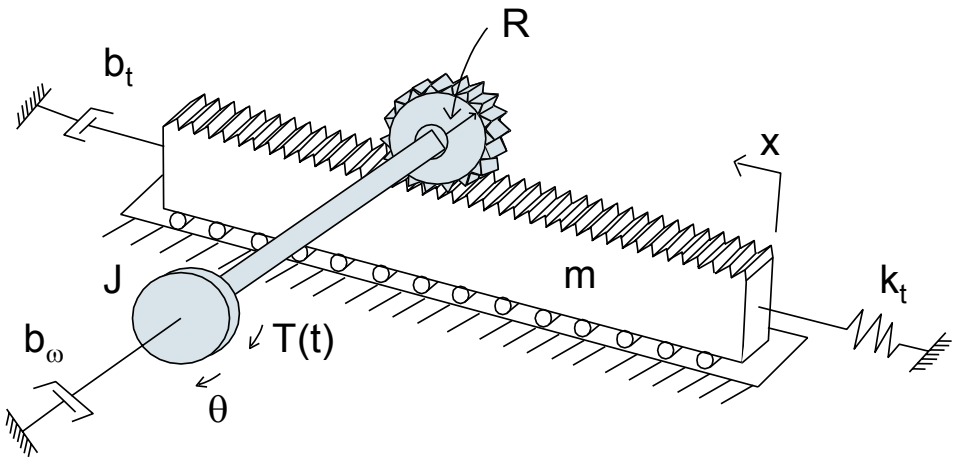


Figure 20.5.1: Rack and Pinion System

A rack and pinion converts rotational motion to translational motion and vice-versa. The tracking mechanism in the CD player passed around in class contained a rack and pinion.
Figure 20.5.1 shows a lumped model where the angular position of the pinion gear is given by θ. The combined shaft and gear inertia is given by J, the rotational damping on the shaft is given by b_ω and the applied torque $T(t)$ is a function of time. The rack position is x, the mass of the rack is m, the linear damping acting on the rack is given by b_t, and the linear spring constant is k_t. Assume that there is adequate room for the rack to roll back and forth.

(a) Draw free body diagrams for the rack and for the pinion.

(b) What is the relationship between x and θ?

(c) Find the second-order differential equation for the system as a function of θ (Use part (b) to eliminate x).

20.6 Recitation 6 Quiz

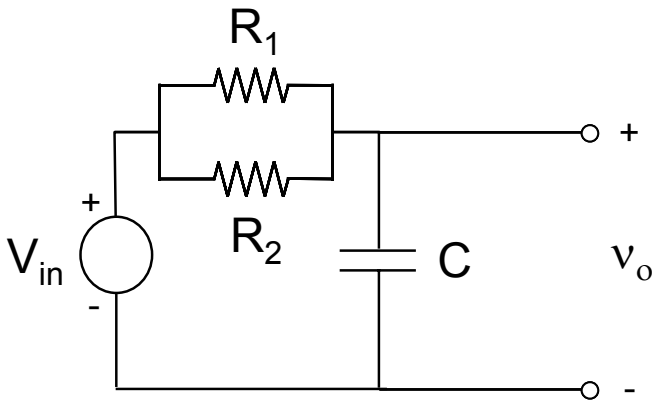


Figure 20.6.1: RC Circuit

The circuit in the figure consists of a voltage source connected to two resistors in parallel and a capacitor. Find the differential equation for this circuit in terms of v_o.

20.7 Recitation 7 Quiz

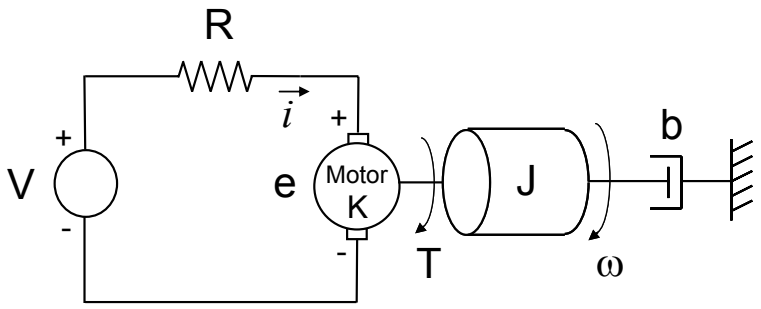


Figure 20.7.1: RC Circuit

Figure 20.7.1 shows an ideal motor connected to a load rotating at angular velocity ω. The motor's resistance is lumped into the element R. The motor is driven by an input voltage V. The motor applies a torque to the load $T = Ki$. The motor's back emf is $e = K\omega$. The combined motor and load inertia is given by J and the load is also subject to a rotational damper b.

Without working through the mathematics, select the differential equation which best represents this system:

(a)

$$\frac{K}{R}V = J\dot{\omega} - \left(b + \frac{K^2}{R}\right)\omega$$

(b)

$$\frac{K}{R}V = J\dot{\omega} + \left(b + \frac{K^2}{R}\right)\omega$$

(c)

$$\frac{K}{R}V = J\ddot{\omega} + \left(b + \frac{K^2}{R}\right)\dot{\omega}$$

Justify your answer (think of transient and steady state response, the physics of the system, etc.)

20.8 Recitation 8 Quiz

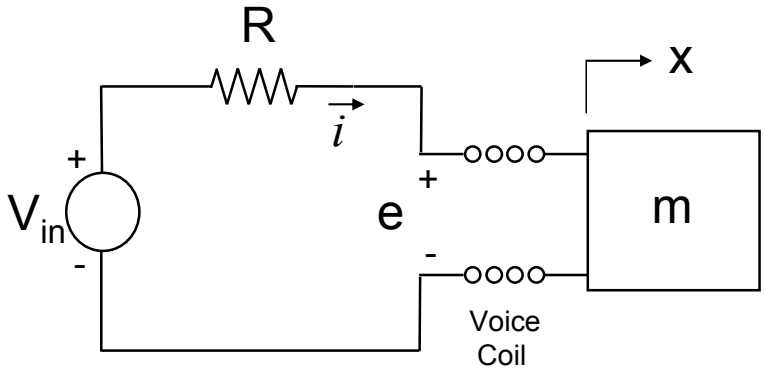


Figure 20.8.1: Voice Coil Circuit

The voice coil shown above is driven with an input voltage V_{in}. We are interested in the motion of the load m, which moves with velocity $v = \dot{x}$.

Consider the voice coil to be ideal, with lumped resistance R. Neglect the voice coil inductance, voice coil inertia, and any translational damping. The voice coil equations are:

$$F = Ki$$
$$e = K_e v$$

where the direction of the applied force F on the mass is the same as x. (Please keep the notation for the back emf constant K_e different than the voice coil constant K.)

Find the differential equation for the system in terms of V_{in} and v (eliminate e and i).

20.9 Recitation 9 Quiz

$$H(s) = \frac{2(s+2)}{s^2+7s+12}$$

The transfer function $H(s)$ given above is the ratio of two polynomials. Use partial fraction expansion to rewrite the transfer function as the sum of the ratio of two smaller polynomials.

20.10 Recitation 10 Quiz

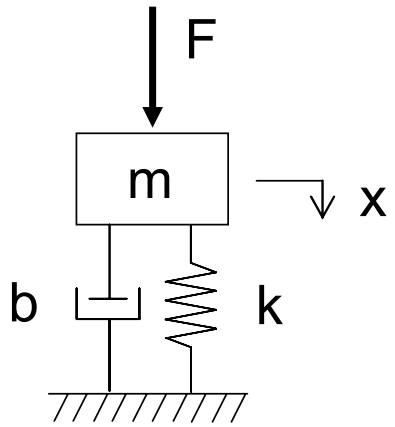


Figure 20.10.1: Mass Spring Damper System

The input force F to the mass spring damper system shown above is $F(t) = \sin \omega t$. The output position steady state response is $x(t) = M \sin \omega t + \phi$. Let $m = 1$, $b = 1$, and $k = 1$.

Find M and ϕ for $\omega = 1$.

20.11 Recitation 10A Quiz

Consider the following feedback control system.

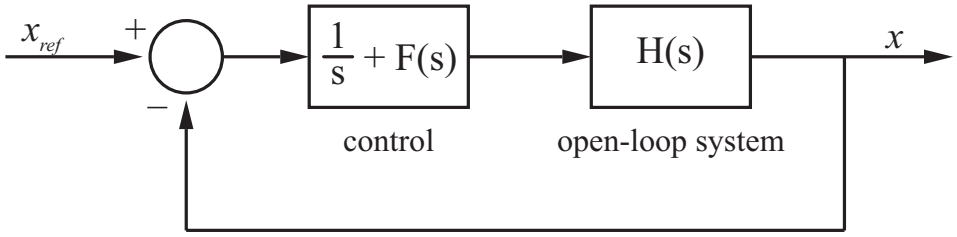


Figure 20.11.1: Feedback Control System Block Diagram

Show that if the open-loop transfer function $H(s)$ has no zeros at the origin, then the DC gain for the closed-loop system is 1.

20.12 Recitation 11 Quiz

Sketch the magnitude and phase Bode plots for the transfer function $G(s)$

$$G(s) = \frac{10}{0.5s + 1}$$

20.13 Recitation 12

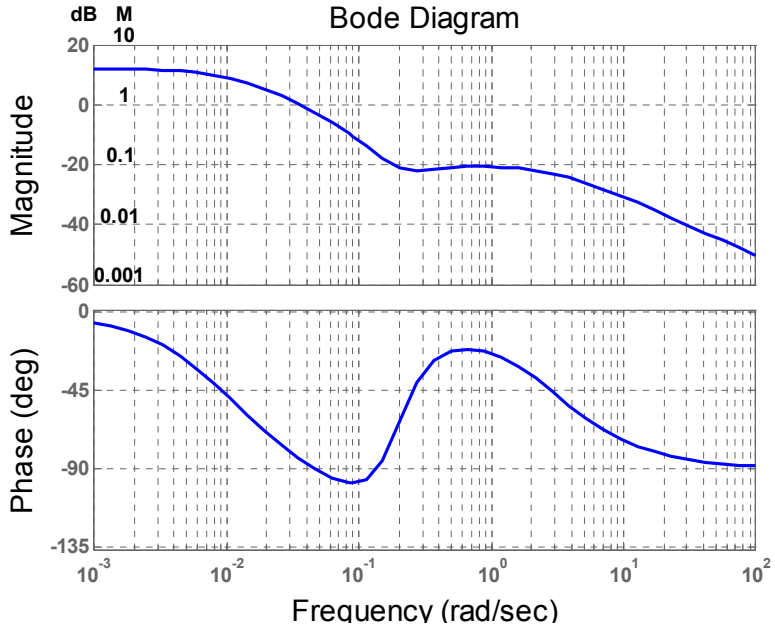


Figure 20.13.1: Bode Diagram for part (a).

(a) The bode plots above were generated from the transfer function $G(s)$. Find the magnitude and phase for $\omega = 1$.

(b) The figure below shows the block diagram for this system. If the input to this linear time-invariant system is $f(t) = \sin \omega t$, what is the output $x(t)$ for $\omega = 1$?

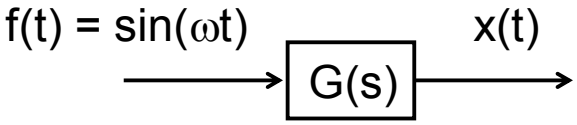


Figure 20.13.2: Block Diagram

20.14 Recitation 12

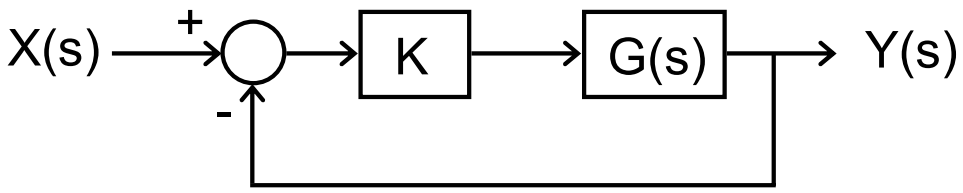


Figure 20.14.1: Block Diagram

Derive the transfer function for the given block diagram.

Chapter

Solutions

1 1st Order Systems

1.1 First Order Time Constant

The solution for this problem is not available.

1.2 Rise and Settling Times

We are given the first-order transfer function

$$H(s) = \frac{1}{\tau s + 1} \tag{1}$$

The response to a unit step with zero initial conditions will be $y(t) = 1 - e^{-t/\tau}$. To determine the amount of time it take y to settle to within Δ of its final value, we want to find the time t_s such that $y(t_s) = 1 - \Delta$. Thus, we obtain

$$\Delta = e^{-t_s/\tau} \tag{2}$$

$$t_s = -\tau \ln \Delta \tag{3}$$

The 10-90% rise time t_r may be thought of as the difference between the 90% settling time ($\Delta = 0.1$) and the 10% settling time ($\Delta = 0.9$).

$$t_r = t_{\Delta=0.1} - t_{\Delta=0.9} \tag{4}$$

Therefore, we find $t_r = 2.2\tau$. For $\Delta = 10^{-6}$, $t_s = 13.82\tau$.

1.3 First Order System Response

PZ: $\dot{y} + 5y = 10u$

a) HOMOGENEOUS SOLUTION:

$\dot{y} + 5y = 0$

$y = A \cdot e^{s \cdot t}$

$\dot{y} = A \cdot s \cdot e^{s \cdot t}$ $\Rightarrow$ $A s e^{st} + 5 A e^{st} = 0$

$s = -5$

$y = A \cdot e^{-5 \cdot t} = A \cdot e^{-t/\tau}$

$\Rightarrow \tau = 1/5 \cdot sec$

$\boxed{\tau = 0.2 \text{ sec}}$

b) $u = 10 \Rightarrow \dot{y} + 5 \cdot y = 100$

STEADY STATE $\left(\frac{dy}{dt} \rightarrow 0\right)$ $\quad y = 0 \Rightarrow \cancel{\dot{y}}^{0} + 5 \cdot y = 100$

$\boxed{y = 20}$

c) $y(t=0) = y_0$

$y(t=0) = A \cdot e^{-5(0)} = A = y_0$

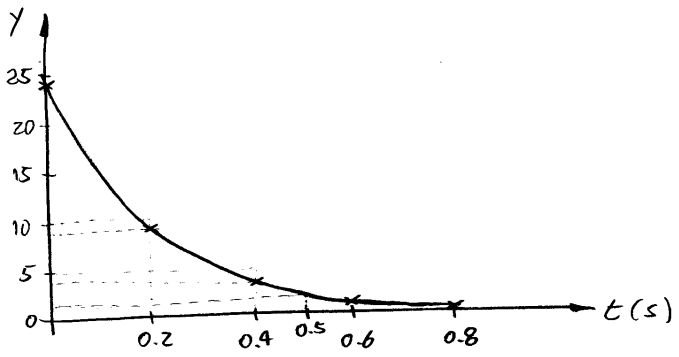
$\Rightarrow y(t) = y_0 \cdot e^{-5 \cdot t}$

$y(t = 0.5 \text{ sec}) = y_0 \cdot e^{-5(0.5)} = y_0 \cdot e^{-2.5} = 2$

$\Rightarrow \boxed{y_0 = 24.36}$

Figure 1.3.1:

d)



t(s)	y/y_0	y
0	1	24.36
0.2	0.37	9.02
0.4	0.13	3.17
0.6	0.05	1.22
0.8	0.02	0.49

e) THE RESPONSE REACHES 2% OF ITS INITIAL VALUE AT $\approx 4\tau$.

PROOF: $y = y_0 \cdot e^{-t/\tau}$ $\quad y/y_0 = 0.02 = e^{-t/\tau}$

$$\ln(0.02) = -t/\tau$$

$$-3.9 = -t/\tau$$

$$\Rightarrow t = 3.9\tau$$

$$\boxed{t \approx 4\tau} = 0.8 \text{ sec.}$$

f) $y = 0.02$

$$y = 24.36\, e^{-5 \cdot t} = 0.02$$

$$e^{-5 \cdot t} = 821 \times 10^{-6}$$

$$-5 \cdot t = \ln|821 \times 10^{-6}|$$

$$-5 \cdot t = -7.105$$

$$\boxed{t = 1.42 \text{ sec.}}$$

Figure 1.3.2:

2 2nd Order Systems

2.1 Second-Order System Response

P3: $20\ddot{y} + 160\dot{y} + 8000y = 0$

$y(t) = A.e^{st}$

$\dot{y}(t) = A.s.e^{st}$

$\ddot{y}(t) = A.s^2.e^{st}$

$20.As^2e^{st} + 160.As.e^{st} + 8000.Ae^{st} = 0$

$20s^2 + 160s + 8000 = 0$

$s_{1,2} = \dfrac{-160 \pm \sqrt{(160)^2 - 4(20)(8000)}}{2(20)}$

$s_{1,2} = -4 \pm j19.6$

$s_{1,2} = -\sigma \pm j\omega d \Rightarrow -\sigma = -4$ AND $\omega d = 19.6$

$y(t) = C.e^{-\sigma.t}.\cos(\omega d.t + \phi)$

$y(t) = C.e^{-4.t}.\cos(19.6t + \phi)$

$\dot{y}(t) = -4C.e^{-4t}\cos(19.6t + \phi) - 19.6Ce^{-4t}\sin(19.6t + \phi)$

$y(t=0) = C.\cos(\phi)$

$\dot{y}(t=0) = -4C.\cos(\phi) - 19.6C.\sin(\phi)$

a) $y(t=0) = 0 \Rightarrow C.\cos(\phi) = 0 \Rightarrow \phi = \pi/2$

$\dot{y}(t=0) = 10 \Rightarrow -4C\cos(\pi/2)^0 - 19.C\sin(\pi/2) = 10$

$C = -10/19.6 = -0.510$

$y(t) = -0.510.e^{-4t}\cos(19.6t + \pi/2)$

Figure 2.1.1:

b) $y(t=0) = 10 \Rightarrow C \cdot \cos(\phi) = 10 \Rightarrow C = 10/\cos(\phi)$

$\dot{y}(t=0) = 0 \Rightarrow -4C\cos(\phi) - 19.6C\sin(\phi) = 0$

$-40 - 19.6C\sin(\phi) = 0$

$-40 - 196\tan(\phi) = 0 \Rightarrow \tan(\phi) = \frac{-40}{196}$

$\phi = -0.2 \text{ rad}$

$C \cdot \cos(-0.2 \text{ rad}) = 10 \Rightarrow C = 10.2$

$\Rightarrow y(t) = 10.2e^{-4t}\cos(19.6t - 0.2)$

c) $y(t=0) = 1 \Rightarrow C \cdot \cos(\phi) = 1$

$\dot{y}(t=0) = 10 \Rightarrow -4C\cos(\phi) - 19.6C\sin(\phi) = 10$

$-4(1) - 19.6\left(\frac{1}{\cos(\phi)}\right)\sin(\phi) = 10$

$\tan(\phi) = \frac{14}{-19.6} = -0.714$

$\phi = -0.620 \text{ rad}$

$C \cdot \cos(-0.620 \text{ rad}) = 1 \Rightarrow C = 1.229$

$\Rightarrow y(t) = 1.229e^{-4t}\cos(19.6t - 0.620)$

d) SEE ATTACHED M-FILE AND PLOTS

e) THE TIME FOR EACH CONDITION OF THE SAME SYSTEM TO REACH STEADY-STATE IS EQUAL AS SHOWN IN THE GRAPHS. THIS CAN BE SEEN ON THE GRAPH BY THE DECAYING OF THE OSCILLATIONS.

Figure 2.1.2:

```
t=[0:0.005:5];
u=ones(1,1001);

XMIN=0;
XMAX=1.5;
YMIN=-10;
YMAX=10;


figure(1);
C=-0.510;
phi=pi/2*u;
y=C*exp(-4*t).*cos(19.6*t+phi);
plot(t,y);
AXIS([XMIN XMAX YMIN YMAX]);
title('y vs. t (Part a)');
xlabel('t');
ylabel('y');


figure(2);
C=10.2;
phi=-0.2*u;
y=C*exp(-4*t).*cos(19.6*t+phi);
plot(t,y);
AXIS([XMIN XMAX YMIN YMAX]);
title('y vs. t (Part b)');
xlabel('t');
ylabel('y');


figure(3);
C=1.229;
phi=-0.62*u;
y=C*exp(-4*t).*cos(19.6*t+phi);
plot(t,y);
AXIS([XMIN XMAX YMIN YMAX]);
title('y vs. t (Part c)');
xlabel('t');
ylabel('y');
```

Figure 2.1.3:

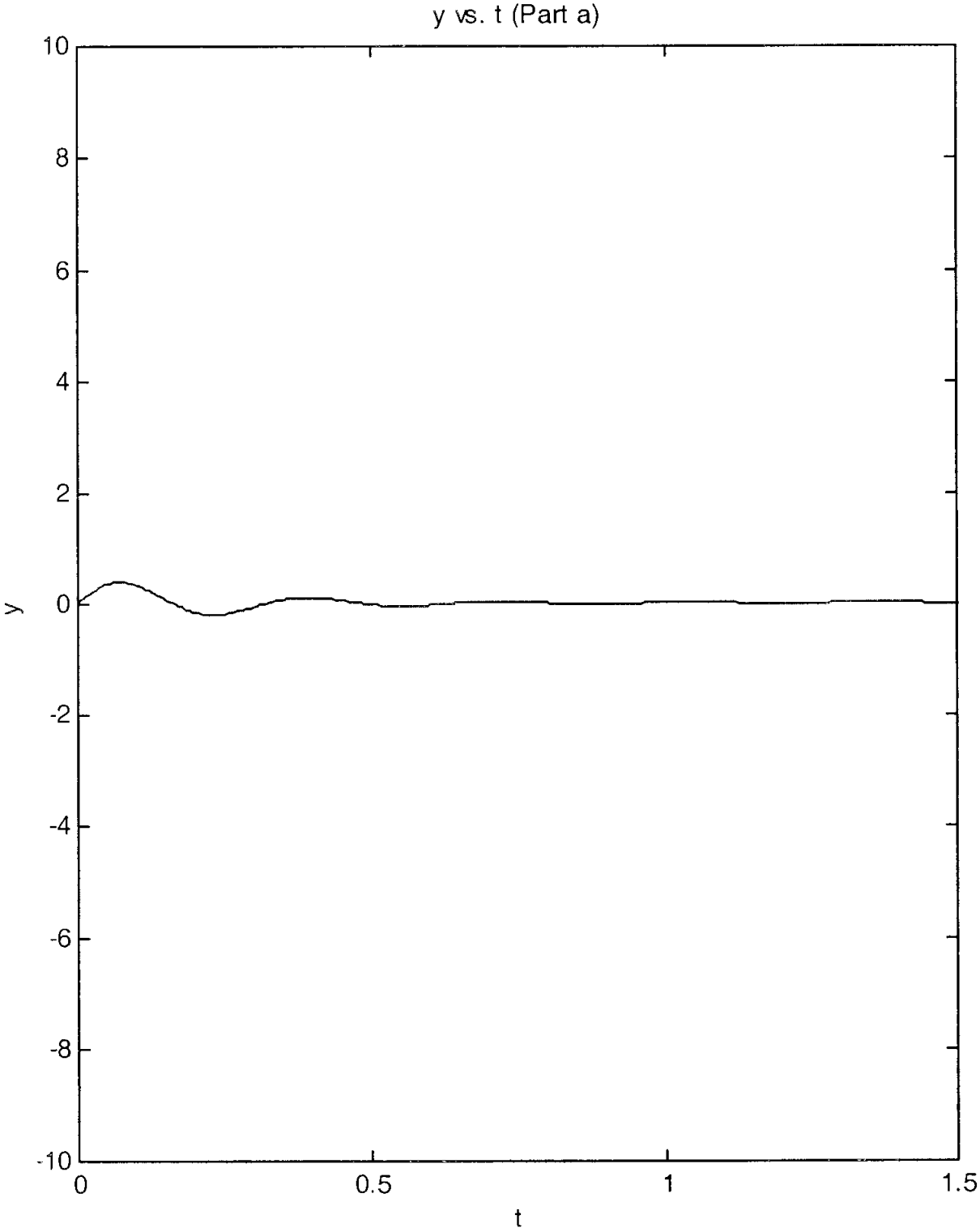


Figure 2.1.4:

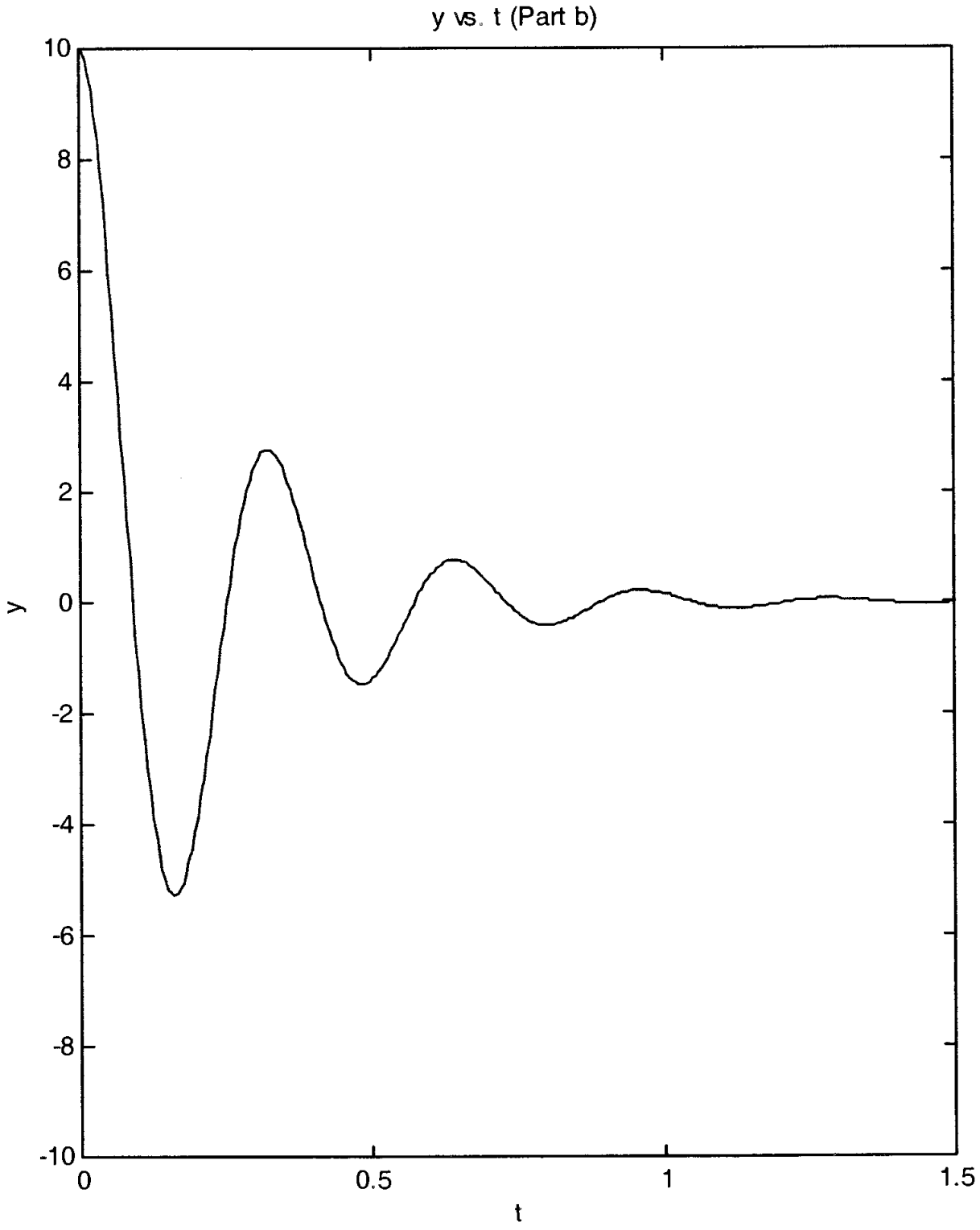


Figure 2.1.5:

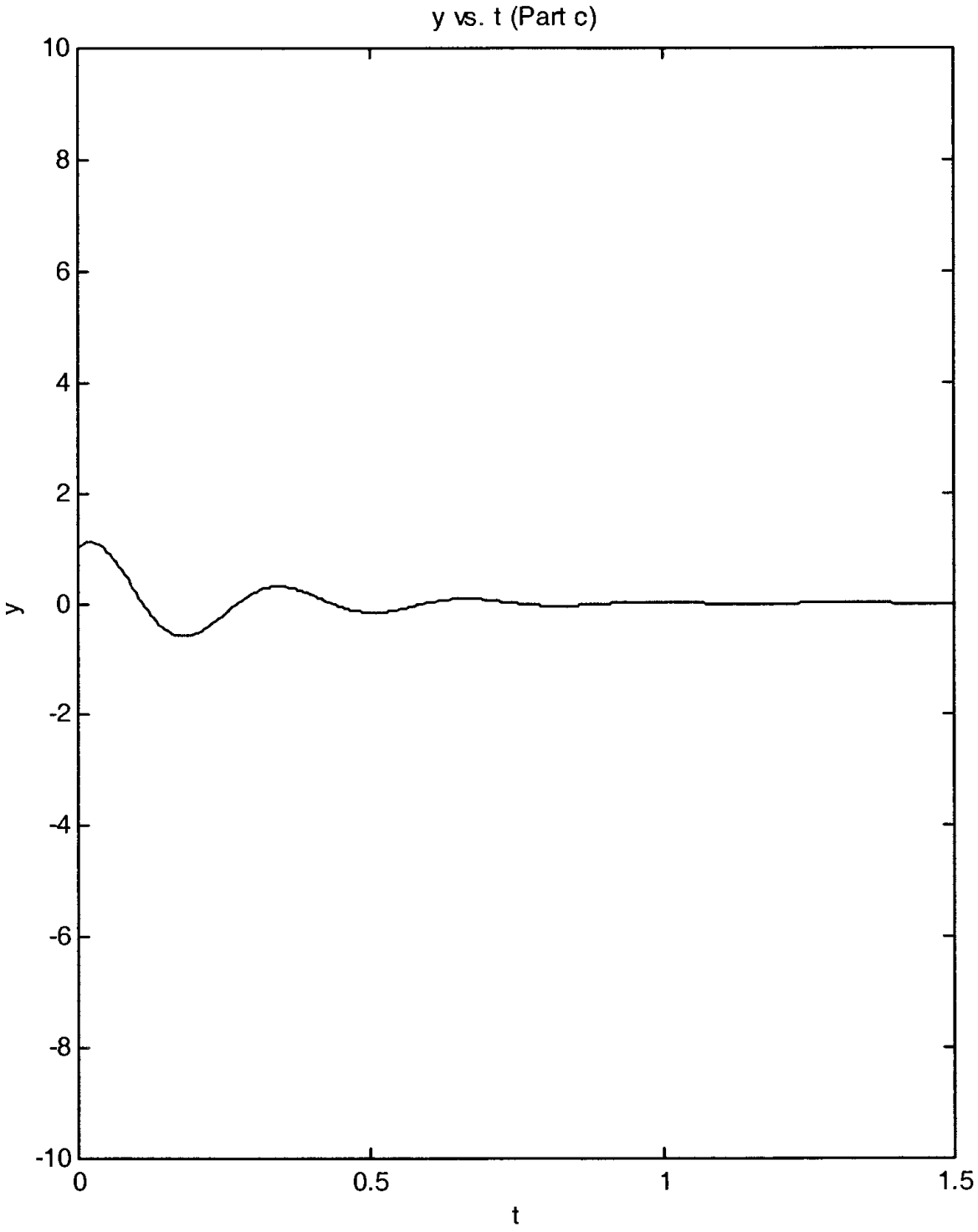


Figure 2.1.6:

2.2 Second Order Responses

(a) From the figures in the problem set, it can be estimated that the period $T = 0.1$sec and the time constant $\tau = 0.35$ sec. $\omega_d = 2\pi/T = 62.8$. Since the system is slightly damped, the estimated damping ratio

$$\zeta = \frac{1}{\tau\omega_n} \approx \frac{1}{\tau\omega_d} = \frac{1}{0.35 \times 63} = 0.045 \tag{1}$$

Further the natural frequency $\omega_n = \omega_d/\sqrt{1-\zeta^2} = 62.8$

(b) A mechanical mass-spring-damper system will give this response. $K = M\omega_n^2$, $C = 2\zeta\omega_n M$. For example, M=1kg, K= 62.8^2 = 3944 N/m, C= $2 * 0.045 * 62.8 * 1$ = 5.6520 N-sec/m. The initial position $x(0) = 0$ mm, $v(0) \approx A\omega_n$ = 300mm∗62.8rad/sec= 18850mm/sec = 18.85m/sec, where A is the initial amplitude of response.

2.3 Second Order Derivation

The solution for this problem is not available.

2.4 Second Order Derivation Continued

The solution for this problem is not available.

3 Higher Order Systems

3.1 Structure of Higher Order System Solutions

The solution for this problem is not available.

3.2 Structure of Higher Order System Solutions

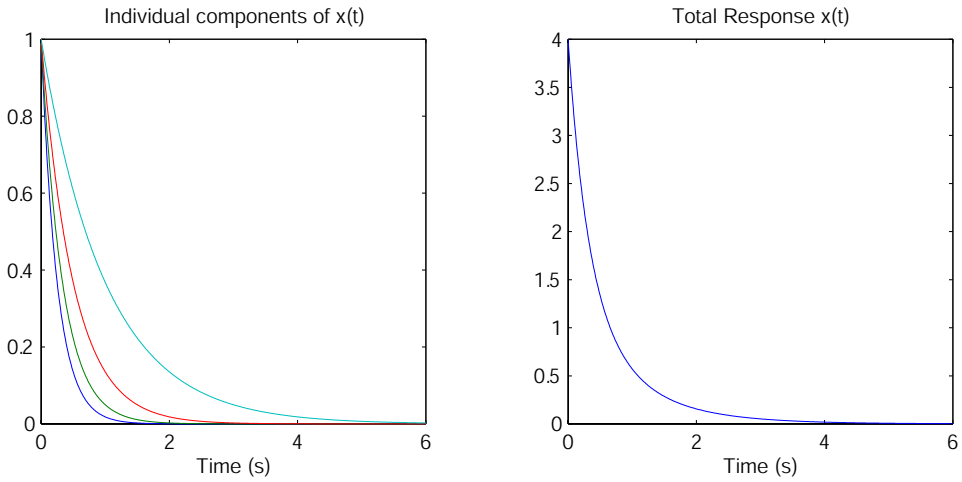


Figure 3.2.1: Sketch Solution for Case I

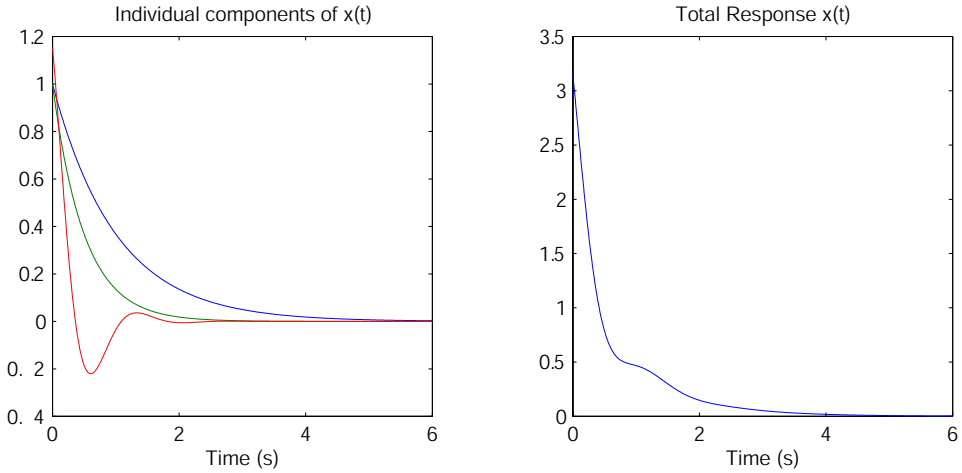


Figure 3.2.2: Sketch Solution for Case II

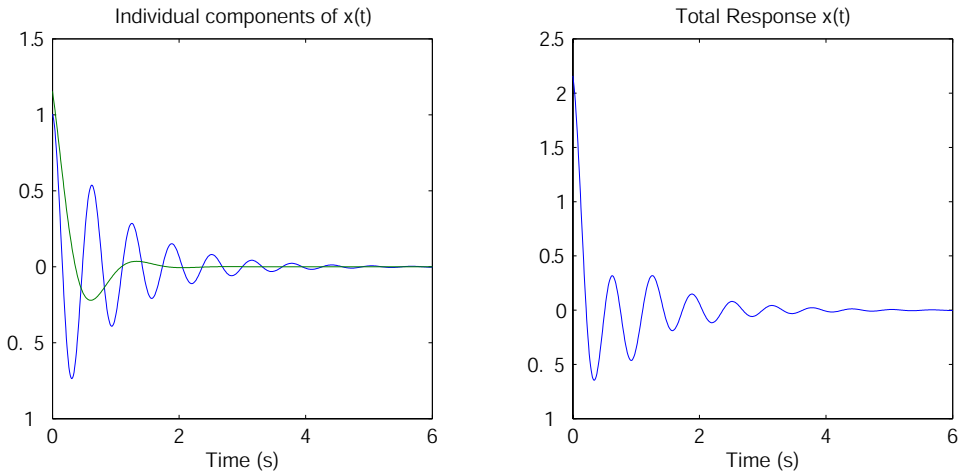


Figure 3.2.3: Sketch Solution for Case III

(a) Case I
$s_1 = -4$, $s_2 = -3$, $s_3 = -2$, and $s_4 = -1$

$$x(t) = C_1 e^{-4t} + C_2 e^{-3t} + C_3 e^{-2t} + C_4 e^{-t}$$

Case II
$s_1 = -1$, $s_2 = -2$, $s_{3,4} = -2.5 \pm 4.33j \rightarrow \omega_n = 5$,

$$x(t) = C_1 e^{-t} + C_2 e^{-2t} + C_3 \frac{1}{\sqrt{1-0.5^2}} e^{-2.5t} \cos(4.33t + \phi_3)$$

Case III
$s_{1,2} = -1 \pm 9.95j \rightarrow \omega_n = 10$, $s_{3,4} = -2.5 \pm 4.33j \rightarrow \omega_n = 5$

$$x(t) = C_1 \frac{1}{\sqrt{1-0.1^2}} e^{-t} \cos(9.95t + \phi_1) + C_2 \frac{1}{\sqrt{1-0.5^2}} e^{-2.5t} \cos(4.33t + \phi_2)$$

(b) Case A - This is a summation of four $1^s t$ order responses. Thus the system does not have an oscillation. Because all the terms are first order, the system will not overshoot the final value.

Case B - This is a summation of 2 first order response and a decaying sinusoid. Since the decay constant of the sinusoid is faster than the slower first order terms, the sinusoid will appear in the initial response but decay to a simple first order response. Depending on the magnitude of the sinusoid it is possible for the system to overshoot.

Case C - This is the summation of two decaying sinusoids. Since the time constant of the decay envelope of the 4.33 r/s sinusoid is faster than the 9.95 r/s sinusoid, the response will initially be a combination of the two sinusoids decaying to just the 9.95 r/s sinusoid. This system can overshoot the final value.

(c) To sketch the responses for each of these systems, we need to make a couple of assumptions. One assumption I made was that all of the coefficients were equal to 1 (Reasonable for this

case). The 2nd assumption I made was that $\phi_n = 0$. Figures 3.2.1-3.2.3 show the responses for the three cases. The left plot shows the individual components, while the right plot shows the total response $(x_1 + x_2 ... + x_n)$. Normally, we are interested in the system response to a set of initial conditions or an input force, in which case the $\phi_n = 0$ and $c_1 = c_2 ....$ For reference, I have included the system response for the initial conditions $x(0) = 1$, $\dot{x}(0) = \ddot{x}(0) = \dddot{x}\ (0)$ (Note: the number of initial conditions for the system is equal to the order of the system thus a first order system needs one initial condition, a 2nd order system needs 2 IC, ect.). Figures 3.2.4-3.2.6 show the results.

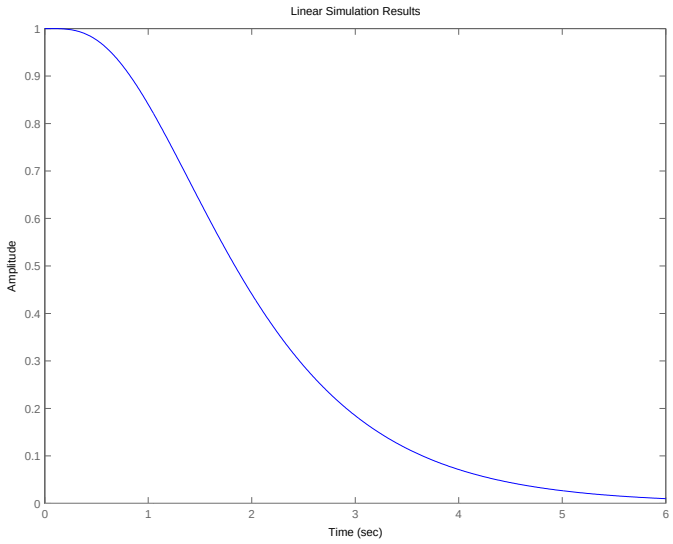


Figure 3.2.4: Solution for Case I

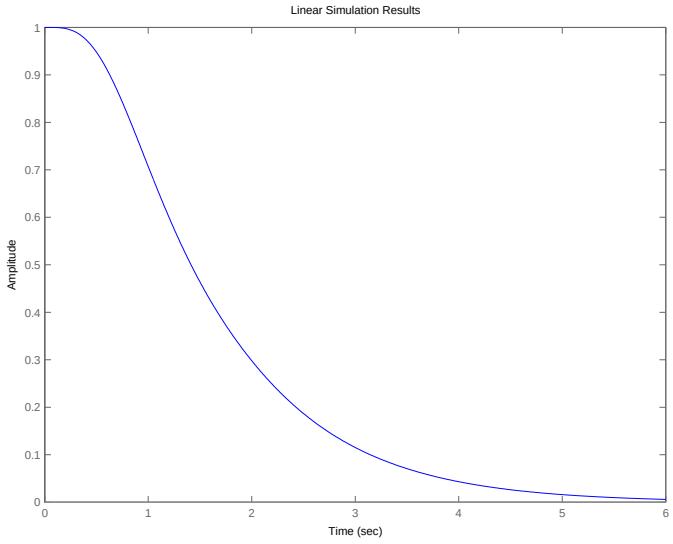


Figure 3.2.5: Solution for Case II

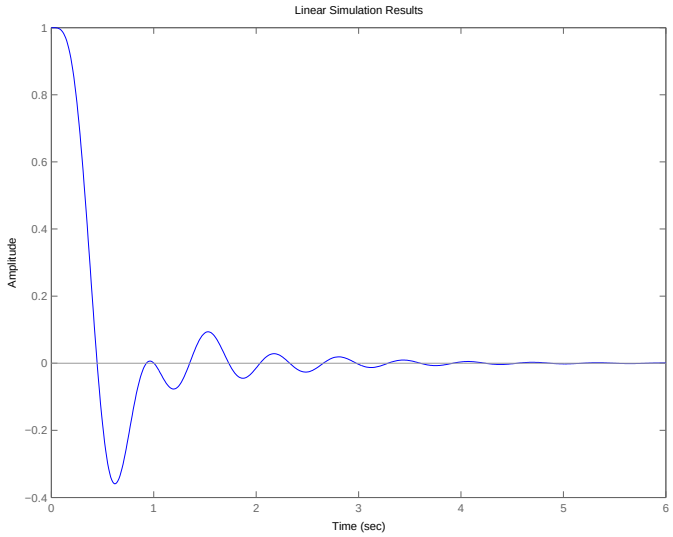


Figure 3.2.6: Solution for Case III

4 Mechanical Systems

4.1 Balloon

Assume hovering balloon has mass m and is subject to a buoyancy force B which tends to lift the balloon. Consider only vertical motion of the balloon. Let the length of the string between the balloon and the floor be y. Assume the string has mass/unit length ρ so that a string mass of ρy must move whenever the balloon moves. Both the balloon mass and the string mass experience gravitational attraction pulling them down. Assume that there is no tension in the string at the point where the string meets the floor. Finally assume there are damping mechanisms (air drag and losses associated with coiling and uncoiling the string on the floor) which can be approximately modeled by a linear damping force $f_d = bdy/dt$ opposing the velocity dy/dyt. These forces are displayed in the free body diagram in Fig. 4.1.1.

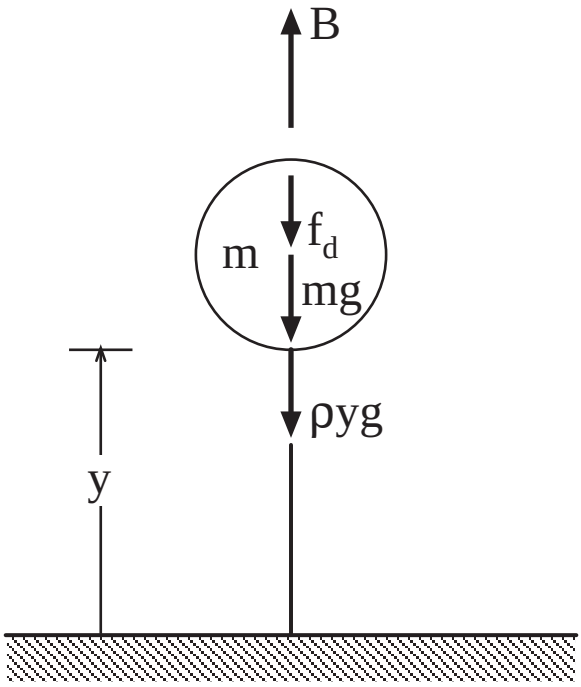


Figure 4.1.1: Free-Body Diagram of Hovering Balloon

To construct a model we consider (i) geometric compatibility requirements, (ii) constitutive equations, and (iii) force-balance requirements.

(i) The displacement of the balloon is $y(t)$. The velocity and acceleration are

$$v = \frac{dy}{dt} \qquad a = \frac{dv}{dt} = \frac{d^2y}{dt^2}$$

(ii) The constitutive equations are

$$f_m = \frac{d}{dt}[(m + \rho y)v] \qquad f_d = bv \qquad f_{gravity} = (m + \rho y)g$$

(iii) The force-balance requirement is

$$f_m = B - f_{gravity} - f_d$$

An equation of motion for the model is obtained by substituting the constitutive equations into the force-balance requirement to get

$$\frac{d}{dt}[(m+\rho y)v] + b\frac{dy}{dt} + (m+\rho y)g = B \tag{1}$$

(a) In a steady state y has a fixed value y_o and all time derivatives. When we set the time derivatives equal to zero in (1) we find that the equation is satisfied if

$$y = y_o = \frac{B - mg}{\rho g}$$

The variable mass term in (1) makes the equation nonlinear. It can be approximated by a linear equation if the varying mass of the string, ρy is replaced by the fixed mass ρy_o of the string in its equilibrium position. This is a reasonable approximation for studying *small* oscillations about the equilibrium position. The dynamics of such oscillations are clarified if we introduce the displacement $z = y - y_o$ of the balloon from its equilibrium position. After inserting $y = y_o + z$ in (1) we find

$$(m+\rho y_o)\frac{dv}{dt} + b\frac{dz}{dt} + \rho g z = B - (m+\rho y_o)g = 0$$

Finally, introducing the total mass $M = m + \rho y_o$ and putting $v = dy/dt = dz/dt$, we obtain

$$M\frac{d^2z}{dt^2} + b\frac{dz}{dt} + \rho g z = 0 \tag{2}$$

This is the standard form for free oscillation of a linear second-order system.

(b) Physically, when the balloon rises above its equilibrium position it lifts additional string whose weight tends to lower the balloon, and when the balloon descends below its equilibrium position it carries a lighter load of string which tends to drive the balloon upward. The equivalent stiffness in the equation of motion (2) is $k = \rho g$. This is the weight per unit length of the string. This example illustrates the fact that restoring-force elements do not always look like springs.

(c) The second-order equation of motion (2) predicts oscillation if the damping is light enough to produce a critical damping factor ζ smaller than unity.

(d) A free vibration governed by (2) will eventually decay to zero if there is a positive damping parameter b.

To obtain the behavioral parameters ζ and ω_o we use the equations

$$\zeta^2 = \frac{LDR^2}{\pi^2 + LDR^2} \qquad \text{and} \qquad \omega_o = \frac{\omega_d}{\sqrt{1-\zeta^2}}, \qquad \text{with} \qquad \omega_d = \frac{2\pi}{T_d}$$

(e) From the given data we see that the ratio of successive peak amplitudes is 0.2. Thus the LDR is ln(0.2) = -1.609, and the damping ratio is

$$\zeta = \left(\frac{(1.609)^2}{\pi^2 + (1.609)^2} \right)^{1/2} = 0.456$$

(f) From the given data we see that the damped natural period T_d is 3.0 seconds, so that the damped natural frquency is $\omega_d = 2\pi/T_d = 2.094$ rad/sec, and

$$\omega_o = \frac{2.094}{\sqrt{1 - (0.456)^2}} = 2.35 \text{ rad/sec}$$

To obtain the model parameters, we begin with the effective stiffness k which is the weight per unit length of the string. This is given as 0.5 ounces per foot which is 0.03125 pounds/foot in the English system or

$$k = 0.03125 \cdot \frac{4.448}{0.3048} = 0.456 \text{ Newtons/meter}$$

(g) The effective mass M of the balloon plus the equilibrium length of string, is given by

$$M = \frac{k}{\omega_o^2} = \frac{0.456}{(2.35)^2} = 0.0826 \text{ kg}$$

in the SI system. In the English system

$$M = \frac{W}{g} = \frac{W}{32.2} = \frac{0.03125}{(2.35)^2} = 0.00566 \text{ pound sec}^2/\text{ft}$$

from which we find $W = 0.1823$ pounds or 2.92 ounces. The weight of 3 feet of string is 1.5 ounces so the weight of the balloon is 1.42 ounces. In the SI system, the mass of 3 feet of string is 3(0.3048)(0.456)/9.81 = 0.0425 kg, so the mass of the balloon is m = 0.0826 - 0.0425 =0.0401 kg.

(h) The effective damping coefficient $b = 2\zeta\omega_o M$ is

$$b = 2(0.456)(2.35)(0.0826) = 0.1770 \text{ N/m/s}$$

in SI units, or

$$b = 2(0.456)(2.35)(0.00566) = 0.0121 \text{ pounds/ft/sec}$$

(i) The effective stiffness was previously obtained as 0.456 Newtons/meter in SI units or 0.03125 pounds/foot in English units.

4.2 Bungee Jumper

Initially the bungee jumper free-falls under the influence of gravity and air resistance, then when the slack is taken up in the elastic cord, the cord exerts an increasing retarding force which reduces her velocity to zero at level A, after which she bobs up and down with decreasing amplitude until she comes to rest at the equilibrium position at level B. For the purposes of a preliminary estimate of the conditions up to instant at which level A is reached, neglect the effects of air resistance and damping in the cord. Under this assumption energy is conserved. In the first jump the equilibrium position B is 20 feet below the point where the cord begins to stretch. The stiffness of the cord is then estimated as

$$k = \frac{W}{\Delta} = \frac{150}{20} = 7.5 \text{ pounds/foot}$$

The undamped natural frequency of the mass-spring system consisting of the jumper and cord is

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{W/g}} = \sqrt{\frac{7.5}{150/32.2}} = 1.269 \text{ rad/sec}$$

Let the elevation of the upper attachment point of the cord be denoted by h_o, the elevation of the point where the slack is taken up and the cord begins to stretch be denoted by h_1, the elevation of the equilibrium point B be denoted by h_B, and the elevation of the point A where the jumper's velocity first vanishes be denoted by h_A.

(a) In the first jump, $h_o - h_1 = 100$ feet, and $h_o - h_B = 120$ feet. The level A can be located by equating the potential energy lost in the fall to the elastic energy in the cord.

$$W(h_o - h_A) = k\Delta(h_o - h_A) = \frac{1}{2}k(h1 - h_A)^2 = \frac{1}{2}k[(h_o - h_A) - 100]^2$$

This reduces to a quadratic equation for $(h_o - h_A)$ whose solutions are

$$(h_o - h_A) = 120 \pm 66.3 \text{ feet}$$

The physically significant root is $(h_o - h_A) = 186.3$ feet. The low point A is 66.3 feet below the equilibrium level B, or 186.3 feet below the upper attachment point.

(b) The maximum downward acceleration is 32.2 feet/sec^2 during the initial free fall.

(c) The maximum upward acceleration occurs at point A and is

$$a_{max} = (\text{Max displacement from equilibrium})\omega_o^2 = (66.3)(1.269)^2 = 106.8\text{ft/sec}^2$$

which is 3.32 times the acceleration of gravity.

(d) The primary assumption made is that dissipation of energy has been neglected. The cord has been assumed to behave like a linear spring.

(e) In the second jump the slack length is doubled which cuts the stiffness in half, and doubles the distance Δ to 40 feet. The natural frequency is reduced to 0.897 rad/sec. The location of the new low point A' is obtained by solving a similar quadratic equation with the new values of $h_0 - h_1 = 200$ feet, $h_o - h_{B'} = 240$ feet, and $\Delta = 40$ feet. The result is

$$(h_o - h_{A'}) = 240 \pm 132.7 \text{ feet}$$

The physically significant root is $(h_o - h_A) = 373$ feet. The low point A is 132.7 feet below the equilibrium level B', or 373 feet below the upper attachment point.

(f) The level of the equilibrium point B' is 240 feet below the attachment point.

(g) The maximum downward acceleration is still 32,2 feet/sec

(h) The maximum upward acceleration is

$$a_{max} = (\text{Max displacement from equilibrium})\omega_o^2 = (132.7)(0.897)^2 = 106.8\text{ft/sec}^2$$

Although the jump involves a longer free fall and a greater extension of the cord the maximum acceleration does not change!

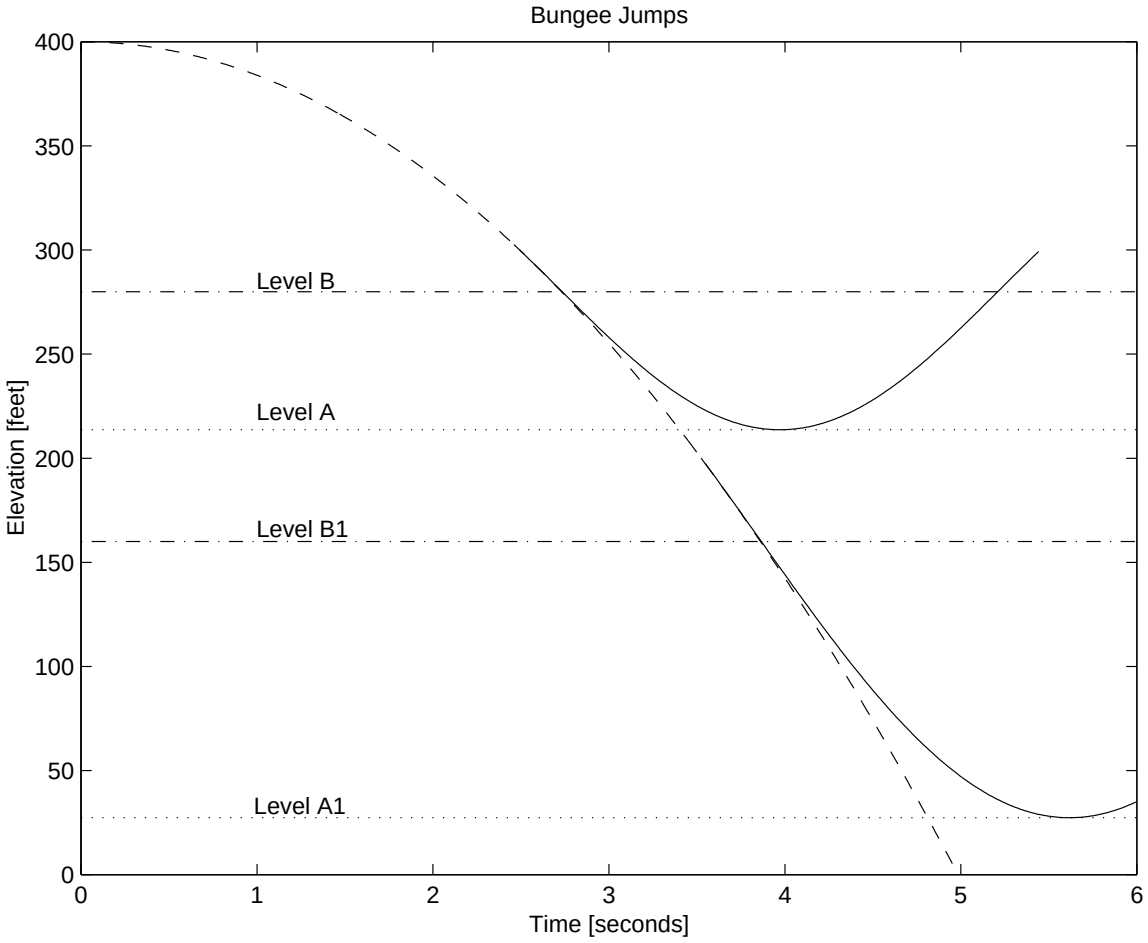


Figure 4.2.1: Time Histories of Bungee Jumps

(i) The time histories of the two jumps are shown in Fig. 4.2.1. For plotting purposes the elevation of the upper attachment point was (arbitrarily) assigned the value 400 feet. The trajectory of free fall is indicated by the dashed-line parabola. With 100 feet of slack the jumper's trajectory departs from the free-fall parabola at $h = 300$ ft and begins to oscillate about the final equilibrium level B. With no damping the maximum excursion (and maximum acceleration) is at Level A. With 200 feet of slack the jumper's trajectory departs from the free-fall parabola at $h = 200$ ft and begins to oscillate about the final equilibrium level B1. With no damping the maximum excursion (and maximum acceleration) is at Level A1.

4.3 Elevator Model

Consider a free-body diagram of the elevator of mass m when it is accelerating upward with acceleration dv/dt. The equation of motion is

$$T - mg = m\frac{dv}{dt}$$

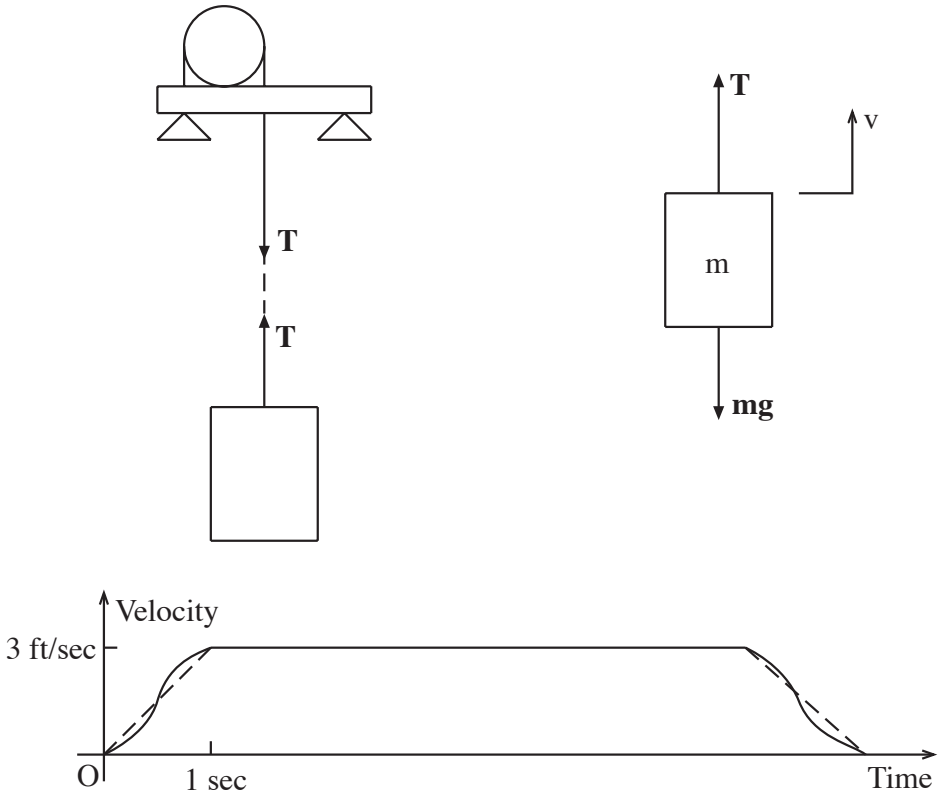


Figure 4.3.1: Elevator Dynamics

When the elevator is at rest, or traveling at constant velocity ($dv/dt = 0$), the tension T in the cable (and the load on the winch) would be simply the weight mg of the elevator and its cargo. Designing the winch structure on the basis of this load would be static design. Including the dynamic effect of accelerating the elevator results in $T = mg + mdv/dt$. To answer the question of whether dynamics matters, one must compare the magnitudes of the static load mg and the dynamic load $mg + mdv/dt$. The problem statement does not specify the explicit time behavior of dv/dt. It merely states that a total change of velocity of 3 feet/second is accomplished in one second. The sketch above shows two possible velocity histories. The slope of the velocity history is the acceleration dv/dt. With the dashed history the acceleration is uniform and has the magnitude

$$\frac{dv}{dt} = \frac{3\text{ft/sec}}{1\text{sec}} = (3)(0.3048) = 0.914\text{m/sec}^2$$

This time history has the smallest maximum acceleration but the sharp corners at the beginning and end of the one second acceleration are hard on the winch motor and are felt as jerks by

the passengers. The time history suggested by the solid curve in the figure produces a smoother transition at the expense of a larger maximum acceleration. If the system is designed to accelerate according to the solid curve the maximum acceleration can be 50% to 100% larger than the uniform acceleration of the dashed history. In the latter case the ratio of the dynamic load to the static load would be

$$\frac{g + (dv/dt)_{max}}{g} = \frac{9.81 + 2(0.914)}{9.81} = 1.186$$

Here, dynamics does indeed matter! The dynamic load is 18.6% larger than the static load.

Engineers often make preliminary design calculations to determine ball-park estimates of size, weight, and power. In these 'back-of-the-envelope' calculations it is common to omit effects with contributions of the order of five to ten percent. However, these effects must be included in final design calculations, which have to be defended in safety investigations.

4.4 Hoisting for Engineers

The solution for this problem is not available.

4.5 Blocked Springs

Let the effective combined stiffness of the four unblocked springs be k_u, and let the effective combined stiffness of the four blocked springs be k_b. In both cases the vehicle weight mg is supported by the springs. Since only half of the spring is active in the blocked case the static deflection $\Delta_b = mg/k_b$ in the blocked case will be half the static deflection $\Delta_u = mg/k_u$ in the unblocked case. Since we are given that $\Delta_u - \Delta_b = 2.5$ inches, we deduce that $\Delta_u = 5.0$ inches, and $\Delta_b = 2.5$ inches.

(a) The effective stiffnesses are

$$k_u = \frac{2500}{5} = 500 \text{ pounds/inch} \qquad \text{and} \qquad k_b = \frac{2500}{2.5} = 1000 \text{ pounds/inch}$$

in British units. In SI units

$$k_u = 500\frac{4.448)}{0.0254} = 87,560 \text{ Newton/meter} \quad \text{and} \quad k_b = 1000\frac{4.448)}{0.0254} = 175,100 \text{ Newton/meter}$$

(b) The undamped natural frequency ω_o is given by $\omega_o^2 = k/m$ where m is the mass of the vehicle. In SI units $m = 2500(0.4536) = 1134$ kg. The natural frequencies are

$$(\omega_o)_u = \sqrt{\frac{87,560}{1134}} = 8.787\text{rad/sec} \qquad \text{and} \qquad (\omega_o)_b = \sqrt{\frac{175,100}{1134}} = 12.437\text{rad/sec}$$

(c) We are given that the unblocked suspension is critically damped ($\zeta_u = 1.0$). The effective damping coefficient is

$$b = 2\zeta_u(\omega_o)_u m = 2(1.0)(8.787)(1134) = 19,930 \text{ Newtons/meter/second}$$

(d) Assuming that the shock absorbers are unchanged by the blocking operation, the damping coefficient remains $b = 19,930$ N/m/s and the new value of ζ for the system with blocked springs is

$$\zeta_b = \frac{b}{2(\omega_o)_b m} = \frac{19,930}{2(12.437)(11340} = 0.7071$$

If y is measured vertically upward from the equilibrium position, and $v = dy/dt$, the constitutive equations are

$$f_m = m\frac{dv}{dt}, \qquad f_d = bv, \qquad \text{and} \qquad f_k = ky$$

with appropriate subscripts u and b for the unblocked and blocked cases, respectively. and the force-balance requirement is

$$f_m = f(t) - f_d - f_k$$

where $f(t)$ is the upward driving force applied to the vehicle. These model requirements can be organized into the following state equations for a state determined system with state variables y and v.

$$\frac{d}{dt}\left\{\begin{array}{c} y \\ v \end{array}\right\} = \left[\begin{array}{cc} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{array}\right]\left\{\begin{array}{c} y \\ v \end{array}\right\} + \left\{\begin{array}{c} 0 \\ \dfrac{f(t)}{m} \end{array}\right\}$$

The output equation for the displacement response $y(t)$ is

$$y(t) = \{\ 1 \quad 0\ \} \left\{ \begin{array}{c} y \\ \\ v \end{array} \right\}$$

The state equations can be integrated by adapting the scripts 'pos.m' and 'eqpos.m' for the Plate on Springs problem of Assignment 5 to apply to the present problem. The adaptation of 'pos.m' involved inputting the specific values of m, b, f_a, x_o, and v_o and eliminating unecessary plots. The adapted version of 'pos.m' is called 'blockedspringssol.m'. It is located at the end of this solution.

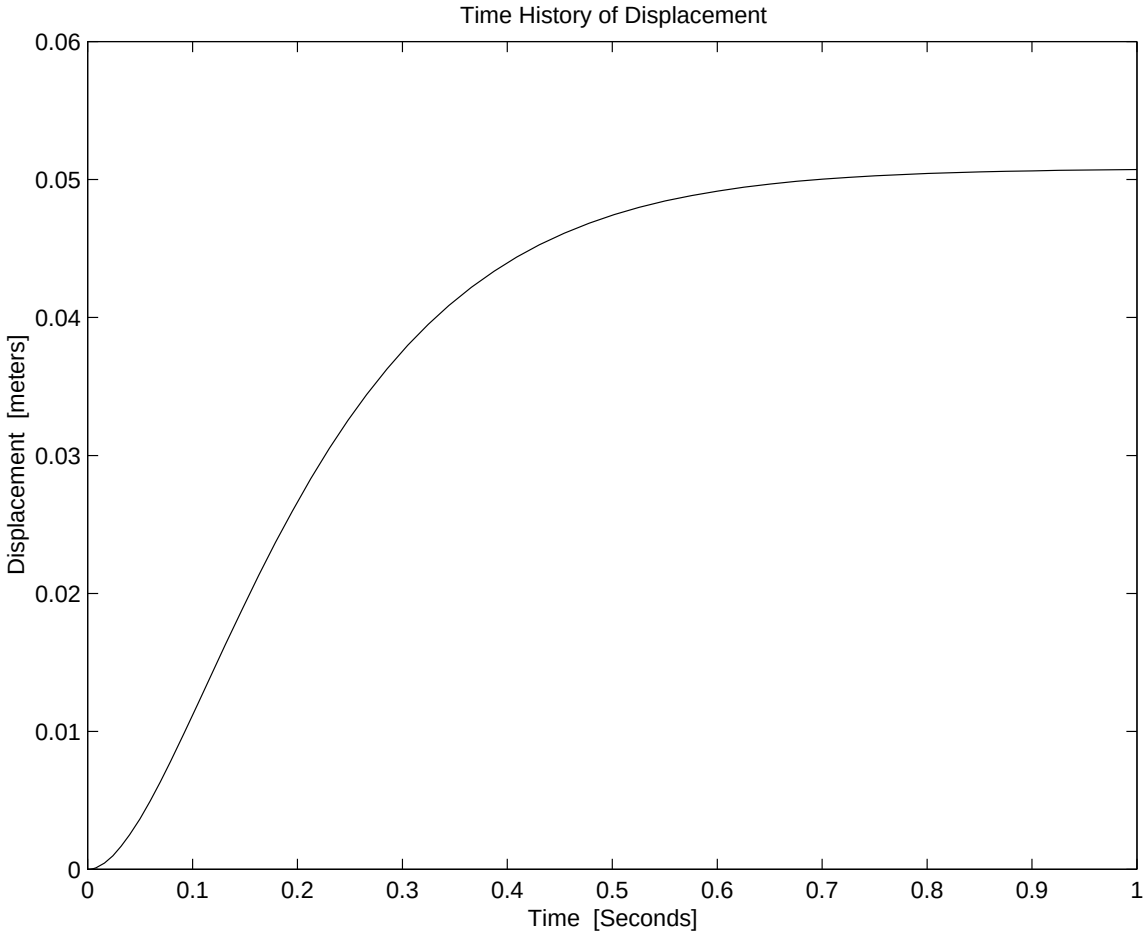


Figure 4.5.1: Response of Unblocked Vehicle

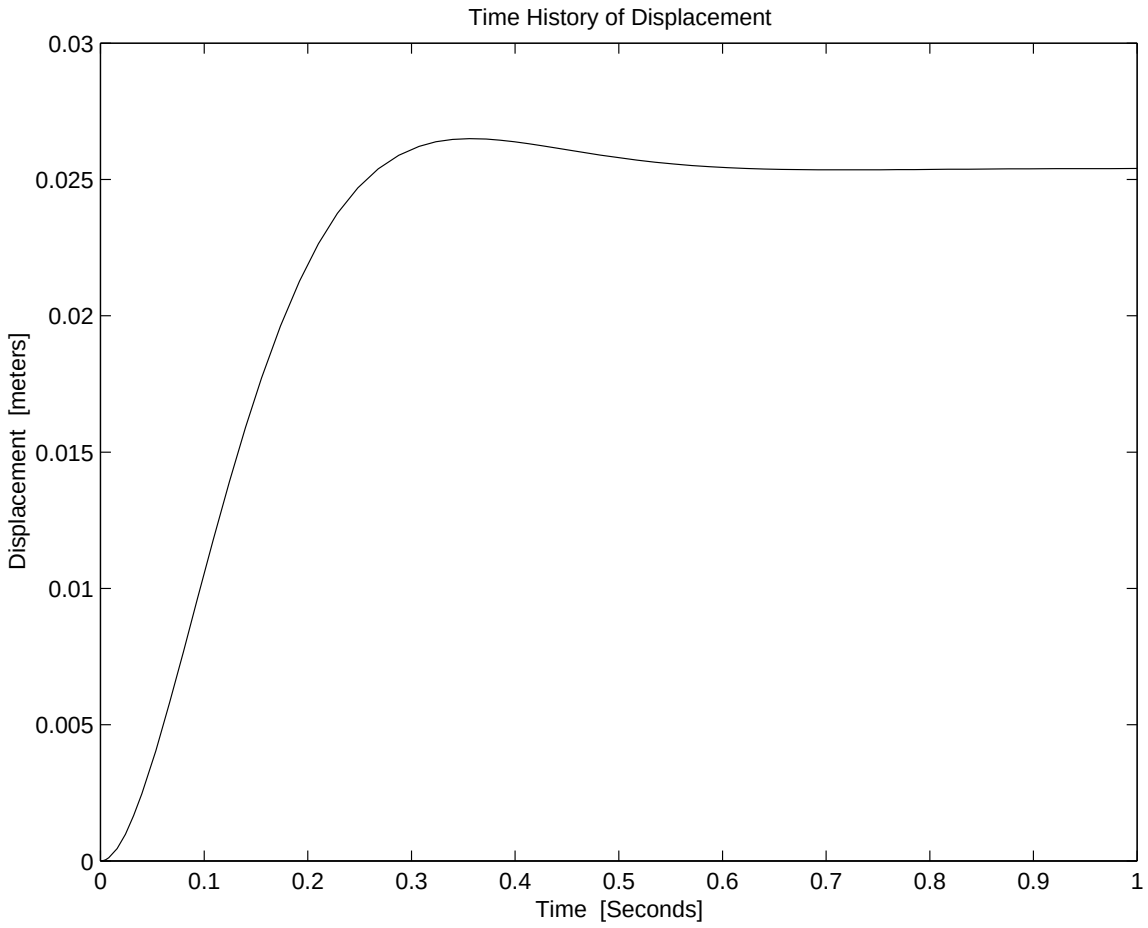


Figure 4.5.2: Response of Blocked Vehicle

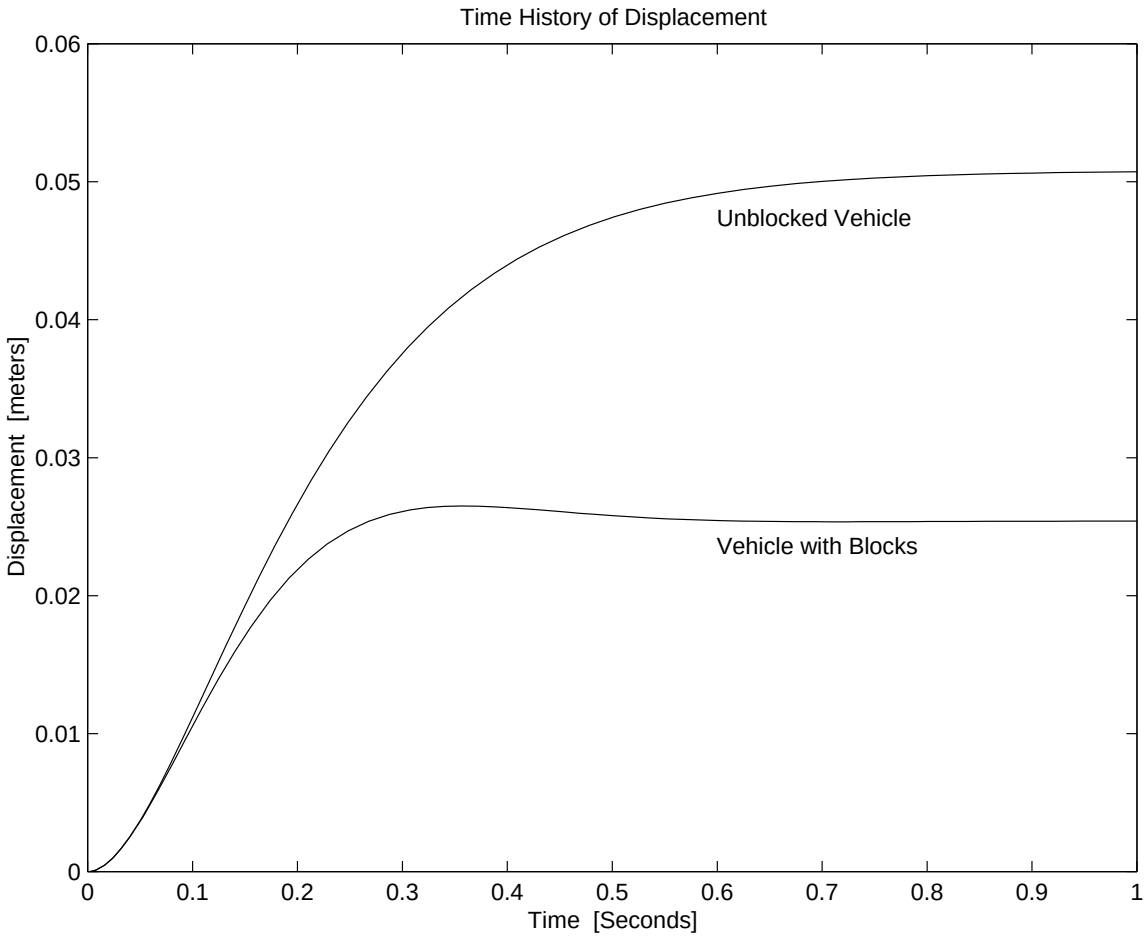


Figure 4.5.3: Comparison of Responses

The adaptation of 'eqpos.m' involved only the name change to 'Eqblockedspringssol.m'.

Using these scripts with the stiffness inputs $k_u = 87560$ N/m and $k_b = 175100$ N/m, and an integration time of 1.0 second produced the response curve shown in Fig. 4.5.1 for the unblocked vehicle, and the response curve shown in Fig. 4.5.2 for the blocked vehicle. The two responses are compared in Fig. 4.5.3. Note the quicker response to a smaller deflection for the blocked vehicle.

4.6 Engine Vibration

The solution for this problem is not available.

4.7 Garage Door

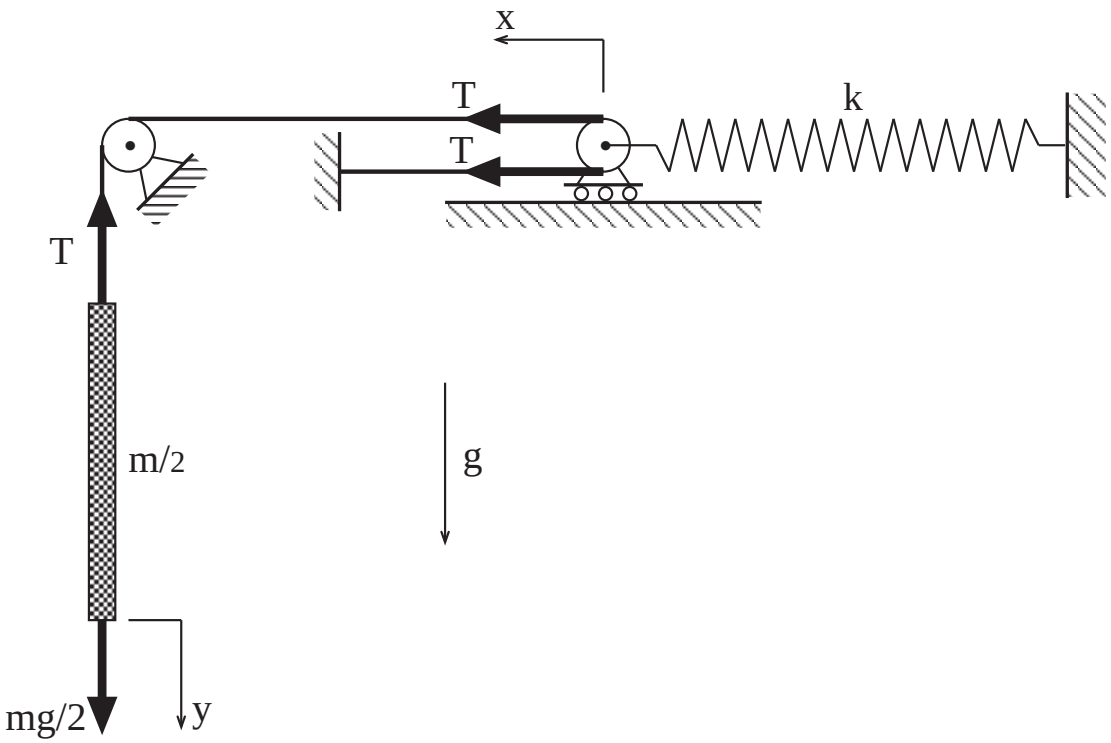


Figure 4.7.1: Garage Door support System

To model the pulley system, we assume that the inertia of the pulleys and cable is negligible in comparison with the mass of the garage door. Furthermore we assume that the cable is flexible (it bends easily) but stretches very little in comparison with the extension of the spring. Let x be the horizontal displacement of the pulley attached to the end of the spring, and let y be the vertical displacement of the door, both measured from the equilibrium position of the door hanging freely under the influence of gravity. In the equilibrium position, the tension T_o in the cables must equal the weight mg of the door. The total force in the spring is $2T_o$ which has produced an initial extension Δ of the springs, where $\Delta = 2T_o/k = 2mg/k$.

In a small oscillation about the equilibrium position, the displacements $x(t)$ and $y(t)$ oscillate about zero and the tension in the cable oscillates about the initial tension $T_o = mg$. When the cable remains taut, the displacements x and y are linked by the geometric compatibility requirement $y = 2x$. This is the crucial insight for this system, and it is not immediately obvious to all. One way to see it is to imagine that the door and the cables are temporarily frozen in position while the pulley attached to the spring is displaced a distance x to the left. An empty loop of cable appears, to the right of the pulley, with a straight length x in the upper cable and a straight length x in the lower cable. Now with the pulley fixed, gradually lower the door to take up the slack in the empty loop. It will be necessary to lower the door a distance $y = 2x$ to make the cable taut again.

(a) To formulate a model to analyze the door oscillations we need to assemble the geometric compatibility requirements, the constitutive equations, and the force balance requirements.

The *geometric compatibility* requirement is

$$y = 2x$$

The *constitutive equations* for the door and the spring are

$$f_m = \frac{m}{2}\frac{d^2y}{dt^2} \qquad \text{and} \qquad 2T = k(\Delta + x) = 2mg + kx$$

and the *force balance* requirement is

$$f_m = mg - T$$

These relations constitute a mathematical model of the garage door system. They can be organized into a single differential equation for either x or y by inserting the constitutive equations for f_m and T in the force balance equation and using the compatibility requirement to eliminate x in favor of y, or *vice versa*. The result is either

$$m\frac{d^2y}{dt^2} + \frac{k}{2}y = 0 \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + \frac{k}{2}x = 0$$

Note that because the origins for the displacements x and y were established at the equilibrium position, the gravity force has canceled out of the equations of motion.

(b) The undamped natural frequency ω_o for the preceding equations is given by

$$\omega_o^2 = \frac{k}{2m} = \frac{kg}{2W}$$

where W is the weight of the door and $g = 386$ in/sec^2 is the acceleration of gravity. The frequency of oscillation f_o in Hz is

$$f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{(5)(386)}{2(200)}} = 0.350 \text{ Hz}$$

(c) It is assumed that the door only moves vertically in the small oscillation. The inertia of the pulleys, cable, and spring is neglected. The elasticity of the cable in tension is neglected, and it is assumed that the tension T is uniform throughout the length of the cable.

4.8 Rotating Damped

A rotor with moment of inertia I and angular velocity ω is subjected to an accelerating torque T and two retarding friction torques, each modeled linearly as $B\omega$.

(a) The equation of motion is

$$I\frac{d\omega}{dt} + 2B\omega = T \qquad \text{or} \qquad \frac{d\omega}{dt} = -\frac{2B}{I}\omega + \frac{1}{I}T \tag{1}$$

(b) The steady-state angular velocity, when $T = 10$ Newton-meters, and $I = 0.001$ kg-m^2, and $B = 0.005$ N-m/r/s is

$$\omega_{ss} = \frac{T}{2B} = \frac{10}{2(0.005)} = 1000 \text{ r/s}$$

(c) Consider the input $T = T_a \sin \Omega t$ to be the imaginary part of the complex input $T_a \exp(i\Omega t)$ and look for a complex solution of the form $\omega = A\exp(i\Omega t)$. After substitution in (2) we find that this is indeed a solution , if

$$A = \frac{T_a}{2B + iI\Omega} = \frac{T_a}{2B}\frac{1}{\sqrt{1 + (\frac{I\Omega}{2B})^2}}\exp(i\phi)$$

where the phase angle ϕ is fixed by the relation

$$\tan\phi = -\frac{I\Omega}{2B}$$

The steady-state solution to the input $T = T_a \sin \Omega t$ is then the imaginary part of the complex solution $A\exp(i\Omega t)$, which is

$$\omega = \frac{T_a}{2B}\frac{1}{\sqrt{1 + (\frac{I\Omega}{2B})^2}}\sin(\Omega t + \phi)$$

The magnitude of the response amplitude is

$$M(\Omega) = \frac{T_a}{2B}\frac{1}{\sqrt{1 + (\frac{I\Omega}{2B})^2}}$$

and the phase angle is

$$\phi = -\tan^{-1}\frac{I\Omega}{2B}$$

For the case where $T_a = 10$ Newton-meters, the values of $M(\Omega)$ and $\phi(\Omega)$ for the frequencies $\Omega = 50$ rad/sec, $\Omega = 5$ rad/sec, and $\Omega = 0.5$ rad/sec are

(i) $\text{M}(50) = \frac{10}{0.01}\frac{1}{\sqrt{1+(\frac{(0.001)(50)}{0.01})^2}} = 196.1$ r/s,
$\phi(50) = \tan^{-1}(0.001)(50)/0.01 = 1.373r = 78.7\,\text{deg}$

(ii) $\text{M}(5) = \frac{10}{0.01}\frac{1}{\sqrt{1+(\frac{(0.001)(5)}{0.01})^2}} = 894$ r/s,
$\phi(5) = \tan^{-1}(0.001)(5)/0.01 = 0.463r = 25.6\,\text{deg}$

(iii) $\mathrm{M}(0.5) = \frac{10}{0.01}\frac{1}{\sqrt{1+(\frac{(0.001)(0.5)}{0.01})^2}} = 999$ r/s,

$\phi(0.5) = \tan^{-1}(0.001)(0.5)/0.01 = 0.0500r = 2.86\,\mathrm{deg}$

(d) The limiting value of the magnitude of the response amplitude $M(\Omega)$ when $\Omega \to 0$ is $A_o = T_a/2B = 1000$ r/s. The ratio $M(\Omega)/A_o$ is 0.1961 when $\Omega = 0.5$ rad/sec, 0.894 when $\Omega = 0.05$ rad/sec, and 0.999 when $\Omega = 0.005$ rad/sec.

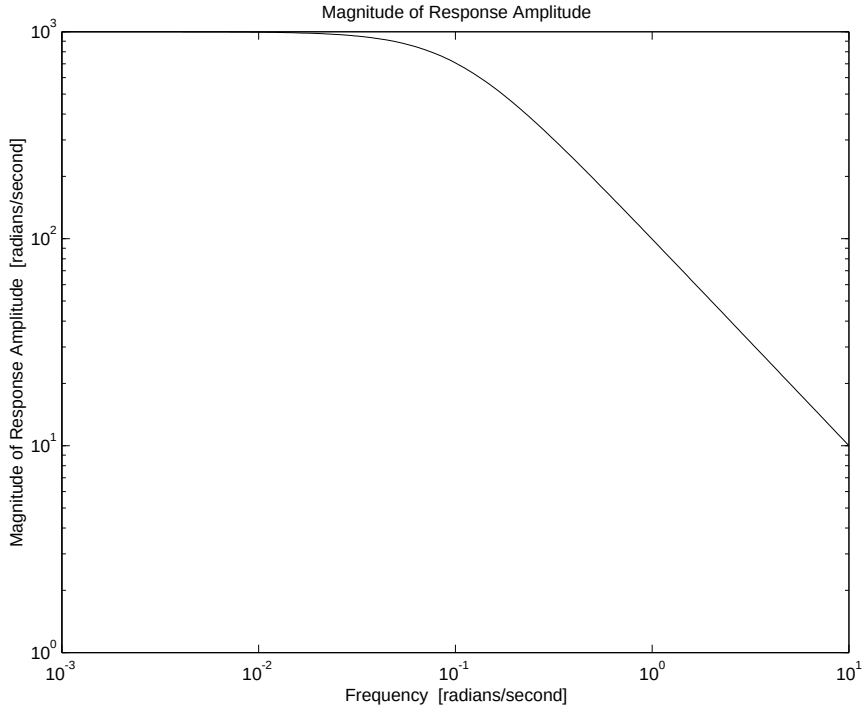


Figure 4.8.1: Magnitude of Response of Rotating Inertia

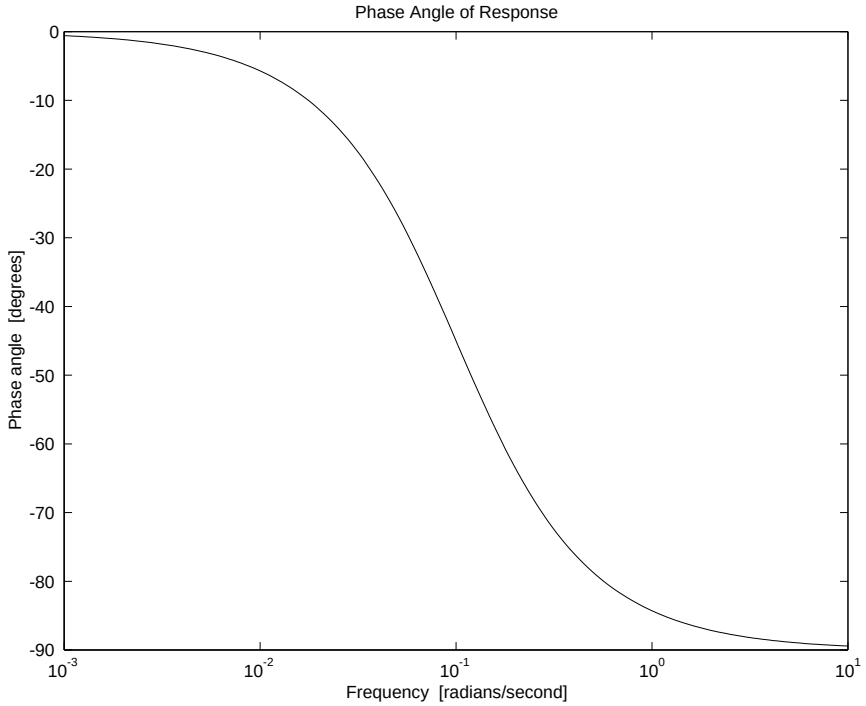


Figure 4.8.2: Phase Angle of Response of Rotating Inertia

(e) The low-frequency asymptote for $M(\Omega)$ is $A_o = T_a/2B$, while the high-frequency asymptote is $T_a/I\Omega$. These asymptotes intersect at the break frequency $\Omega_{break} = 2B/I = 10$ rad/sec.

(e) The Bode plots shown in Fig. 4.8.1 and Fig. 4.8.2 were obtaining by running a MATLAB script similar to that used in Problem 4.21.

4.9 Car Suspension 1

The differential equation for the system is

$$m\ddot{x} + c\dot{x} + kx = c\dot{x_0} + kx_0, \tag{1}$$

where x_0 stands for the ground waveform input. For step input, the particular solution will be:

$$x_p(t) = 1. \tag{2}$$

The homogenous solution has the form:

$$x_h(t) = Ae^{-\zeta\omega_n t}sin(\omega_d t) + Be^{-\zeta\omega_n t}cos(\omega_d t) \tag{3}$$

To find the initial velocity, integrate equation 1 from 0^- to 0^+,

$$m(\dot{x}(0^+) - \dot{x}(0^-)) + c(x(0^+) - x(0^-)) + k \times 0 = cx_0(0^+) - x_0(0^-) \tag{4}$$
$$m\dot{x}(0^+) = cx_0(0^+) \tag{5}$$
$$\dot{x}(0^+) = \frac{c}{m}x_0(0^+) = \frac{c}{m} \tag{6}$$

Plugging the initial conditions into the full solution

$$x(t) = Ae^{-\zeta\omega_n t}sin(\omega_d t) + Be^{-\zeta\omega_n t}cos(\omega_d t) + 1 \tag{7}$$
$$x(0^+) = 0 = B + 1 \Rightarrow B = -1 \tag{8}$$
$$\dot{x}(0^+) = \frac{c}{m} = A\omega_d - \zeta\omega_n B \Rightarrow A = (\frac{c}{m\omega_d} - \frac{\zeta\omega_n}{\omega_d}) \tag{9}$$

For $m = 500$kg, $k = 5 \times 10^4$N/m and $c = 2 \times 10^3$ Ns/m,

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \tag{10}$$
$$\zeta = \frac{c}{2\sqrt{km}} = 0.2 \tag{11}$$
$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 9.8 \tag{12}$$
$$\sigma = \zeta\omega_n = 2; \tag{13}$$
$$A = 0.2 \tag{14}$$
$$x(t) = 0.2e^{-2t}sin(9.8t) - e^{-2t}cos(9.8t) + 1 \tag{15}$$

The result is plotted in Figure 4.9.1. The passengers will not like this ride because there are too many oscillations. when $c = 2 \times \sqrt{km} = 10000$, the system will be critically damped, in this case the complete solution has the form:

$$x(t) = 1 + Ae^{-\omega_n t} + Bte^{-\omega_n t} \tag{16}$$
$$x(0^+) = 0 = A + 1 \Rightarrow A = -1 \tag{17}$$
$$\dot{x}(0^+) = \frac{c}{m} = -\omega_n A + B \Rightarrow B = \frac{c}{m} - \omega_n = 10 \tag{18}$$
$$x(t) = 1 - e^{-10t} + 10te^{-10t} \tag{19}$$

In order to find the acceleration for the under-damped case, reformulate equation 16,

$$x(t) = 0.2e^{-2t}sin(9.8t) - e^{-2t}cos(9.8t) + 1 \tag{20}$$
$$= Re\{1 - 1.02e^{-2t+j(9.8t+0.197)}\} \tag{21}$$
$$\ddot{x}(t) = Re\{-1.02(-2 + 9.8j)^2 e^{-2t+j(9.8t+0.197)}\} \tag{22}$$
$$\ddot{x}(t) = 102e^{-2t}cos(9.8t + 0.6) \tag{23}$$

The maximum acceleration could happen at t=0+ and t=$(\pi - 0.6)/9.8 = 0.26$sec,

$$\ddot{x}(t = 0+) = 102cos(0.6) = 84 \tag{24}$$
$$\ddot{x}(t = 0.26) = -102e^{-2*0.26} = -60 \tag{25}$$

So the maximum acceleration is $84\text{m}/sec^2$.
Similarly, for the critically damped case, differentiating equation 25 twice,

$$\ddot{x}(t) = -300e^{-10t} + 1000te^{-10t} \tag{26}$$
$$\ddot{x}(0+) = -300 \tag{27}$$

So the passenger will have bigger acceleration in the critically damped case. From the step response, the 5% settling time is 1.35sec in the under-damped case and 0.4sec in the critically-damped case.

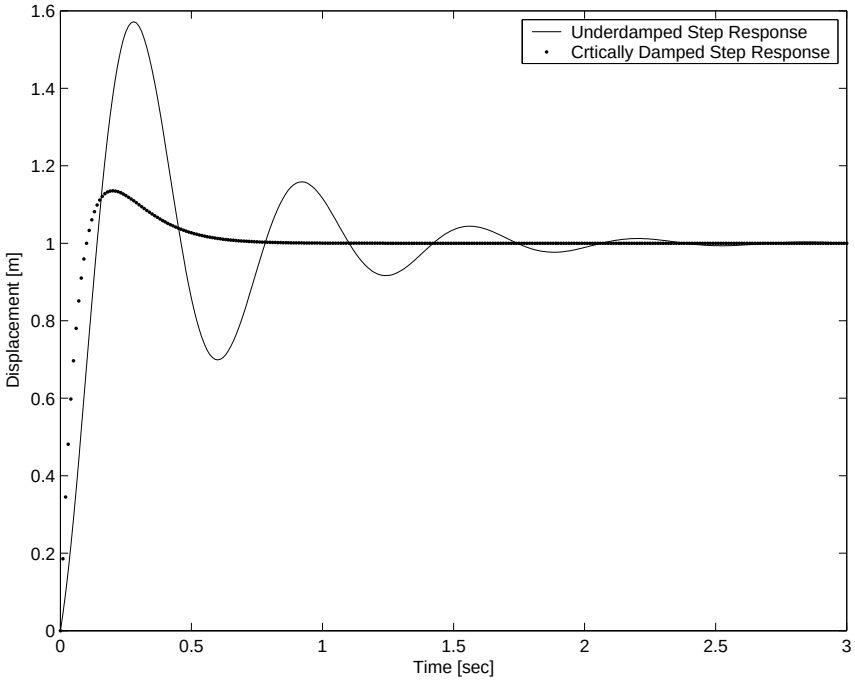


Figure 4.9.1: Step Response

4.10 Disk Drive

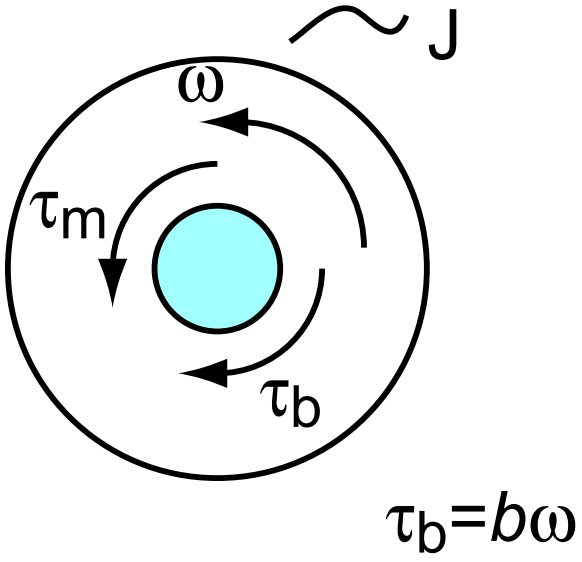


Figure 4.10.1: Free body diagram of disk drive spindle

(a) Figure 4.10.1 shows the free body diagram for the spindle. Note: that all constitutive equations are included with the diagram.

(b)

$$\begin{aligned} J\dot{\omega} &= \tau_m - \tau_b \\ J\dot{\omega} &= \tau_m - b\omega \\ \dot{\omega} + \frac{b}{J}\omega &= \frac{\tau_m}{J} \end{aligned}$$

(c) When the disk spins down $\tau_m = 0$ therefore our equation of motion becomes

$$\dot{\omega} + \frac{b}{J}\omega = 0$$

We know that the solution for this must take the form $\omega(t) = Ae^{st} = Ae^{-t/\tau}$. If we substitute into the equation of motion we get

$$\begin{aligned} Ase^{st} + A\frac{b}{J}e^{st} &= 0 \\ s &= -\frac{b}{J} \\ \tau &= \frac{J}{b} \rightarrow b = \frac{J}{\tau} \\ b &= \frac{0.001}{0.5}\frac{\text{kg m}^2}{\text{s}} = 0.002\frac{\text{N m s}}{\text{rad}} \end{aligned}$$

Please note that angular velocity must always be in rad/s (the unit label rad=1) unless otherwise noted.

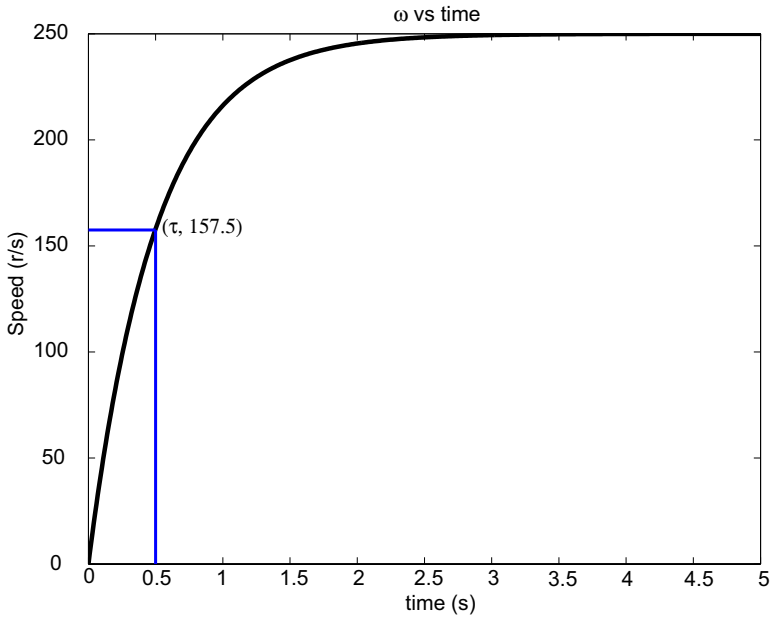


Figure 4.10.2: Plot ω vs time

(d) To determine $\omega(t)$, we need to determine the final velocity of the disk. When $\dot{\omega} = 0$ the characteristic equation become

$$b\omega = \tau_m \ \rightarrow \ \omega_f = \frac{\tau_m}{b} = 250r/s$$

Since the initial velocity of the system is 0, we know the solution must be of the form

$$\omega(t) \quad = \quad \omega_f(1 - e^{-t/\tau}) = 250(1 - e^{-2t})$$

Figure 4.10.2 shows the $\omega(t)$

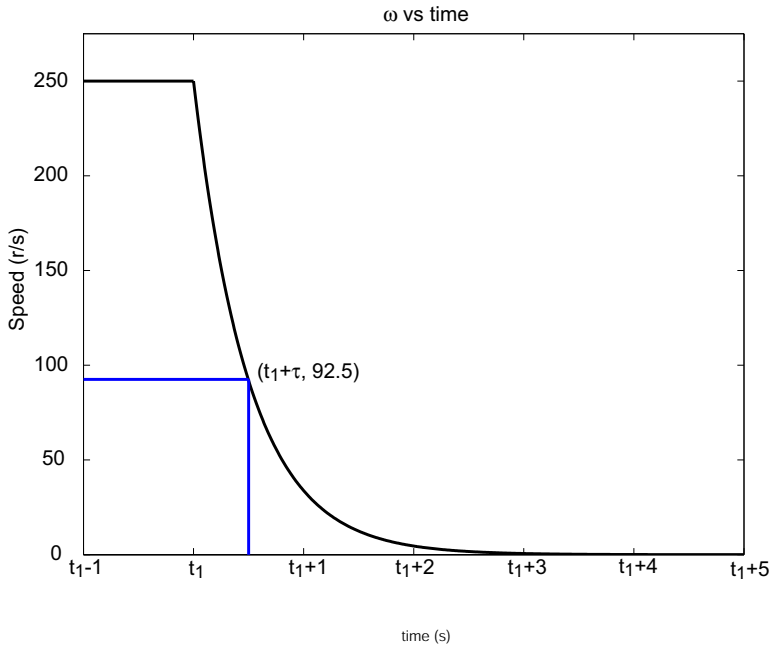


Figure 4.10.3: Plot ω vs time

(e) $\omega(t)$ for $t > t_1$ is

$$\omega(t) \quad = \quad \omega_f e^{-(t-t_1)/\tau} = 250e^{-2(t-t_1)}$$

Figure 4.10.2 shows the time response of the disk.

4.11 Crashworthiness

The sketch in Fig. 4.11.1 shows the linear model used to study crashworthiness. The mass m of the vehicle is (2000 pounds)(0.4536 kg/pound) = 907.2 kg and the vehicle speed at impact v_o is (10 mi/hr)(5280 ft/mi)(0.3048 meters/ft)/(3600 secs/hr) = 4.470 meters/sec. When there is no energy dissipation ($b = 0$) the displacement of the mass goes through a half-cycle of undamped vibration of the form

$$x(t) = A \sin \omega_o t$$

The vehicle then retreats from the barrier with velocity $-v_o$. The amplitude A of the oscillation is given as 6 inches which is (6 in)(0.0254 meters/in) = 0.1524 meters, but the undamped natural frquency ω_o is unknown.

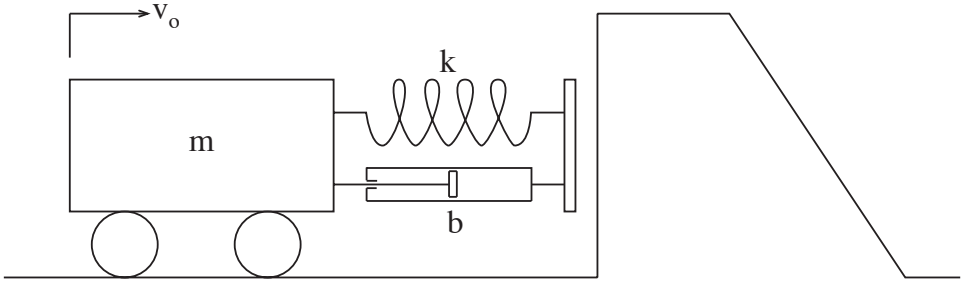


Figure 4.11.1: Model of Vehicle Impacting a Barrier

One way to calculate ω_o is to set the velocity obtained by differentiating $x(t)$ equal to v_o at $t = 0$. This leads to

$$\omega_o = \frac{v_o}{A} = \frac{4.470}{0.1524} = 29.33 \text{ rad/sec}$$

Another way to obtain ω_o is to use conservation of energy,

$$KE = \frac{1}{2} m v_o^2 = \frac{1}{2} k A^2$$

to get k, and then use $\omega_o^2 = k/m$ to evaluate ω_o.

(a) If the value of ω_o is already known, then the effective stiffness k is obtained from

$$k = m\omega_o^2 = (907.2)(29.33)^2 = 780,400 \text{ N/m}$$

If ω_o is not available, then the stiffness can be obtained directly from conservation of energy.

$$k = m \left(\frac{v_o}{A} \right)^2 = 907.2 \left(\frac{4.470}{0.1524} \right)^2 = 780,500 \text{ N/m}$$

(b) During the half-cycle of undamped sinusoidal motion $x(t) = A \sin \omega_o t$ the acceleration is $d^2x/dt^2 = -A\omega_o^2 \sin \omega_o t$. The peak deceleration occurs at the instant of peak deflection of the spring and has the magnitude

$$Decel_{peak} = A\omega_o^2 = (0,1524)(29.33)^2 = 131.1 \text{ meters/sec}^2$$

The geometric compatibility requirements are satisfied by taking x to represent the displacement of the mass *and* the deformation of the spring, and $v = dx/dt$ to represent the velocity of the mass *and* the relative velocity across the damper. The constitutive equations are

$$f_m = m\frac{dv}{dt} \qquad f_d = bv \qquad \text{and} \qquad f_k = kx$$

and the force-balance requirement is

$$f_m = -f_d - f_k$$

These model requirements can be used to construct the state equations for the state variables x and v

$$\frac{d}{dt}\left\{\begin{array}{c} x \\ v \end{array}\right\} = \left[\begin{array}{cc} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{array}\right]\left\{\begin{array}{c} x \\ v \end{array}\right\}$$

The desired outputs are (i) the deflection $disp = x$ of the fender, (ii) the deceleration of the vehicle $decel = (k/m)x + (b/m)v)$, and (iii) the total force on the barrier $f_{barr} = kx + bv$. In matrix notation

$$\left\{\begin{array}{c} disp \\ decel \\ f_{barr} \end{array}\right\} = \left[\begin{array}{cc} 1 & 0 \\ k/m & b/m \\ k & b \end{array}\right]\left\{\begin{array}{c} x \\ v \end{array}\right\}$$

The state equations can be integrated by adapting the scripts 'pos.m' and 'eqpos.m'. The adaptation of 'pos.m' involved inputting the specific values of m, k, x_o, and v_o and letting the program ask for the desired value of ζ instead of the desired value of b. The adapted version of 'pos.m' is called 'crashworthinesssol.m' and is found at the end of this solution.

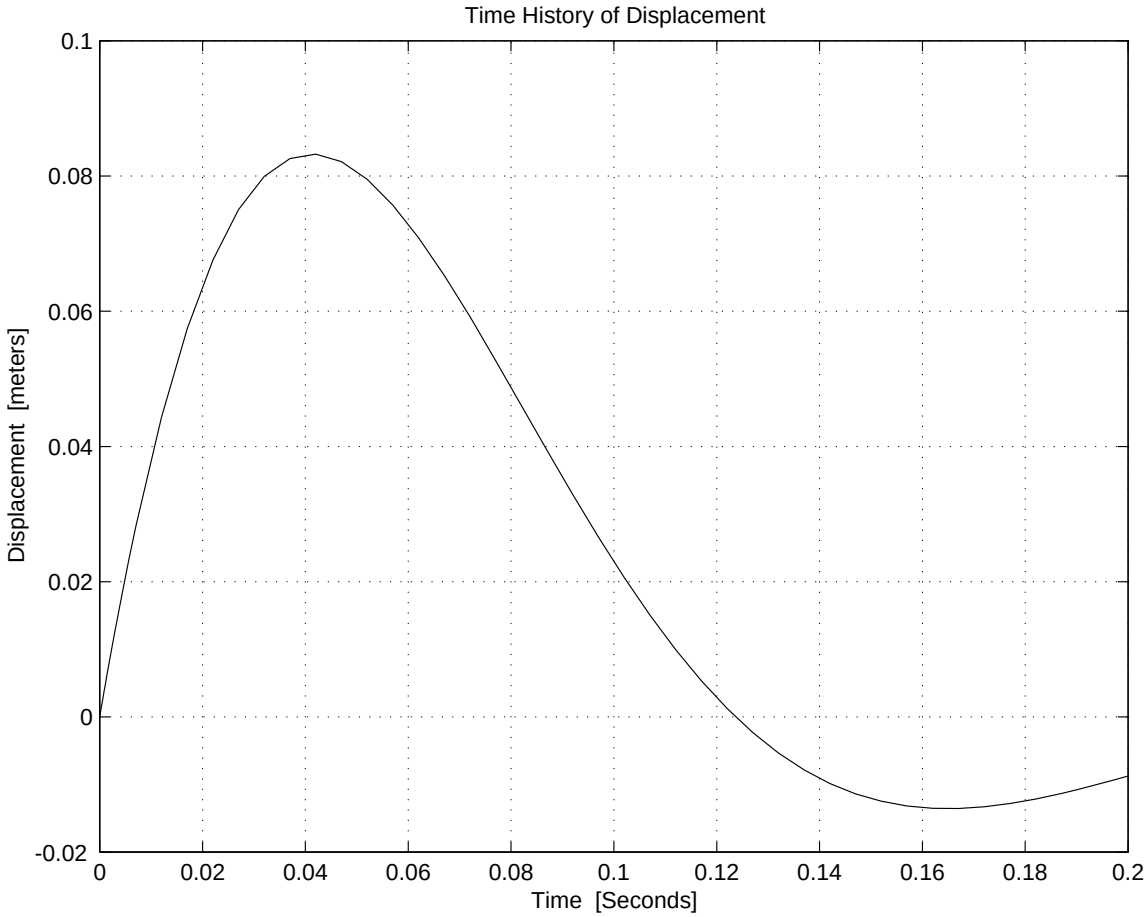


Figure 4.11.2: Displacement Response for $\zeta = 0.5$

The script for the equation called for by 'crashworthinesssol.m' is called 'Eqcrashworthinesssol.m'. It is similar to'Eqblockedspringssol.m' except that here there is no applied force f_a.

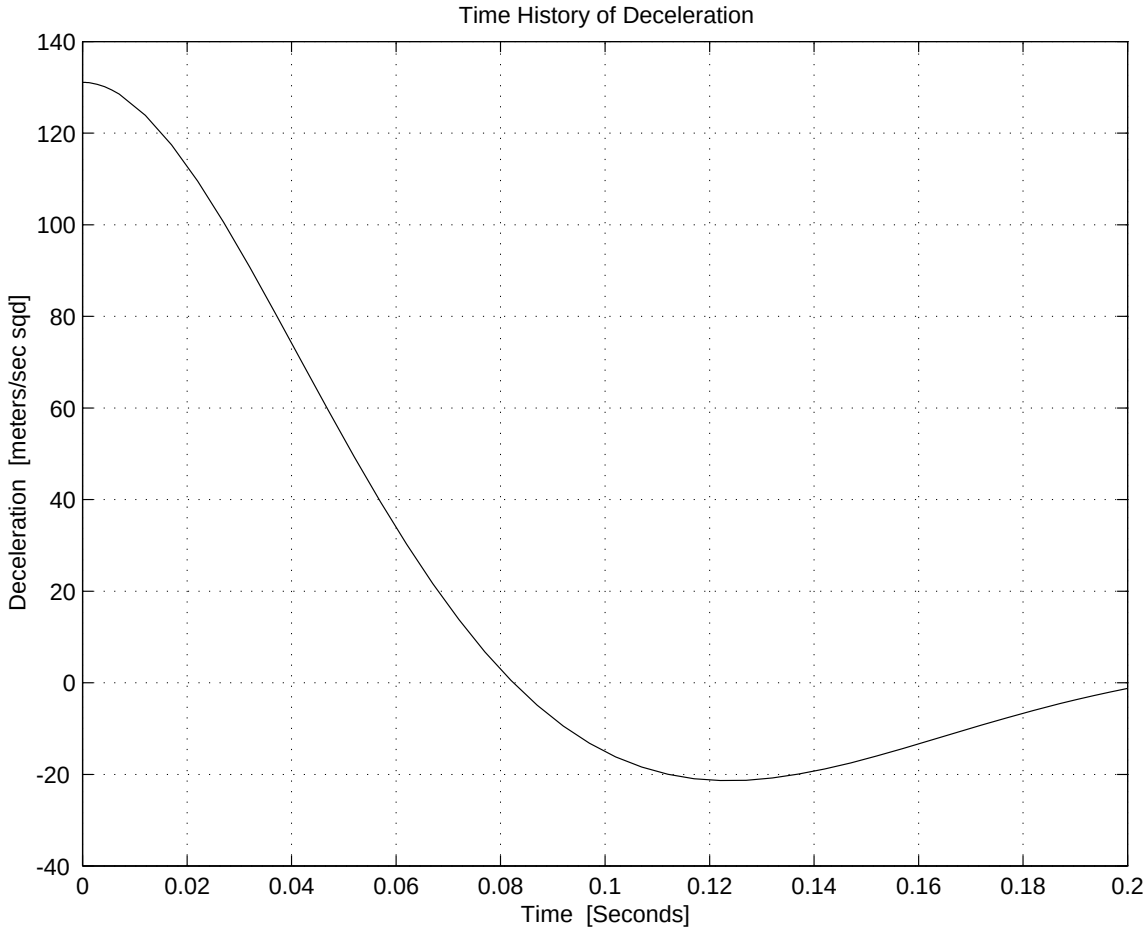


Figure 4.11.3: Deceleration Response for $\zeta = 0.5$

Typical plots produced by 'crashworthiness.m' are shown in Figures 4.11.2-4.11.4, which were obtained for $\zeta = 0.5$.

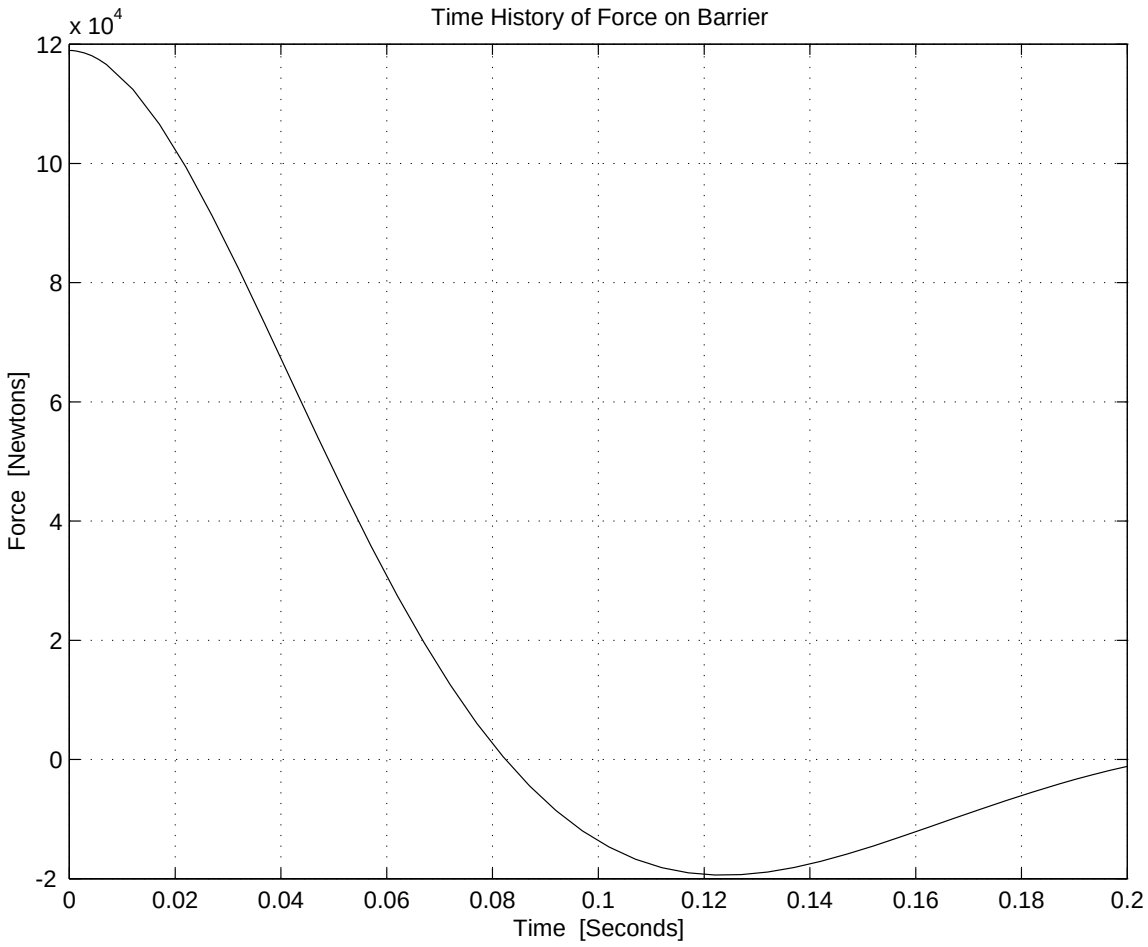


Figure 4.11.4: Barrier Force Response for $\zeta = 0.5$

(g) The plots obtained are only valid during the interval in which the fender remains in contact with the barrier. In the undamped case, this period was a half cycle of sinusoidal oscillation. In the damped cases, the model results are valid only as long as the barrier force is compressive. If tension is required the fender separates from the barrier. For the case $\zeta = 0.5$ we see that the model predicts tensile barrier force after $t = 0.082$ seconds. In *all four* cases there is separation of the vehicle from the barrier after the impact.

(h) Peak decelerations are proportional to peak barrier forces (this is a consequence of Newton's law $F = ma$). In the undamped case, the barrier force is delivered by the spring. Its magnitude is $f_{barr,u} = mA\omega_o^2 = mv_o\omega_o$. In the damped case, the barrier force is delivered by both the spring and the damper. At the initial instant of impact, when the fender begins to move with velocity v_o, but hasn't yet deflected the spring, the entire barrier force is delivered by the damper. Its magnitude is $f_{barr,d} = bv_o$ which can be made as large as desired by increasing the parameter b. This initial damping force will be the *same* as the peak undamped force if $bv_o = mv_o\omega_o$. This occurs when $\zeta = 0.5$.

(i) The peak deceleration is greatest in Case (f), $\zeta = 1.0$.

(j) By trial and error with 'crashworthinesssol.m' one finds that 'decel' has a fairly flat minimum value of slightly more than 106 meters/second^2 over the range $0.25 < \zeta < 0.30$.

4.12 Nonlinear Rotational System

The solution for this problem is not available.

4.13 Developing Differential Equations

We are interested in modeling the rotation system in Figure 4.13.1 and the translational system in Figure 4.13.2. The free-body diagram for the rotational system is given in Figure 4.13.3. The free-body diagram for the translation system is given in Figure 4.13.4.

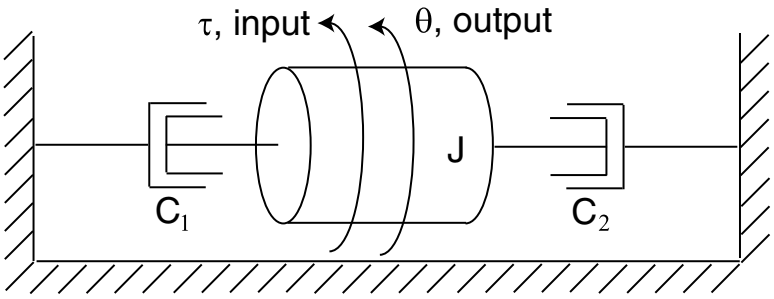


Figure 4.13.1: SDOF Rotational System

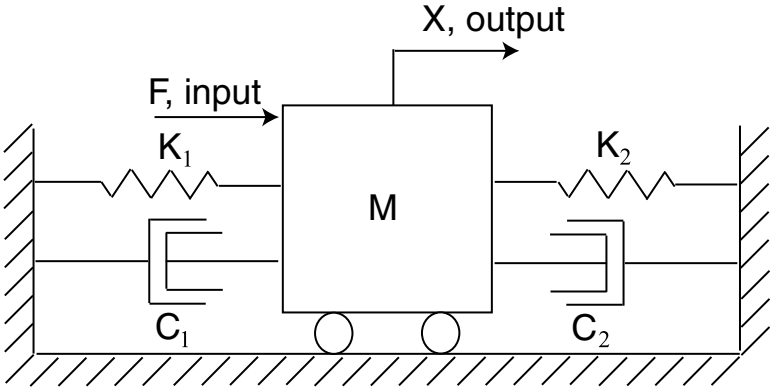


Figure 4.13.2: SDOF Translational System

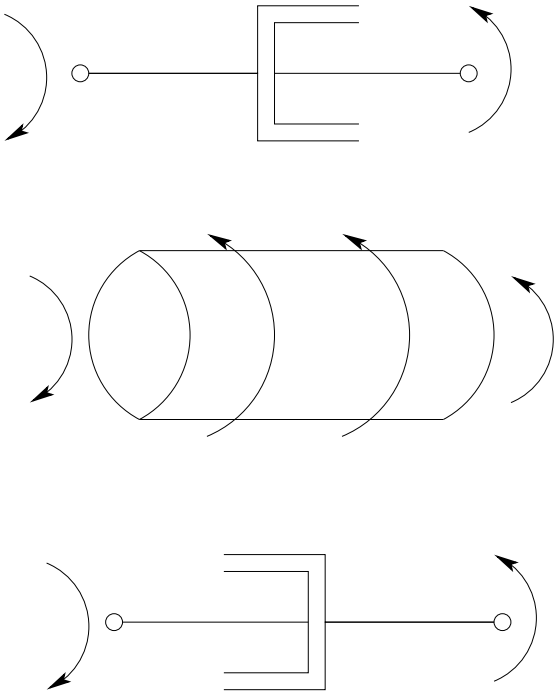


Figure 4.13.3: Free-body diagram for the Rotational System

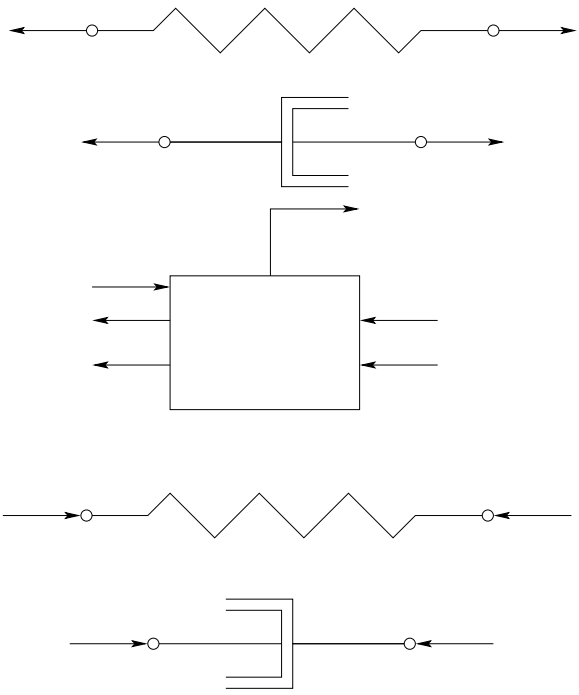


Figure 4.13.4: Free-body diagram for the Translational System

Using the free-body diagram, the equation of motion for the rotational system is

$$J\ddot{\theta} + (c_1 + c_2)\dot{\theta} = \tau$$

The rotational system is a first-order system because we can rewrite the governing equation using the velocity ($\omega = \dot{\theta}$)such that there is only a single derivative.

$$J\dot{\omega} + (c_1 + c_2)\omega = \tau$$

The governing equation for the translational system is

$$M\ddot{x} + (c_1 + c_2)\dot{x} + (k_1 + k_2)x = F$$

This system is second-order.

4.14 Mass Spring System Frequency Response

(a) The differential equation for this system is

$$m\ddot{x} + kx = F$$

Take the Laplace transform and rearrange to obtain the transfer function

$$X(s)(ms^2 + k) = F(s)$$
$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + k}$$

M is simply the magnitude of the transfer function evaluated at $s = j\omega$.

$$M = \left|\frac{X(s)}{F(s)}\right| = \frac{1}{|100 - \omega^2|}$$

The phase is

$$\phi = \tan^{-1}\frac{0}{1} - \tan^{-1}\frac{0}{100 - \omega^2}$$
$$= 0°, \omega < 10 \text{ and } -180°, \omega > 10$$

(d) The plot is shown in Figure 4.14.1.

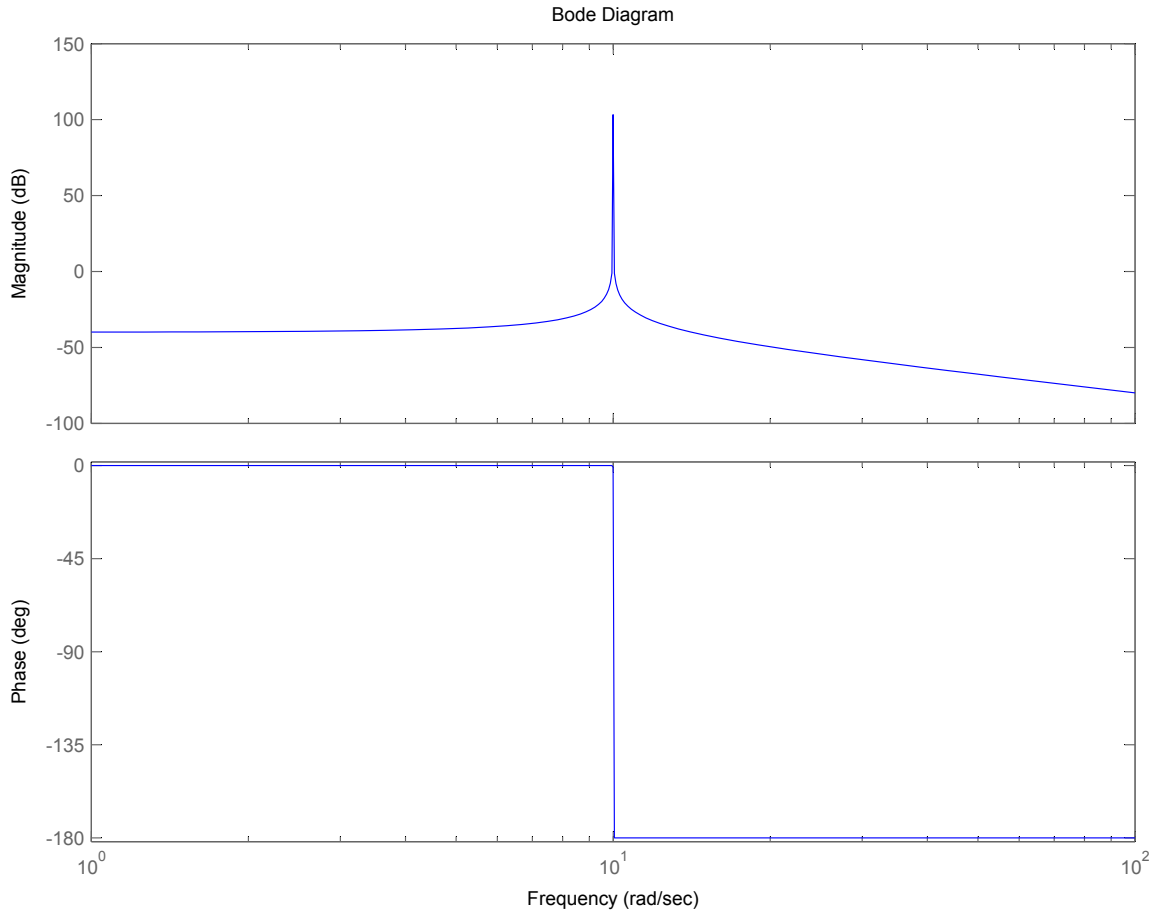


Figure 4.14.1: Bode plot

4.15 Modeling Practice

The solution for this problem is not available.

4.16 Small Motion Transfer Function

The solution for this problem is not available.

4.17 Mass Spring Damper System Frequency Response 1

(a) The differential equation for this system is

$$m\ddot{x} + b\dot{x} + kx = F$$

Take the Laplace transform and rearrange to obtain the transfer function

$$\begin{aligned} X(s)(ms^2 + bs + k) &= F(s) \\ \frac{X(s)}{F(s)} &= \frac{1}{ms^2 + bs + k} \end{aligned}$$

M is simply the magnitude of the transfer function evaluated at $s = j\omega$.

$$M = \left| \frac{X(s)}{F(s)} \right| = \frac{1}{\sqrt{(100 - \omega^2)^2 + \omega^2}}$$

The phase is

$$\begin{aligned} \phi &= \tan^{-1} \frac{0}{1} - \tan^{-1} \frac{\omega}{100 - \omega^2} \\ &= -\tan^{-1} \frac{\omega}{100 - \omega^2} \end{aligned}$$

(d) The plot is shown in Figure 4.17.1.

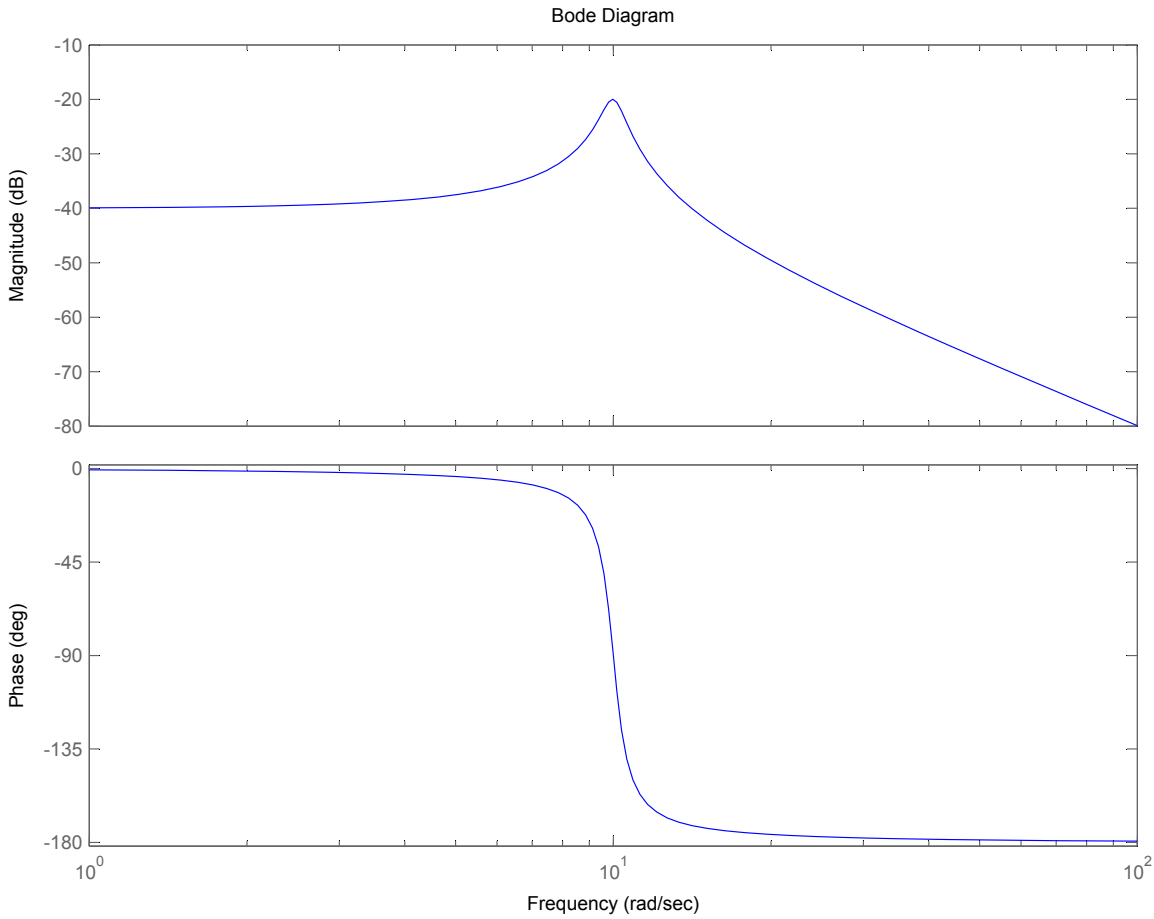


Figure 4.17.1: Bode plot

4.18 Mass Spring Damper System Frequency Response 2

(a) Begin by writing out two force balance equations, one for each mass. This will yield two differential equations involving x_1, x_2 and F. These simplified equations are

$$m_1\ddot{x}_1 + (b_1 + b_2)\dot{x}_1 + (k_1 + k_2)x_1 = b_2\dot{x}_2 + k_2x_2 + F$$
$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 = b_2\dot{x}_1 + k_2x_1$$

Next take the Laplace transform of each equation

$$X_1(s)\left(m_1s^2 + (b_1 + b_2)s + (k_1 + k_2)\right) = X_2(s)\left(b_2s + k_2\right) + F(s)$$
$$X_2(s)\left(m_2s^2 + b_2s + k_2\right) = X_1(s)\left(b_2s + k_2\right)$$

To obtain the desired transfer functions, simply substitute one equation into the other. After a lot of messy algebra, this yeilds

$$\frac{X_1(s)}{F(s)} = \frac{m_2s^2 + b_2s + k_2}{m_1m_2s^4 + (m_2(b_1 + b_2) + m_1b_2)s^3 + (m_1k_2 + m_2(k_1 + k_2) + b_1b_2)s^2 + (b_1k_2 + b_2k_1)s + k_1k_2}$$
$$\frac{X_2(s)}{F(s)} = \frac{b_2s + k_2}{m_1m_2s^4 + (m_2(b_1 + b_2) + m_1b_2)s^3 + (m_1k_2 + m_2(k_1 + k_2) + b_1b_2)s^2 + (b_1k_2 + b_2k_1)s + k_1k_2}$$

(b) The bode plots and pole/zero plots are shown in Figures 4.18.1-4.18.4.

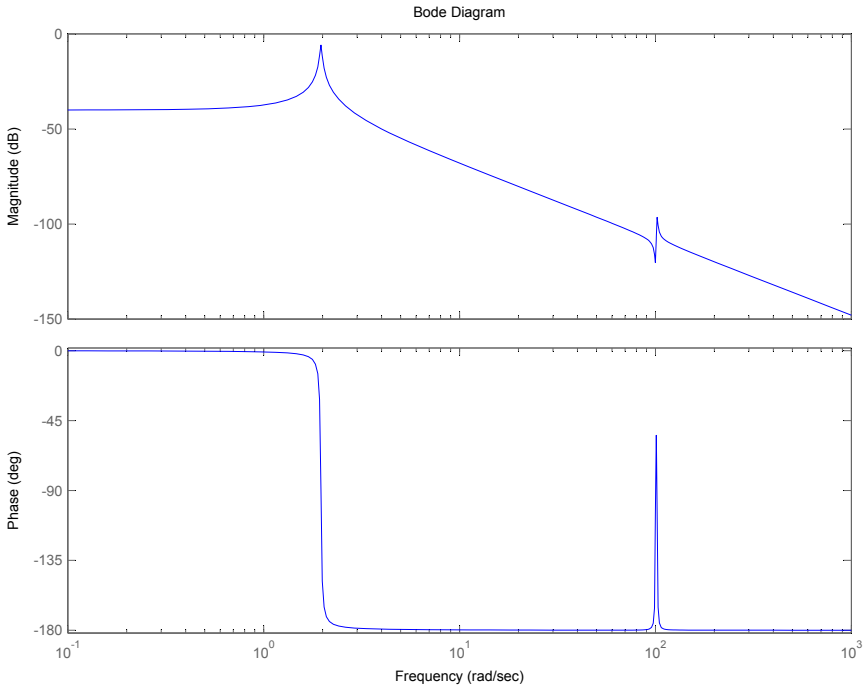


Figure 4.18.1: Bode plot for $H_1(s)$

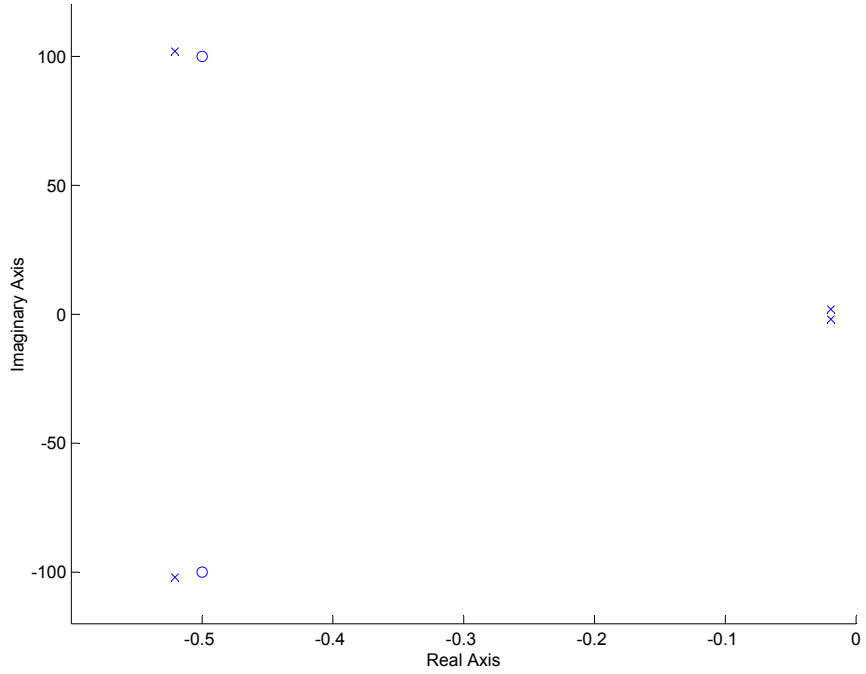


Figure 4.18.2: Pole/zero plot for $H_1(s)$

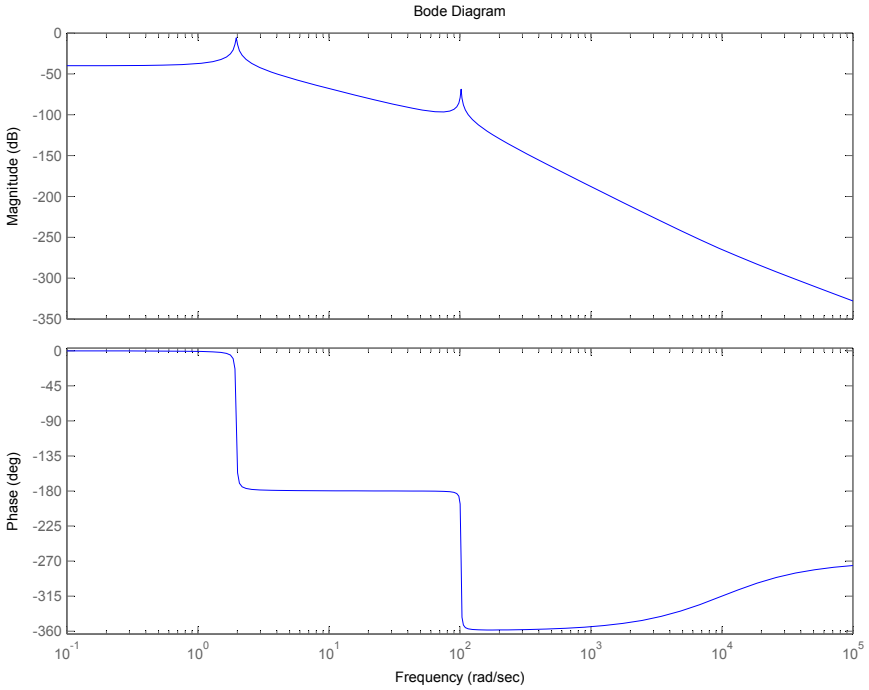


Figure 4.18.3: Bode plot for $H_2(s)$

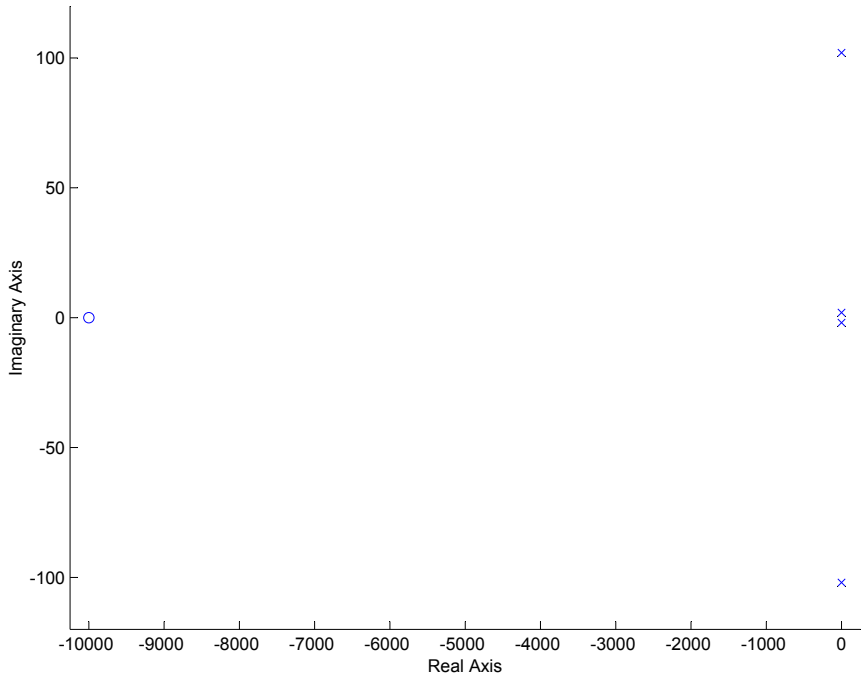


Figure 4.18.4: Pole/zero plot for $H_1(s)$

For both bode plots, peaks appear at the frequencies of the pole locations, that is, the imaginary value of each pole. The imaginary value of a pole is simple the undamped natural frequency, and it makes sense that peaks would occur at damped natural frequencies. m_1 is at a relative maximum at about 1.96 [rad/sec]. Both masses oscillate with a magnitude of -5.85 [dB]. m_1 is at a relative minimum at about 100 [rad/sec], oscillating at -120 [dB]. m_2 is oscillating at -80[dB], which is 100 times the amplitude of m_1.

4.19 Propeller Shaft Vibration

Torsional vibration of coupler shaft between engine and propellor. Moment of inertia of propellor is estimated as $I = mL^2/12$, where m is the mass of an aluminum rod, 2 inches (0.0508 meters) in diameter and six feet (1.829 meters) in length. The density of aluminum is 2720 kg/m^3, so

$$m = \rho\pi r^2 L = 2720\pi(0.0254)^2(1.829) = 10.08 \text{ kg}$$

and

$$I = \frac{mL^2}{12} = \frac{(10.08)(1.829)^2}{12} = 2.811 \text{ kg-m}^2$$

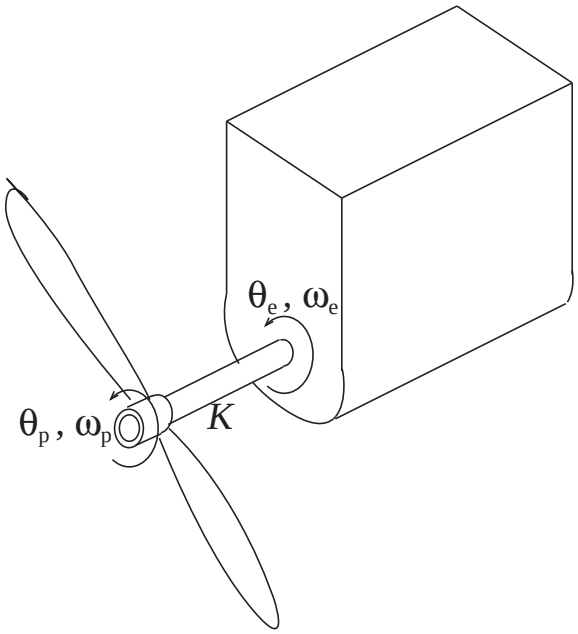


Figure 4.19.1: Torsional Vibration of Propellor Shaft

(a) The input is the given angular displacement of the engine end of the shaft

$$\theta_e = \Omega t + \epsilon \sin \frac{\Omega}{2} t$$

Take the output to be the angular displacement θ_p and angular velocity $\omega_p = d\theta_p/dt$ of the propellor end of the shaft. Assume that the that the torque T_{shaft} transmitted to the propellor from the shaft and the torque T_{fric} applied to the propellor by the air resistance can be modelled by the *linear* relations

$$T_{shaft} = K(\theta_p - \theta_e) \qquad \text{and} \qquad T_{fric} = B\theta_p$$

The torque balance requirement is

$$-T_{shaft} - T_{fric} = T_I = I\frac{d\omega_p}{dt}$$

These relations can be combined in the standard form for a state-determined system

$$\frac{d}{dt}\left\{\begin{array}{c}\theta_p\\ \omega_p\end{array}\right\} = \left[\begin{array}{cc}0 & 1\\ -K/I & -B/I\end{array}\right]\left\{\begin{array}{c}\theta_p\\ \omega_p\end{array}\right\} + \left\{\begin{array}{c}0\\ 1\end{array}\right\}\frac{K}{I}(\Omega t + \epsilon \sin\frac{\Omega}{2}t)$$

or in the form of a single second-order equation

$$I\frac{d^2\theta_p}{dt^2} + B\frac{d\theta_p}{dt} + K\theta_p = K(\Omega t + \epsilon \sin\frac{\Omega}{2}t)$$

The input motion consists of a uniform rotation at angular rate Ω plus a sinusoidal oscillation at frequency $\Omega/2$. Because the equation is *linear* the steady-state responses to these two inputs can be obtained separately and superposed to obtain the complete solution. For the uniform rotation, try a response of the form

$$\theta_{p,unif} = \Omega t + \text{Constant}$$

and find that it satisfies

$$I\frac{d^2\theta_{p,unif}}{dt^2} + B\frac{d\theta_{p,unif}}{dt} + K\theta_{p,unif} = K\Omega t$$

if the Constant $= -B\Omega/K$. this means that, after an initial transient, uniform engine speed results in a uniform propellor speed with a constant twist in the shaft just large enough for the shaft torque to balance the friction torque. For the sinusoidal oscillation, the response $\theta_{p,osc} = \psi(t)$ must satisfy the equation

$$I\frac{d^2\psi}{dt^2} + B\frac{d\psi}{dt} + K\psi = K\epsilon \sin\frac{\Omega}{2}t$$

A solution is sought in the form

$$\psi = \text{Im}\{A\ \exp(i\frac{\Omega}{2}t)\}$$

The complex amplitude A must satisfy the relation

$$\frac{A}{\epsilon} = \frac{\frac{K}{I}}{\frac{K}{I} - \frac{\Omega^2}{4} + i\frac{B\Omega}{2I}} = \frac{\omega_o^2}{\omega_o^2 - \frac{\Omega^2}{4} + i\zeta\omega_o\Omega}$$

where

$$\omega_o^2 = \frac{K}{I} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{B}{I}$$

This result is very similar to the result obtained in Problem 16.9, with the exception that here the firing frequency $\Omega/2$ plays the role of the driving frequency Ω in Problem 16.9. The ratio of the complex amplitude at frequency $\Omega/2$ to the response when $\Omega \to 0$ is

$$Z = \frac{A(\Omega/2)}{A(0)} = \frac{1}{[1 - \frac{(\Omega/2)^2}{\omega_o^2}] + i2\zeta\frac{(\Omega/2)}{\omega_o}}$$

The peak value of $|Z|$ is

$$|Z|_{peak} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

and that the peak amplitude is realized when the forcing frequency is

$$\Omega_{peak} = \omega_o\sqrt{1-2\zeta^2}$$

(b) To use the results of Problem 16.9, we need the ratio, $|Z|_{peak}$, of the peak response to the zero-frequency response and the magnitude of the forcing frequency which produces the peak response. Here we know that peak response occurs when the engine speed is 2200 rpm. The corresponding forcing frequency is $\Omega/2 = 1100$ rpm or 115.2 rad/sec. We do not know the zero-frequency response, but we do know that the ratio of the peak response, at 2200 rpm, to the response at 500 rpm is 4. We also know from Fig.2 that the response magnitude for low frequencies, well below the natural frequency, is not much greater than the zero-frequency response. Therefor we can take the given ratio of 4 as an estimate of the ratio $|Z|_{peak}$. With these estimates, we can use the results of Problem 16.9 to obtain estimates of the behavioral parameters ω_o and ζ.

$$\omega_o = \frac{\Omega_{peak}}{\sqrt{1-2\zeta^2}} = \frac{115.2}{\sqrt{1-2\zeta^2}} \qquad \text{and} \qquad |Z|_{peak} = 4 = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

The solution of these equations yields the estimates

$$\zeta = 0.1260 \qquad \text{and} \qquad \omega_o = 117.1 \text{ rad/sec}$$

from which follow

$$K = I\omega_o^2 = 2.811(117.1)^2 = 38,500 \text{ N-m/radian}$$

and

$$B = 2\zeta\omega_o I = 2(0.126)(117.1)(2.811) = 83.0 \text{ N-m/rad/sec}$$

as estimates of the model parameters.

4.20 Safe Packaging

The first two figures for this problem are completely irrelevant to the actual solution, they are only there to show the thought process you would go through if you were solving this problem on your own.

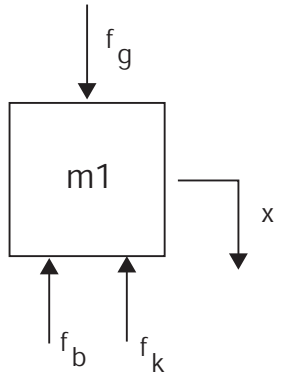


Figure 4.20.1: Free Body Diagram of instrument.

(a) From the FDB,

$$\begin{aligned} \Sigma F &= m_1\ddot{x} = f_g - f_k - f_b \\ \text{Where } f_g &= m_1 g \\ f_k &= kx \\ f_b &= b\dot{x} \\ m_1\ddot{x} &= m_1 g - b\dot{x} - kx \\ \ddot{(x)} + \frac{b}{m_1}\dot{x} + \frac{k}{m_1}x &= g \\ \text{with } x_0 &= 0,\ \dot{x}_0 = v_0 \end{aligned}$$

(b) For the system to oscillate ζ must be less than 1.

$$\begin{aligned} \omega_n^2 &= \sqrt{\frac{k}{m}} \\ 2\zeta\omega_n &= \frac{b}{m} \\ \zeta &= \frac{1}{2}\frac{b}{m}\sqrt{\frac{m}{k}} = \frac{b}{2\sqrt{km}} \\ \zeta &< 1 \\ b &< 2\sqrt{km} \end{aligned}$$

(c) Since this is a pretty straight forward 2nd order system we can just add the forced and free

responses

$$
\begin{aligned}
x(t) &= x_{free}(t) + x_{forced}(t) \\
x_{forced}(t) &= \frac{g}{k}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(t\omega_n\sqrt{1-\zeta^2}+\phi\right)+1\right] \\
x_{free}(t) &= \frac{v_0}{\omega_n\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(t\omega_n\sqrt{1-\zeta^2}\right)
\end{aligned}
$$

4.21 Sliding Damped

A mass m is subjected to an accelerating force f and a retarding friction force, modeled linearly as bv.

(a) The equation of motion is

$$m\frac{dv}{dt} + bv = f \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{b}{m}v + \frac{1}{m}f \tag{1}$$

(b) The steady-state velocity, when $f = 10$ Newtons, and $m = 1000$ kg, and $b = 100$ N/m/s is

$$v_{ss} = \frac{f}{b} = \frac{10}{100} = 0.10 \text{ m/s}$$

(c) Consider the input $f = f_a \sin \Omega t$ to be the imaginary part of the complex input $f_a \exp(i\Omega t)$ and look for a complex solution of the form $v = A\exp(i\Omega t)$. After substitution in (1) we find that this is indeed a solution , if

$$A = \frac{f_a}{b + im\Omega} = \frac{f_a}{b}\frac{1}{\sqrt{1 + (\frac{m\Omega}{b})^2}} \exp(i\phi)$$

where the phase angle ϕ is fixed by the relation

$$\tan\phi = -\frac{m\Omega}{b}$$

The steady-state solution to the input $f = f_a \sin \Omega t$ is then the imaginary part of the complex solution $A\exp(i\Omega t)$, which is

$$v = \frac{f_a}{b}\frac{1}{\sqrt{1 + (\frac{m\Omega}{b})^2}} \sin(\Omega t + \phi)$$

The magnitude of the response amplitude is

$$M(\Omega) = \frac{f_a}{b}\frac{1}{\sqrt{1 + (\frac{m\Omega}{b})^2}}$$

and the phase angle is

$$\phi = -\tan^{-1}\frac{m\Omega}{b}$$

For the case where $f_a = 10$ Newtons, the values of $M(\Omega)$ and $\phi(\Omega)$ for the frequencies $\Omega = 0.5$ rad/sec, $\Omega = 0.05$ rad/sec, and $\Omega = 0.005$ rad/sec are

(i) $\text{M}(0.5) = \frac{10}{100}\frac{1}{\sqrt{1+(\frac{(1000)(0.5)}{100})^2}} = 0.01961$ m/s,
$\phi(0.5) = \tan^{-1}(1000)(0.5)/100 = 1.373r = 78.7\,\text{deg}$

(ii) $\text{M}(0.05) = \frac{10}{100}\frac{1}{\sqrt{1+(\frac{(1000)(0.05)}{100})^2}} = 0.0894$ m/s,
$\phi(0.05) = \tan^{-1}(1000)(0.05)/100 = 0.463r = 25.6\,\text{deg}$

(iii) $\text{M}(0.005) = \frac{10}{100}\frac{1}{\sqrt{1+(\frac{(1000)(0.005)}{100})^2}} = 0.0999$ m/s,
$\phi(0.005) = \tan^{-1}(1000)(0.005)/100 = 0.050r = 2.86\,\text{deg}$

(d) The limiting value of the magnitude of the response amplitude $M(\Omega)$ when $\Omega \to 0$ is $A_o = f_a/b = 0.10$ m/s. The ratio $M(\Omega)/A_o$ is 0.1961 when $\Omega = 0.5$ rad/sec, 0.894 when $\Omega = 0.05$ rad/sec, and 0.999 when $\Omega = 0.005$ rad/sec.

(e) The low-frequency asymptote for $M(\Omega)$ is $A_o = f_a/b$, while the high-frequency asymptote is $f_a/m\Omega$. These asymptotes intersect at the break frequency $\Omega_{break} = b/m = 0.10$ rad/sec.

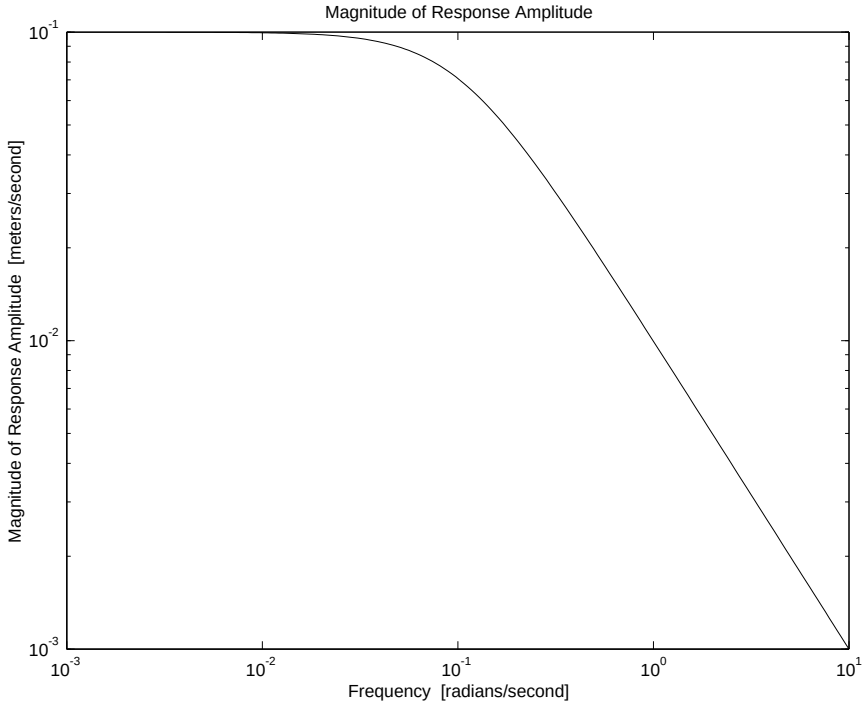


Figure 4.21.1: Magnitude of Response of Sliding Mass

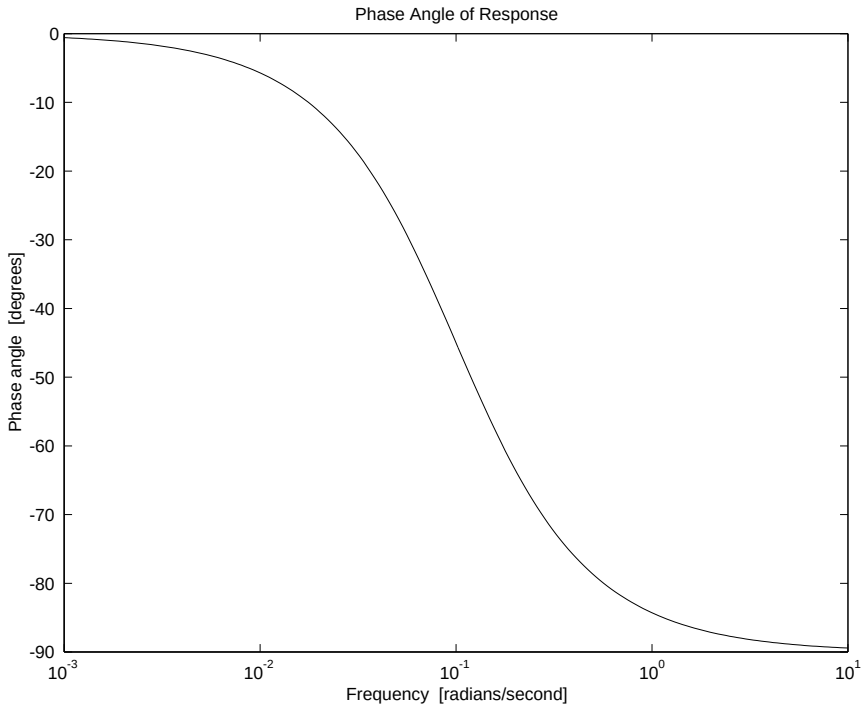


Figure 4.21.2: Phase Angle of Response of Sliding Mass

(f) The Bode plots shown in Fig. 4.21.1 and Fig. 4.21.2 were obtained by running the MATLAB script 'slidingsol.m'

4.22 Toy Flywheel

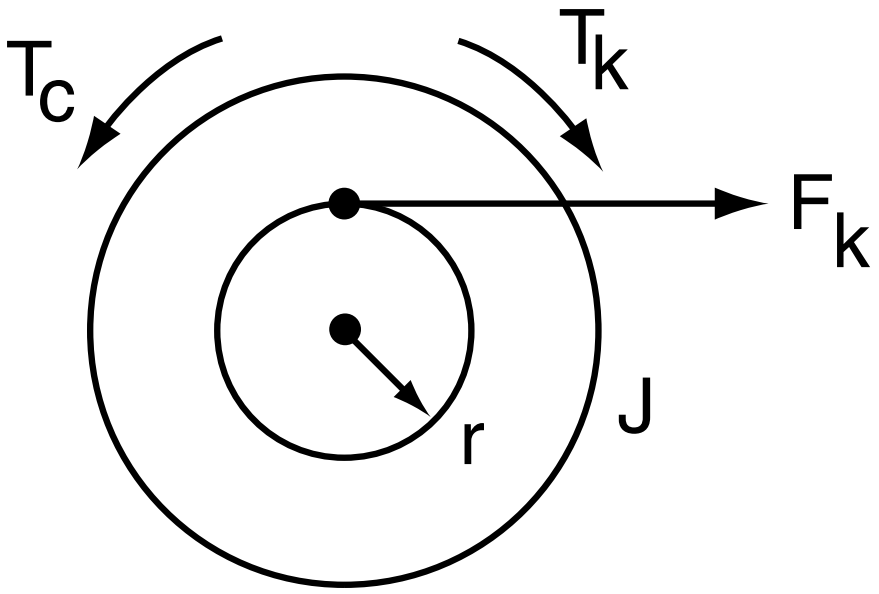


Figure 4.22.1: Cross-Sectional and Top view

a) Figure 4.22.1 shows the free body diagram for this system. Summing the torques

$$\begin{aligned} \Sigma T &= J\ddot{\theta} = T_k - Tc \\ T_k &= rF_k \\ F_k &= k(x_s - x) \\ T_c &= c_t\dot{\theta} \\ \text{so } J\ddot{\theta} + c_t\dot{\theta} &= rk(x_s - x) \\ J\ddot{\theta} + c_f\dot{\theta} + rkx &= rkx_s \\ \text{Recall that } r\theta &= x \Rightarrow \theta = \frac{x}{r} \end{aligned}$$

$$\text{so } \boxed{\tfrac{J}{r}\ddot{x} + \tfrac{c_t}{r}\dot{x} + rkx = rkx_s(t)}$$

This may be rewritten as

$$\ddot{x} + \frac{c_t}{J}\dot{x} + \frac{r^2k}{J}x = \frac{r^2k}{J}x_s(t)$$

(b) For this part we note the following relationships

$$2\zeta\omega_n \equiv \frac{c_t}{J}$$

$$\omega_n^2 \equiv \frac{r^2k}{J}$$

For the string not to go slack the system must not overshoot x_0, this means that $\zeta \geq 1$. For $\zeta \geq 1$ to be true

$$c_t \geq 2\sqrt{r^2kJ}$$

(c) For the string to go slack the system must be allowed to overshoot or $\zeta < 1$. In the case of $\zeta < 1$

$$\begin{aligned} x(t) &= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{\zeta\omega_n t} \cos(\omega_d t - \psi) \\ \omega_d &= \omega_n \sqrt{1-\zeta^2} \\ \psi &= \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \end{aligned}$$

This equation can be expressed in terms of the system variables by substituting for ω_n and ζ from 3. Figure 4.22.2 shows the time response for this system where t^* indicates the time after which the response is not valid.

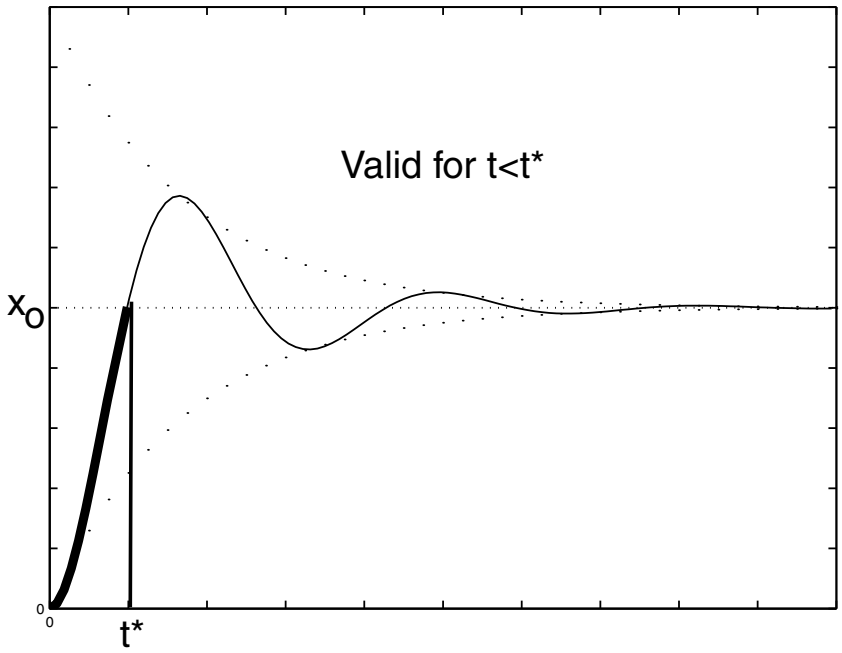


Figure 4.22.2: Time Response

(d) When $c_t = 0$, the expression for $x(t)$ becomes

$$x(t) = x_0(1 - \cos\omega_n t)$$

As we saw in part 3, t^* is the time at which the response passes x_0. In this case, it takes a quarter cycle to get to x_0, thus

$$t^* = \frac{1}{4}\frac{2\pi}{\omega_n}$$

Figure 4.22.3 shows the response for this system.

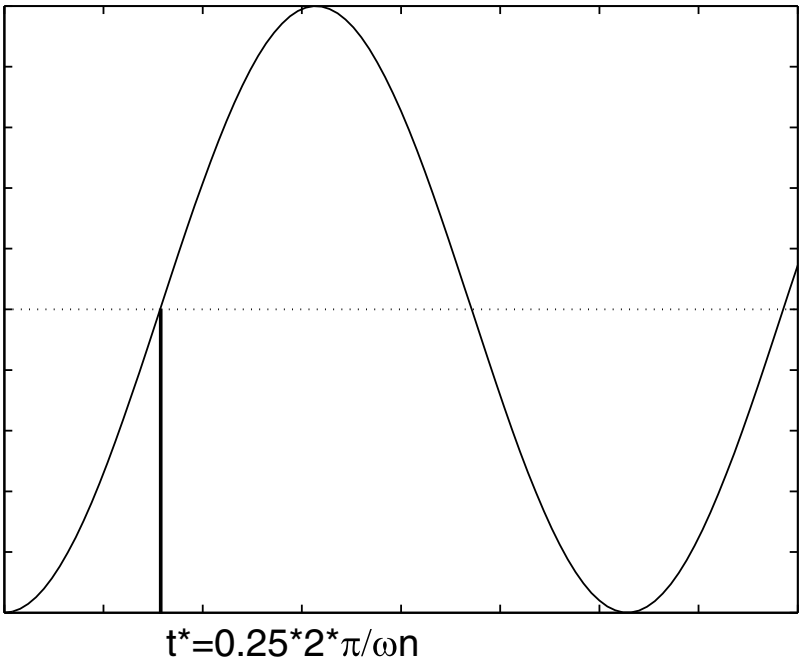


Figure 4.22.3: Step Response

4.23 Truck Shocks

The basic assumption here is that the back of the truck can be modeled as a plate of mass M supported by an equivalent spring of stiffness k and damping parameter b, onto which hops Uncle Massive, of mass m, which applies a step-function of force to the system, while changing the total mass of the system to $M + m$. Under these assumptions the theory developed in class can be used to estimate the effective mass M of the back of the truck from the observed frequency of oscillation and static deflection.

(a) The statement that the oscillation contained more than five clearly defined cycles is a tip-off that the system is lightly damped, which suggests that damping might be neglected in making a preliminary estimate. The static deflection Δ under a load of $w = mg$ tells us that the equivalent stiffness k can be estimated from $w = k\Delta$, or

$$k = \frac{w}{\Delta} = \frac{250 \text{ pounds}}{1.0 \text{ inch}} = 250 \text{ pounds/inch}$$

The observed oscillation frequency of 2 Hz can be taken as the undamped natural frequency, $\omega_o/2\pi$, of the system with mass $M + m$ and stiffness k. The truck mass M can be estimated by using the relation

$$\omega_o^2 = \frac{k}{M + m}$$

in combination with the preceding expression for k to obtain

$$M = \frac{mg}{\Delta\omega_o^2} - m \tag{1}$$

In the British system of units, mass is subordinate to force, and M is replaced by W/g where W is the equivalent weight of the back of the truck. To use British units, Eq.(1) is multiplied through by $g = 386$ in/sec^2 to get

$$W = \frac{wg}{\Delta\omega_o^2} - w = \frac{(250)(386)}{(1)(4\pi)^2} - 250 = 361 \text{ pounds}$$

Alternatively the given data can be converted to SI units ($\Delta = 0.0254$ m, and $m = 113.4$ kg),and Eq.(1) used to obtain $M = 163.8$ kg whose weight, $Mg = (163.8)(9.81) = 1607$ Newtons, is equivalent to the British weight of 361 pounds.

(b) It is assumed that the truck oscillation is a primarily vertical motion of the effective mass of the rear end of the truck plus Uncle Massive on an effective vertical spring (the rear tires plus suspension). It is also assumed that the observed *damped* natural frequency is a good approximation for the *undamped* natural frequency required in the formulas above.

4.24 Car Suspension 2

(a),(b) The differential equation for the system is

$$m\ddot{x} + b\dot{x} + kx = F, \tag{1}$$

The particular solution will be:

$$x_p(t) = F/k. \tag{2}$$

The homogenous solution has the form:

$$x_h(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t) + Be^{-\zeta\omega_n t}\sin(\omega_d t) \tag{3}$$

Plugging the initial conditions into the full solution

$$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t) + Be^{-\zeta\omega_n t}\sin(\omega_d t) + F/k \tag{4}$$
$$0 = A + F/k \tag{5}$$
$$A = -F/k \tag{6}$$
$$\dot{x}(t) = e^{-\zeta\omega_n t}\ (-A\omega_d \sin\omega_d t - A\zeta\omega_n \cos\omega_d t) + (B\omega_d \cos\omega_d t - B\zeta\omega_n \sin\omega_d t) \tag{7}$$
$$0 = -A\zeta\omega_n + B\omega_d \tag{8}$$
$$B = -\frac{A\zeta\omega_n}{\omega_d} \tag{9}$$

For $m = 500$kg, $k = 5 \times 10^4$N/m and $b = 2 \times 10^3$ Ns/m,

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \tag{10}$$
$$\zeta = \frac{c}{2\sqrt{km}} = 0.2 \tag{11}$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 9.8 \tag{12}$$
$$\sigma = \zeta\omega_n = 2; \tag{13}$$
$$A = -0.01 \tag{14}$$
$$B = 2.04 \cdot 10^{-3} \tag{15}$$
$$x(t) = -0.01e^{-2t}\cos(9.8t) - .00204e^{-2t}\sin(9.8t) + .01 \tag{16}$$

The result is plotted in Figure 4.24.1.

(c) The passengers will not like this ride because there are too many oscillations. when $b = 2 \cdot \sqrt{km} = 10000$, the system will be critically damped, in this case the complete solution has the form:

$$x(t) = Ae^{-\omega_n t} + Bte^{-\omega_n t} + F/k \tag{17}$$
$$x(0) = A + F/k \tag{18}$$
$$A = -F/k \tag{19}$$
$$A = -0.01 \tag{20}$$
$$\dot{x}(0) = -\omega_n A + -\omega_n Bte^{-\omega_n t} + Be^{-\omega_n t} \tag{21}$$
$$0 = -\omega_n A + B \tag{22}$$
$$B = \omega_n A \tag{23}$$
$$B = -0.1 \tag{24}$$
$$x(t) = -0.01e^{-\omega_n t} - 0.1te^{-\omega_n t} + 0.01 \tag{25}$$

In order to find the acceleration for the under-damped case, differentiate equation 16 twice with respect to time,

$$x(t) = -0.01e^{-2t}\cos(9.8t) - .00204e^{-2t}\sin(9.8t) + .01 \tag{26}$$
$$\dot{x}(t) = .10208e^{-2t}\sin 9.8t \tag{27}$$
$$\ddot{x}(t) = e^{-2t}\cos 9.8t - .20416^{-2t}\sin 9.8t \tag{28}$$

The maximum acceleration will happen at $t = 0^+$

$$\ddot{x}(t = 0+) = 1 \tag{29}$$
$$\tag{30}$$

So the maximum acceleration is $1\text{m}/sec^2$.

Similarly, for the critically damped case, differentiating equation 25 twice,

$$x(t) = -0.01e^{-\omega_n t} - 0.1te^{-\omega_n t} + 0.01 \tag{31}$$
$$\ddot{x}(t) = (1 - 10t)e^{-10t} \tag{32}$$
$$\ddot{x}(0+) = 1 \tag{33}$$

So the accelerations will be the same.

(d) From the step response, the 5% settling time is 1.35 sec in the under-damped case and 0.47 sec in the critically-damped case.

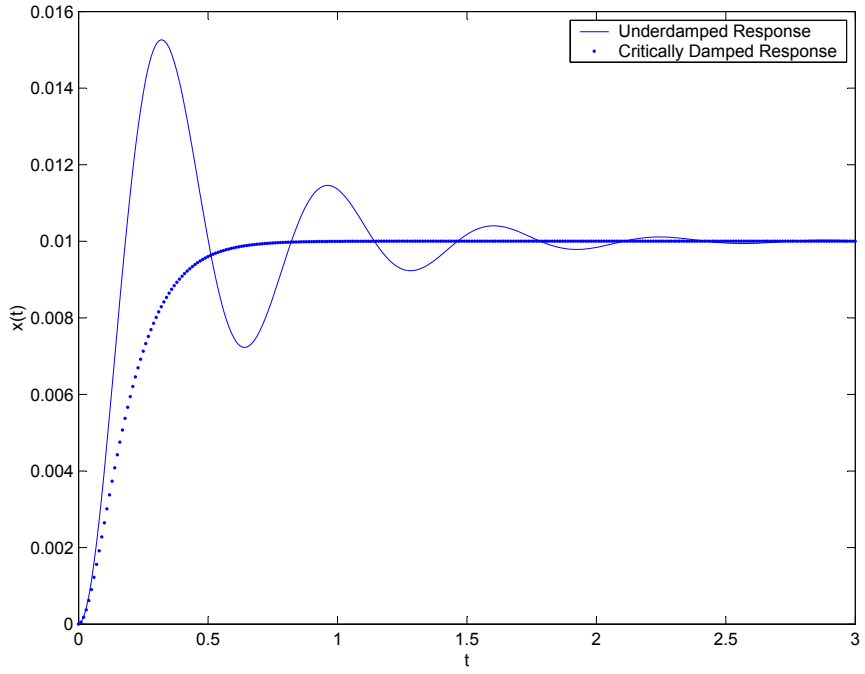
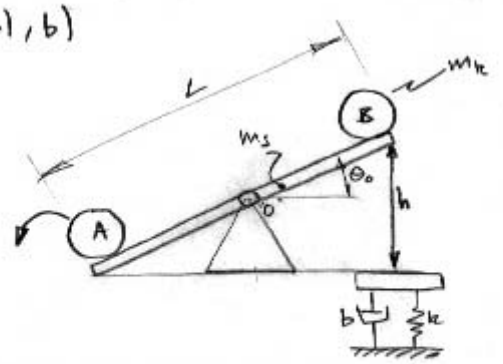


Figure 4.24.1: Step Response

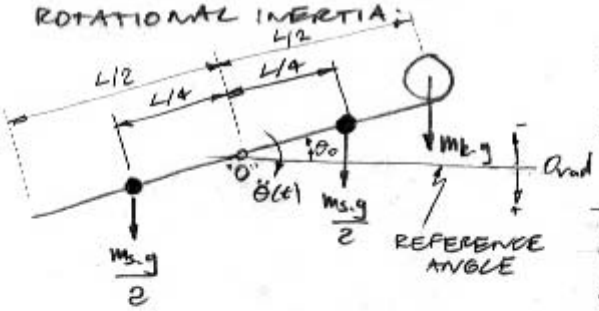
4.25 Kid-Skid

P4: a), b)



IF THE KID "A" JUMPS OFF THE SEE-SAW THEN THE KID "B" WILL FALL DOWN FROM REST POSITION. THE KID "B" WILL REACH A FINAL VELOCITY JUST BEFORE LANDING ON THE SKID. THIS WILL BE FOUND AS FOLLOWS:

THE KID "B" AND SEE-SAW CAN BE MODELED AS A ROTATIONAL INERTIA:



$$J_o = 2m_s\left(\frac{L}{4}\right)^2 + m_k\left(\frac{L}{2}\right)^2$$

$$J_o = \frac{L^2}{8}(m_s + 2m_k)$$

THE INITIAL CONDITIONS ARE AS FOLLOWS:

$$\dot{\theta}(t=0) = 0$$

$$\theta_o = \theta(t=0) = -\sin^{-1}\left(\frac{h}{L}\right)$$

$$\sum M_{o\,z} = J_o.\ddot{\theta}(t) = \left(\frac{m_s.g.\frac{L}{4}}{2} - \frac{m_s.g.\frac{L}{4}}{2} + m_k.g.\frac{L}{2}\right)\cos\theta(t)$$

WE ASSUME THAT $L >> h$ SO THAT $\cos\theta(t) \cong 1$ AND $\sin\theta(t) \cong \theta(t)$

SO $J_o.\ddot{\theta}(t) = m_k.g.\frac{L}{2}$ AND $\theta(t=0) = -h/L$

$$\dot{\theta}(t) = \frac{m_k.g.L}{2J_o}.t + C_1 \;;\; \dot{\theta}(t=0) = 0 \Rightarrow C_1 = 0$$

$$\theta(t) = \frac{m_k.g.L}{4J_o}.t^2 + C_2 \;;\; \theta(t=0) = -h/L \Rightarrow C_2 = -h/L$$

$$\theta(t) = \frac{m_k.g.L}{4J_o}.t^2 - \frac{h}{L}$$

KID "B" WILL REACH THE SKID WHEN $\theta = h/L$

$$\Rightarrow \theta(t_1) = \frac{m_k.g.L}{4J_o}.t^2 - \frac{h}{L} = \frac{h}{L} \Rightarrow t_1 = \sqrt{\frac{8J_o h}{m_k.g.L^2}}$$

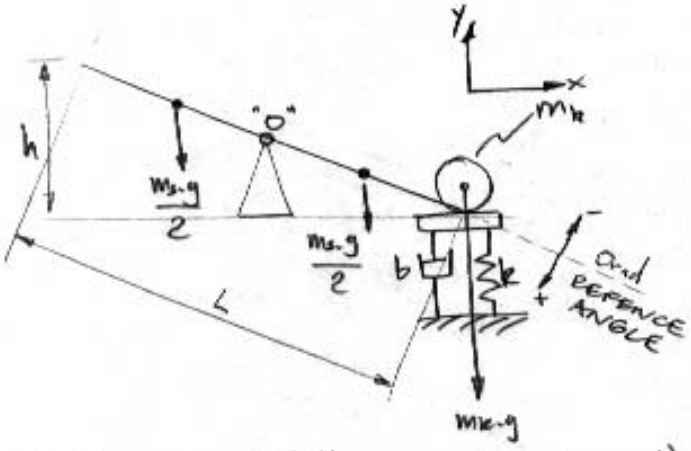
Figure 4.25.1:

$$t_1 = \frac{2}{L}\sqrt{\frac{2J_o \cdot h}{m_k \cdot g}}$$

$$\dot{\theta}(t_1) = \frac{m_k \cdot g \cdot L}{2J_o}\left(\frac{2}{L}\sqrt{\frac{2J_o h}{m_k g}}\right) = \sqrt{\frac{2m_k g \cdot h}{J_o}}$$

$$\dot{\theta}(t_1) = \sqrt{\frac{2m_k \cdot g \cdot h}{J_o}}$$

NOW WE CONSIDER A MODEL WITH THE KID-SKIA:



FOR THIS MODEL TIME WILL BE TAKEN AS:

$$t' = t - t_1$$

SINCE $L >> h$ THEN:

$$y(t') = \theta(t') \cdot \frac{L}{2}$$

$$\sum M_{o+} = J_o \cdot \ddot{\theta}(t') = m_k \cdot g \cdot \frac{L}{2}$$

$$\sum M_{o+} = J_o \cdot \ddot{\theta}(t') = m_k g \frac{L}{2} - k \cdot y(t') - b \cdot \dot{y}(t')$$

$$J_o \ddot{\theta}(t') + b \cdot \frac{L}{2}\dot{\theta}(t') + k \cdot \frac{L}{2} \cdot \theta(t') = \frac{m_k \cdot g \cdot L}{2}$$

EQUATION OF MOTION:

$$\ddot{\theta}(t') + \frac{b \cdot L}{2J_o}\dot{\theta}(t') + \frac{k \cdot L}{2J_o}\theta(t') = \frac{m_k \cdot g \cdot L}{2J_o}$$

INITIAL CONDITIONS:

$$\dot{\theta}(t = t_1) = \dot{\theta}(t' = 0) = \sqrt{\frac{2m_k \cdot g \cdot h}{J_o}}$$

c) RELATING y_{max} TO OUR EQUATION OF MOTION WILL GIVE US:

$$\theta_{max} = \frac{2 \cdot y_{max}}{L}$$

WE NEED COMPARE OUR EQUATION OF MOTION WITH THE FOLLOWING

$$\ddot{\theta}(t') + 2\zeta\omega_n \cdot \dot{\theta}(t') + \omega_n^2 x = \frac{T}{J_o}$$

Figure 4.25.2:

WE RECOGNIZE $\omega_n^2 = \frac{k \cdot L}{2J_0} \Rightarrow \omega_n = \sqrt{\frac{k \cdot L}{2J_0}}$

$T = \frac{m_k \cdot g \cdot L}{2}$ AND $2\zeta\omega_n = \frac{b \cdot L}{2J_0}$

$\Rightarrow \zeta = \frac{bL}{4J_0}\sqrt{\frac{2J_0}{kL}} \Rightarrow \zeta = \frac{b}{4}\sqrt{\frac{2 \cdot L}{k \cdot J_0}}$

θ OVERSHOOT $\approx 1 - \frac{\zeta}{0{,}6} \Rightarrow \theta_{MAX} = \frac{T}{k \cdot L/2}\left(1 + 1 - \frac{\zeta}{0{,}6}\right)$

$\theta_{MAX} = \frac{m_k \cdot g}{k}\left(2 - \frac{\zeta}{0{,}6}\right)$ WITH $\zeta = \frac{b}{4}\sqrt{\frac{2 \cdot L}{k \cdot J_0}}$

SO $\boxed{y_{MAX} = \frac{m_k \cdot g \cdot L}{2k}\left(2 - \frac{b}{2A}\sqrt{\frac{2 \cdot L}{k \cdot J_0}}\right)}$

TO AVOID EXCEEDING THE KID SKID y_{MAX} IS A MATTER OF DESIGNING THE SKID WITH AN APPROPIATE STIFFNESS "k" AND DAMPING "b". IF THE STIFFNESS IS GIVEN AS A FIXED PARAMETER THEN INCREASING THE DAMPING WILL DECREASE THE KID-SKID MAXIMUM DISPLACEMENT.

Figure 4.25.3:

4.26 Rolling Machine

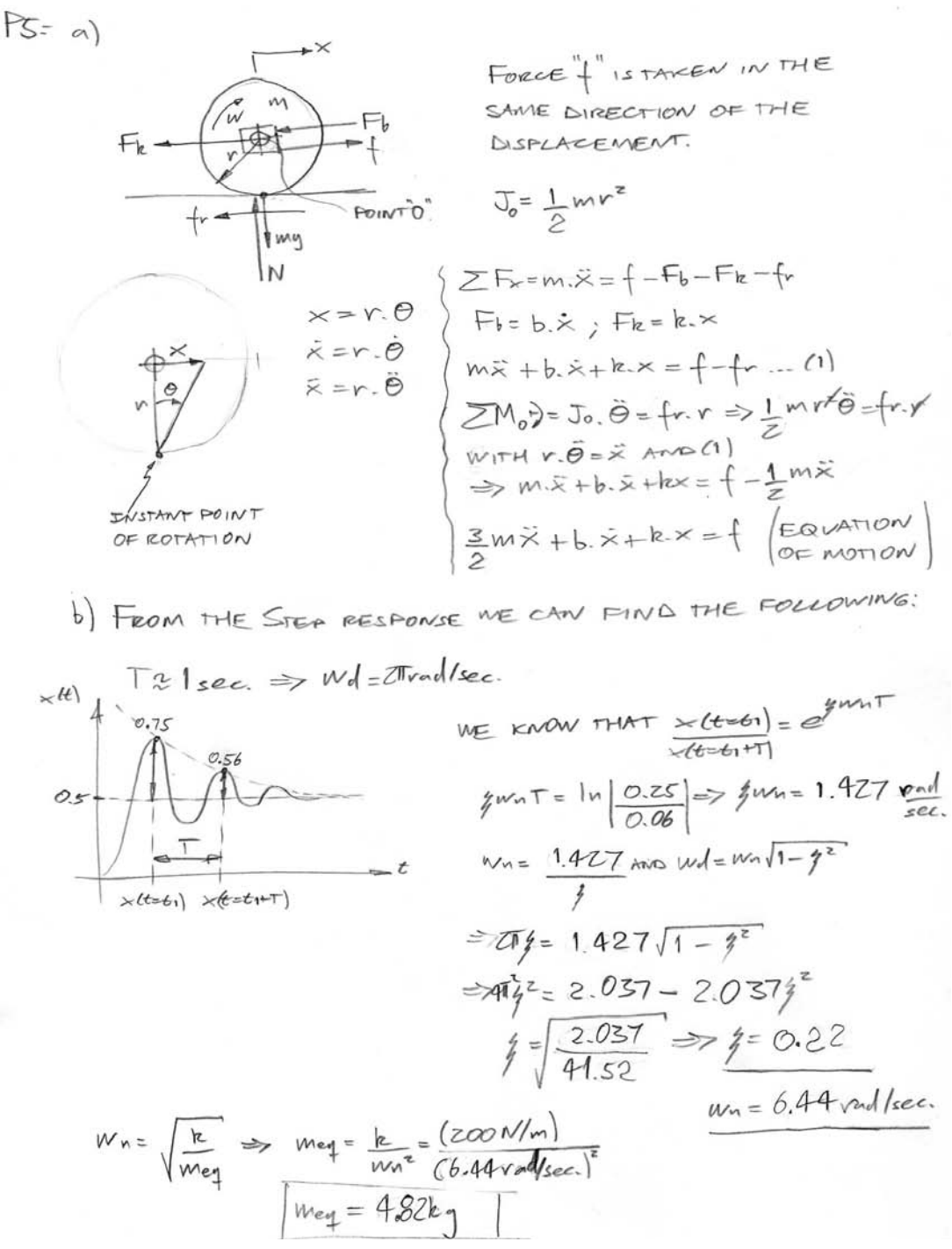


Figure 4.26.1:

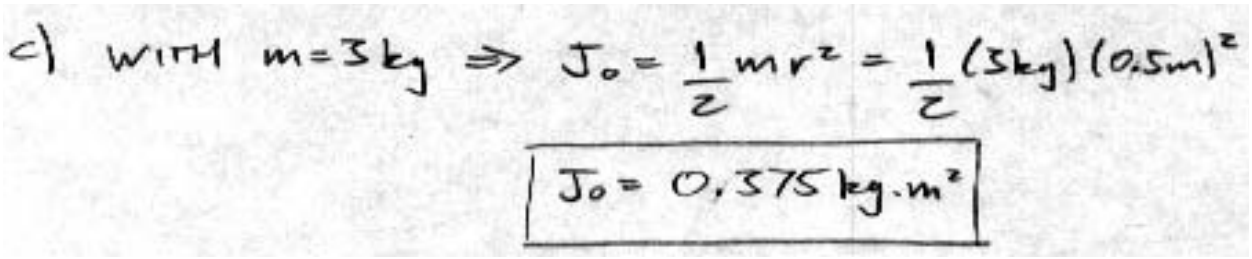
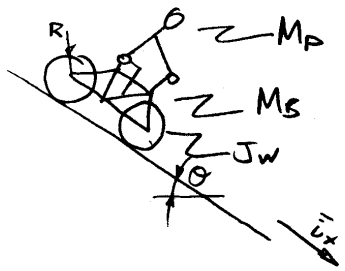


Figure 4.26.2:

4.27 Sunday Bike Ride

2.003 MODELING DYNAMICS AND CONTROL I
SPRING 2004
PROBLEM SET 1

P1: a), b), c), d)



M_P: MASS OF PERSON [kg]

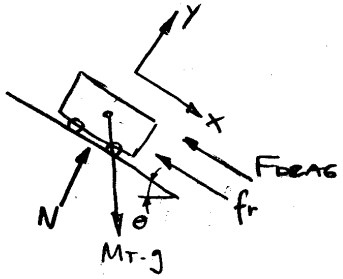
M_B: MASS OF BIKE [kg]

J_W: WHEEL INERTIA [kg.m]

WHEELS INERTIA IS CONSIDERED EQUAL FOR BOTH RIDERS, AND CONSIDERED WITHIN THE TOTAL INERTIA MASS AS FOLLOWS:

$M_{JW} = f(J_W, R) \Rightarrow M_T = M_P + M_B + M_{JW}$

BIKE MODELED AS A MASS GOING DOWNHILL OVER ROLLING ELEMENTS



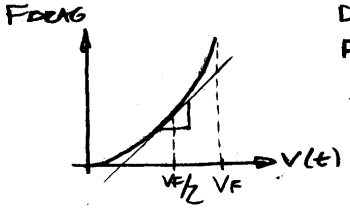
$M_P + M_B = M_T$

$F_{DRAG} = B \cdot v(t)^2$

$B = \frac{1}{2} \cdot \rho \cdot A$ (ASSUMED EQUAL FOR BOTH PERSONS)

(ρ: AIR DENSITY; A: PROJECTED AREA)

WE MAKE THE FINAL VELOCITY V_F AND LINEARIZE THE DRAG FORCE AS FOLLOWS:



$F_{DRAG} = B \cdot V_F \cdot v(t)$

So $\sum F_x = M_T \cdot \dot{v}(t) = M_T \cdot g \cdot \sin\theta - f_r - F_{DRAG}$

(EQUATION OF MOTION) $\Rightarrow M_T \cdot \dot{v}(t) + B \cdot V_F \cdot v(t) - M_T \cdot g \cdot \sin\theta - f_r = 0$

$$\dot{v}(t) + \frac{B \cdot V_F}{M_T} \cdot v(t) - g \cdot \sin\theta - \frac{f_r}{M_T} = 0$$

$$\dot{v}(t) + K_1 \cdot v(t) - K_2 = 0$$

(WITH: $K_1 = \frac{B \cdot V_F}{M_T}$; $K_2 = g \cdot \sin\theta + \frac{f_r}{M_T}$)

Figure 4.27.1:

HOMOGENEOUS EQUATION: $\ddot{x}(t) + K_1 \cdot \dot{x}(t) = 0$

$v(t) = A \cdot e^{s \cdot t}$

$\dot{v}(t) = A \cdot s \cdot e^{s \cdot t}$

$\Rightarrow A \cdot s \cdot e^{s \cdot t} + K_1 \cdot A \cdot e^{s \cdot t} = 0$

$s_1 = -K_1$; $s_2 = 0$ OR DOESN'T EXIST

NOW WE COULD GUESS OR THINK?

LET'S THINK.

SHOULDN'T THE VELOCITY START INCREASING UNTIL A MAXIMUM VALUE AT WHICH THE DRAG FORCE PREVENTS THE BIKE FROM ACCELERATING.

WELL THEN THE VELOCITY VS. TIME SHOULD BE AS FOLLOWS:

SO $v(t) = C \cdot (1 - e^{s_1 \cdot t})$

$\dot{v}(t) = -C \cdot s_1 \cdot e^{s_1 \cdot t}$

REPLACING IN THE EQUATION OF MOTION:

$-C \cdot s_1 \cdot e^{s_1 \cdot t} + K_1 \cdot C \cdot (1 - e^{s_1 t}) - K_2 = 0$

$C \cdot K_1 \cdot e^{-K_1 \cdot t} + K_1 \cdot C \cdot (1 - e^{-K_1 \cdot t}) - K_2 = 0$

INITIAL CONDITIONS: $v(t=0) = 0$

$v(t=0) = C \cdot (1 - e^0) = 0$ OK

NOW $C \cdot K_1 e^{-K_1(0)} + K_1 \cdot C \cdot (1 - e^{-K_1(0)}) - K_2 = 0$

$C \cdot K_1 = K_2 \Rightarrow C = \frac{K_2}{K_1}$

SO $v(t) = \frac{K_2}{K_1} \cdot (1 - e^{-K_1 \cdot t})$

REPLACING K_1 AND K_2

$$v(t) = \frac{M_T \cdot g \cdot \sin\theta - f_r}{B \cdot V_F} \left(1 - e^{-\left(\frac{B \cdot V_F}{M_T}\right) \cdot t}\right)$$

Figure 4.27.2:

NOW LET'S SAY THAT THE BIKERS HAVE ACCELERATED CLOSE ENOUGH TO THEIR MAXIMUM VELOCITY SO $\left(1-e^{-\left(\frac{B\cdot V_F}{M_T}\right)\cdot 6}\right)\longrightarrow 1$

$$V_F \approx MAX\{v(t)\} = \frac{M_T\cdot g\cdot \sin\theta - fr}{B\cdot V_F} \Rightarrow V_F \approx \sqrt{\frac{M_T\cdot g\cdot \sin\theta - fr}{B}}$$

FRICTION COMES FROM ROLLING AND CAN BE ASSUMED ALMOST EQUAL FOR BOTH BIKERS.

SO IT IS CLEAR THAT THE BIKER WITH GREATER TOTAL MASS WILL APPROACH A GREATER VELOCITY AT THE BOTTOM OF THE HILL.

WHY WOULD YOU BUY A LIGHTER BIKE?

WELL DO ALL THIS ANALYSIS GOING UP THE HILL AND YOU WILL KNOW THE ANSWER.

YOU WANT A FASTER ANSWER TO THE ORIGINAL QUESTION? WELL AT A MAXIMUM VELOCITY THE ACCELERATION IS ZERO, SO LOOK AT YOUR EQUATION OF MOTION

$$M_T\cdot \cancel{\dot{v}(t_1)}^{\,0} + B\cdot V_F\cdot v(t_1) - M_T\cdot g\cdot \sin\theta - fr = 0$$

$$\Rightarrow v(t_1) = V_F = \frac{M_T\cdot g\cdot \sin\theta - fr}{B\cdot V_F} \Rightarrow V_F = \sqrt{\frac{M_T\cdot g\cdot \sin\theta - fr}{B}}$$

YOU COULD FIND AN ANSWER WITHOUT MAKING THE F_{DRAG} LINEAR ALSO.

Figure 4.27.3:

4.28 Compound Mass Spring Damper System 1

(a) The free body diagram for the mass m is shown in Figure 4.28.1 and the free body diagrams for nodes n_1 and n_2 are shown in Figure 4.28.2.

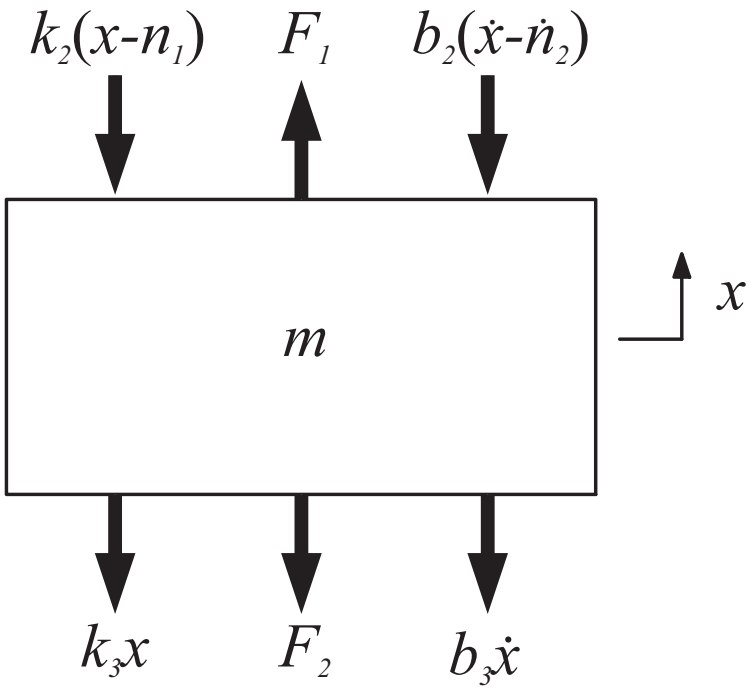


Figure 4.28.1: Free body diagram for mass m

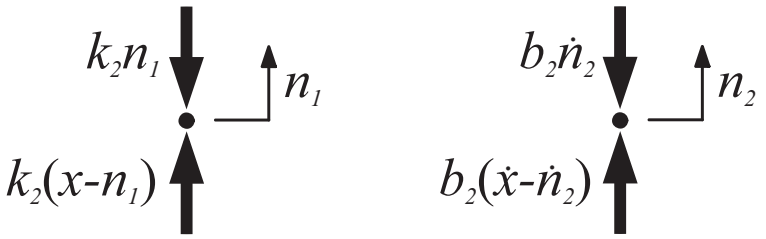


Figure 4.28.2: Free body diagrams for nodes n_1 and n_2

(b) We begin by writing the force balance equation for node n_1, keeping in mind that the "effective

mass" for this node is 0.

$$\begin{aligned} k_2(x - n_1) - k_1 n_1 &= m_{n_1}\ddot{n}_1 \\ k_2 x - (k_2 + k_1)n_1 &= 0 \\ k_2 x &= (k_2 + k_1)n_1 \\ \frac{k_2}{k_2 + k_1}x &= n_1 \end{aligned} \tag{1}$$

We do the same for node n_2

$$\begin{aligned} b_2(\dot{x} - \dot{n}_2) - b_1\dot{n}_2 &= m_{n_2}\ddot{n}_2 \\ b_2\dot{x} - (b_2 + b_1)\dot{n}_2 &= 0 \\ b_2\dot{x} &= (b_2 + b_1)\dot{n}_2 \\ \frac{b_2}{b_2 + b_1}\dot{x} &= \dot{n}_2 \end{aligned} \tag{2}$$

Now we write out the force balance equation for the mass m.

$$\begin{aligned} F_1 - F_2 - k_2(x - n_1) - b_2(\dot{x} - \dot{n}_2) - k_3 x - b_3\dot{x} &= m\ddot{x} \\ F_1 - F_2 - (k_2 + k_3)x - (b_2 + b_3)\dot{x} + k_2 n_1 + b_2\dot{n}_2 &= m\ddot{x} \end{aligned} \tag{3}$$

We have equations for the positions $n_1(t)$ and $n_2(t)$, so we substitute equations (1) and (2) into (3)

$$\begin{aligned} F_1 - F_2 - (k_2 + k_3)x - (b_2 + b_3)\dot{x} + k_2\frac{k_2}{k_2 + k_1}x + b_2\frac{b_2}{b_2 + b_1}\dot{x} &= m\ddot{x} \\ m\ddot{x} + \left(b_2 + b_3 - \frac{b_2^2}{b_1 + b_2}\right)\dot{x} + \left(k_2 + k_3 - \frac{k_2^2}{k_1 + k_2}\right)x &= F_1 - F_2 \\ m\ddot{x} + \left(\frac{b_2(b_1 + b_2)}{b_1 + b_2} + b_3 - \frac{b_2^2}{b_1 + b_2}\right)\dot{x} + \left(\frac{k_2(k_1 + k_2)}{k_1 + k_2} + k_3 - \frac{k_2^2}{k_1 + k_2}\right)x &= F_1 - F_2 \\ m\ddot{x} + \left(\frac{b_1 b_2}{b_1 + b_2} + b_3\right)\dot{x} + \left(\frac{k_1 k_2}{k_1 + k_2} + k_3\right)x &= F_1 - F_2 \\ m\ddot{x} + b_{eq}\dot{x} + k_{eq}x &= F_1 - F_2 \end{aligned} \tag{4}$$

where

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2} + k_3 \text{ and } b_{eq} = \frac{b_1 b_2}{b_1 + b_2} + b_3$$

Equation (4) is our final 2^{nd} order differential equation, but it was a lot of algebra to get to. There is a simpler way to derive the equation if you first simplify the Free Body Diagrams. Looking back to the original problem diagram, we can simplify the two springs above the mass into one equivalent spring. Remember that the effective spring constant for springs in parallel is their summation, and the effective spring constant for springs in series is the reciprocal of of the summation of their inverses, simplified to be their product over their summation:

$$k_{parallel} = k_1 + k_2 \text{ and } k_{series} = \frac{k_1 k_2}{k_1 + k_2} \tag{5}$$

This equivalent model is also valid for damping coefficients. We use equation (5) to redraw the free body diagram as shown below in Figure 4.28.3.

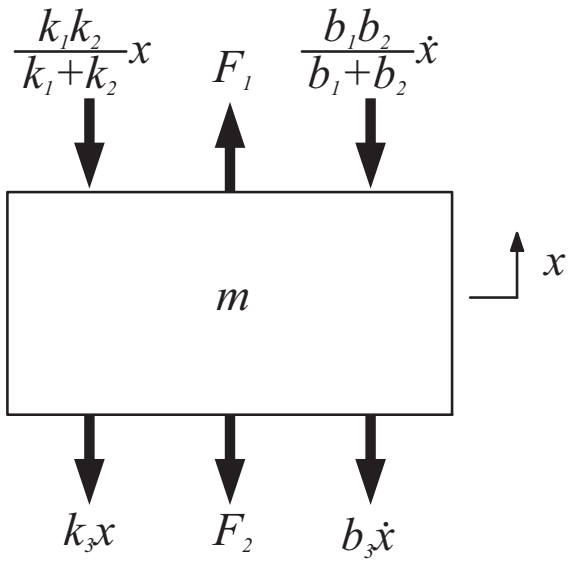


Figure 4.28.3: Force balance for mass m with equivalent constants

We now use this diagram to write out the force balance equation:

$$\begin{aligned} F_1 - F_2 - \frac{k_1k_2}{k_1+k_2}x - \frac{b_1b_2}{b_1+b_2}\dot{x} + k_3x + b_3\dot{x} &= m\ddot{x} \\ m\ddot{x} + \left(\frac{b_1b_2}{b_1+b_2} + b_3\right)\dot{x} + \left(\frac{k_1k_2}{k_1+k_2} + k_3\right)x &= F_1 - F_2 \\ m\ddot{x} + b_{eq}\dot{x} + k_{eq}x &= F_1 - F_2 \end{aligned} \tag{6}$$

Equations (4) and (6) are identical, but equation (6) is much easier to derive.

(c) Notice that the differential equation in (6) is almost in a standard form. We can devide equation (6) through by the effective spring constant of the system to arrive at the standard form in equation (7), where $f(t)$ is some set of forces.

$$\begin{aligned} \frac{m}{k_{eq}}\ddot{x} + \frac{b_{eq}}{k_{eq}}\dot{x} + x &= \frac{F_1 - F_2}{k_{eq}} \\ \frac{1}{\omega_n^2}\ddot{x} + \frac{2\zeta}{\omega_n}\dot{x} + x &= f(t) \end{aligned} \tag{7}$$

By comparing these equations, we can come up with expressions for ω_n and ζ, and plug in the appropiate system values for the numberical answers.

$$\begin{aligned} \omega_n &= \sqrt{\frac{k_{eq}}{m}} = 100 \text{ [rad/sec]} \\ \zeta &= \frac{b_{eq}}{2\sqrt{k_{eq}m}} = .05 \\ \omega_d &= \omega_n \sqrt{1-\zeta^2} = 99.875 \text{ [rad/sec]} \end{aligned}$$

The poles for underdamped second order systems are located at $s = -\zeta\omega_n \pm \omega_d$. The s-plane plot of this system is shown in Figure 4.28.4.

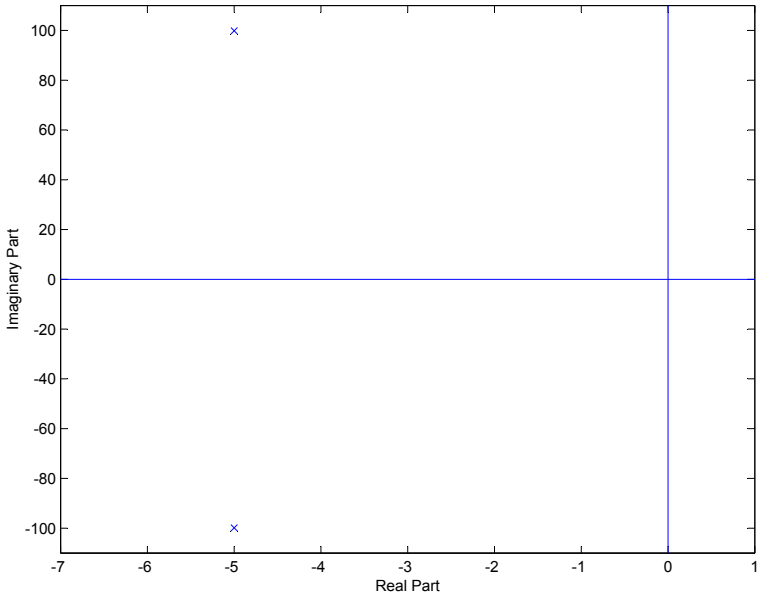


Figure 4.28.4: S-plane plot of system poles

(d) The general form for the solution $x(t)$ of equation (6) is

$$x(t) = Ae^{-\sigma t}\cos\omega_d t + Be^{-\sigma t}\sin\omega_d t$$

where $\sigma = \zeta\omega_n$. We find A by plugging in the initial condition $x(0) = 10^{-2}$:

$$\begin{aligned} Ae^{-\sigma 0}\cos\omega_d 0 + Be^{-\sigma 0}\sin\omega_d 0 &= 10^{-2} \\ A &= 10^{-2} \end{aligned}$$

B is found by plugging in the intial condition $\dot{x}(0) = 0$:

$$\begin{aligned} Ae^{-\sigma t}\cos\omega_d t + Be^{-\sigma t}\sin\omega_d t &= x(t) \\ e^{-\sigma t}\big((-A\omega_d\sin\omega_d t - A\sigma\cos\omega_d t) + (B\omega_d\cos\omega_d t - B\sigma\sin\omega_d t)\big) &= \dot{x}(t) \\ -A\sigma + B\omega_d &= 0 \\ B = \frac{A\sigma}{\omega_d} &= 10^{-2}\frac{\sigma}{\omega_d} \end{aligned}$$

Use these results to plot $x(t)$ in matlab. The plot is shown in Figure 4.28.5.

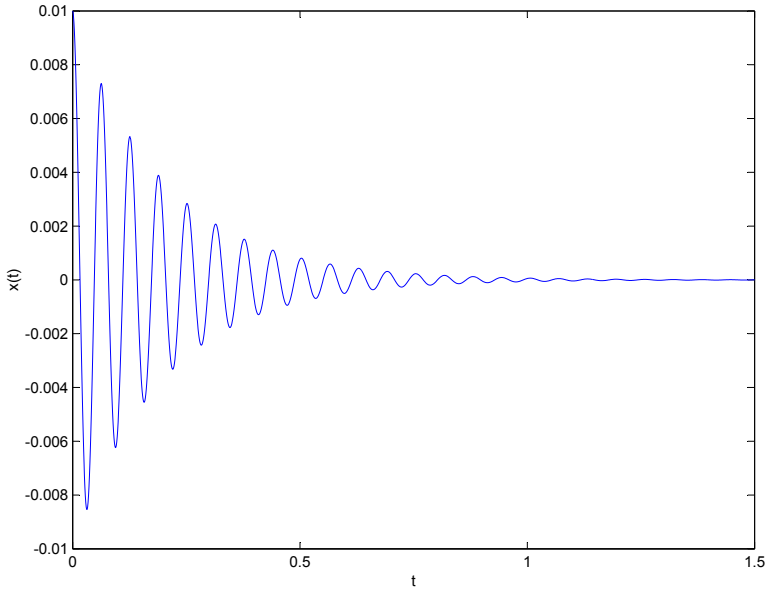


Figure 4.28.5: Plot of $x(t)$ when $x(0) = 10^{-2}$, $\dot{x}(0) = 0$

4.29 Compound Mass Spring Damper System 2

Begin by taking equation (4) from the solution of Problem 4.28 as the force balance in this problem:

$$m\ddot{x} + b_{eq}\dot{x} + k_{eq}x = F_1 - F_2$$

where

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2} + k_3 \text{ and } b_{eq} = \frac{b_1 b_2}{b_1 + b_2} + b_3$$

The problem states that $F_1 = 10$ [N], $F_2 = 0$ [N], and $x(0) = \dot{x}(0) = 0$. Rewrite the force balance to be

$$m\ddot{x} + b_{eq}\dot{x} + k_{eq}x = 10$$

The solution to a driven second order differential equation will be the sum of the particular solution and the homogeneous solution. Start by finding the particular solution. Guess that $x_p(t)$ will have the form $x_p(t) = C$, $\dot{x}_p(t) = 0$.

$$\begin{aligned} m\ddot{x}_p + b_{eq}\dot{x}_p + k_{eq}x_p &= 10 \\ 0 + 0 + k_{eq}x_p &= 10 \\ x_p &= \frac{10}{k_{eq}} \end{aligned}$$

We know the homogeneous solution of this equation from the solution of Problem 4.28:

$$x_h = Ae^{-\sigma t}\cos\omega_d t + Be^{-\sigma t}\sin\omega_d t$$

The total solution is the sum of the particular and homogeneous solutions.

$$x(t) = Ae^{-\sigma t}\cos\omega_d t + Be^{-\sigma t}\sin\omega_d t + \frac{10}{k_{eq}}$$

we find the constants A and B by plugging in the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$:

$$\begin{aligned} Ae^{-\sigma 0}\cos\omega_d 0 + Be^{-\sigma 0}\sin\omega_d 0 + \frac{10}{k_{eq}} &= 0 \\ A + \frac{10}{k_{eq}} &= 0 \\ A &= -\frac{10}{k_{eq}} \end{aligned}$$

and

$$\begin{aligned} Ae^{-\sigma t}\cos\omega_d t + Be^{-\sigma t}\sin\omega_d t + \frac{10}{k_{eq}} &= x(t) \\ e^{-\sigma t}\big((-A\omega_d\sin\omega_d t - A\sigma\cos\omega_d t) + (B\omega_d\cos\omega_d t - B\sigma\sin\omega_d t)\big) &= \dot{x}(t) \\ -A\sigma + B\omega_d &= 0 \\ B = \frac{A\sigma}{\omega_d} &= -\frac{10}{k_{eq}}\frac{\sigma}{\omega_d} \end{aligned}$$

A sketch of this solution is found in Figure 4.29.1

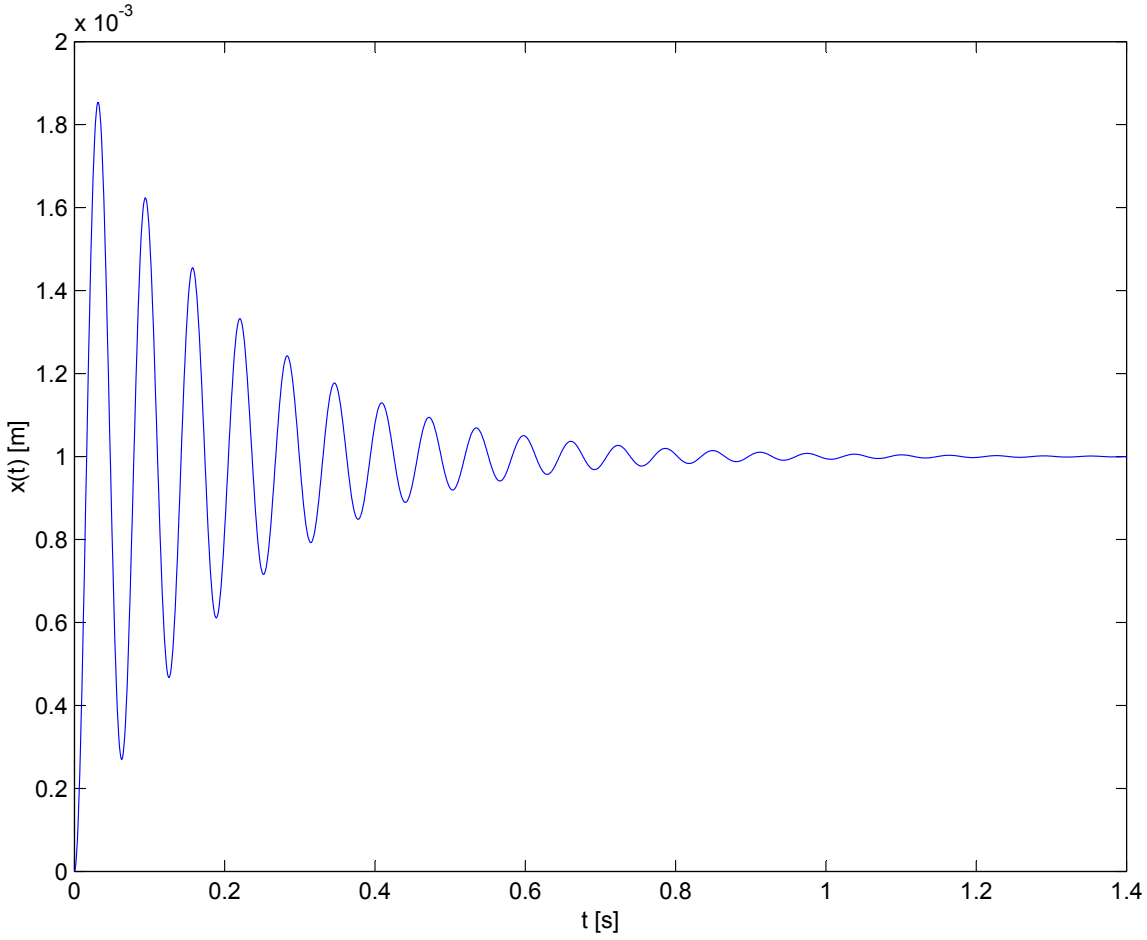
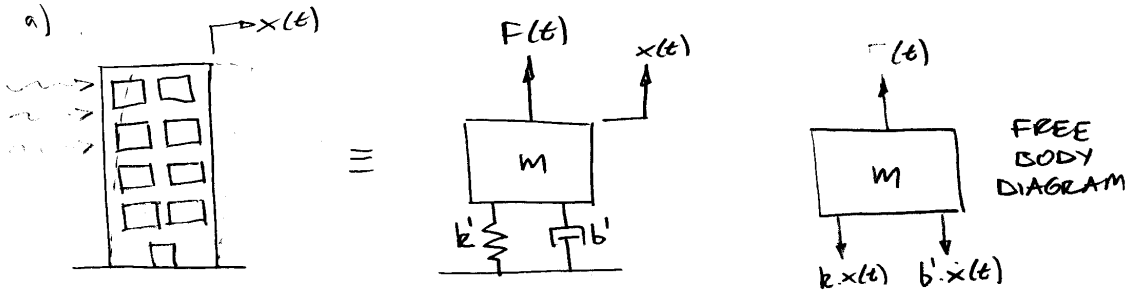


Figure 4.29.1: Plot of $x(t)$ when $x(0) = 0$, $\dot{x}(0) = 0$, $F_1 = 10$ [N]

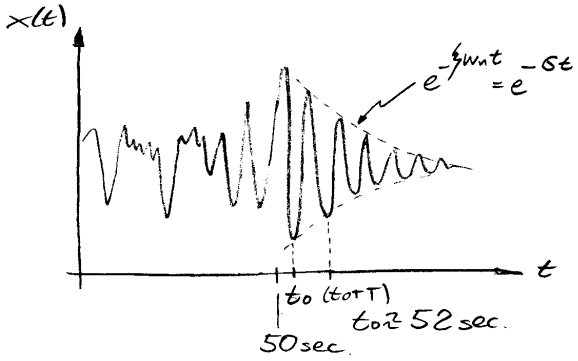
4.30 Wind Induced Building Vibrations

P4: a)



EQUATION OF MOTION: $m\ddot{x}(t) + b'\dot{x}(t) + k \cdot x(t) = F(t)$

b), c) FROM THE GRAPH WE SEE THAT WHEN THE WIND FORCE SUDDENLY STOPS THE BUILDING HAS A DECAYING OSCILLATION.



$x(t=t_0) = -0.45$

$\dot{x}(t=t_0) = 0$

FROM THE GRAPH WE CAN SEE THAT THE PERIOD OF OSCILLATION IS $T \approx 6$ sec.

So $\omega_d = T^{-1} = 0.17$ Hz

$\omega_d = 1.068$ rad/sec.

$$x(t-t_0) = x_0 \cdot \frac{1}{\sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n(t-t_0)} \cos(\omega_d \cdot (t-t_0) + \phi)$$

$$\phi = -\tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$\frac{x(t=t_0)}{x(t=t_0+T)} = \frac{x_0 \cdot \frac{1}{\sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n(t_0-t_0)}}{x_0 \cdot \frac{1}{\sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n(T)}} = e^{\zeta\omega_n T}$$

$x(t=t_0+T) = -0.34 \Rightarrow e^{\zeta\omega_n T} = -0.45/(-0.34)$

$e^{\zeta\omega_n T} = 1.324$

$\zeta\omega_n T = \ln(1.324) = 0.280$

Figure 4.30.1:

$\zeta\omega_n = \frac{0.280}{T} = \frac{0.280}{6} \Rightarrow \zeta.\omega_n = 0.0467$ rad/sec.

$\omega_n = \frac{0.0467}{\zeta}$

$\omega_d = \omega_n.\sqrt{1-\zeta^2} = \frac{0.0467}{\zeta}.\sqrt{1-\zeta^2}$

$\omega_d.\zeta = 0.0467.\sqrt{1-\zeta^2} \Rightarrow 1.068\zeta = 0.0467\sqrt{1-\zeta^2}$

$\frac{1.141\zeta^2}{2.181\times10^{-3}} = 1-\zeta^2 \Rightarrow 523\zeta^2 = 1-\zeta^2$

$524\zeta^2 = 1 \Rightarrow \zeta = \sqrt{\frac{1}{524}}$

$\zeta = 0.044 \Rightarrow \omega_n = \frac{0.0467}{0.044}$

$\omega_n = 1.061$ rad/sec.

So $\omega_n = 1.061$ rad/sec.

$\zeta = 0.044$

$\sigma = \zeta.\omega_n = 0.0467$ rad/sec.

$\omega_d = 1.068$ rad/sec.

EQUATION OF MOTION: $\frac{1}{\omega_n^2}\ddot{x}(t) + \frac{2\zeta}{\omega_n}.\dot{x}(t) + x(t) = \frac{F(t)}{k}$

$\frac{1}{\omega_n^2} = \frac{1}{(1.061)^2} = 0.888$; $\frac{2\zeta}{\omega_n} = \frac{2(0.044)}{1.061} = 0.083$

$\Rightarrow 0.888\ddot{x}(t) + 0.083\dot{x}(t) + x(t) = \frac{F(t)}{k}$

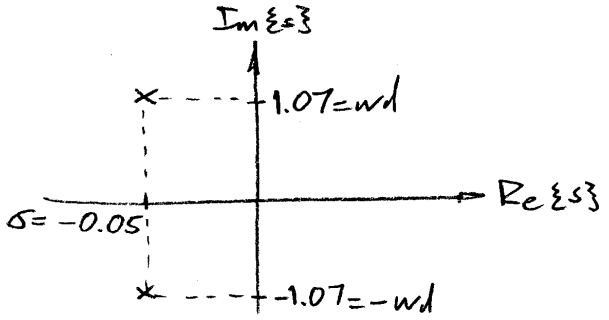
TO FIND k WE WILL NEED THE MASS PARTICIPATING IN THE DYNAMICS, OR THE DEFLECTION OF THE BUILDING AT A CONSTANT AIR SPEED.

Figure 4.30.2:

d) Roots: $s_1 = -\sigma + j\omega_d$; $s_2 = -\sigma - j\omega_d$

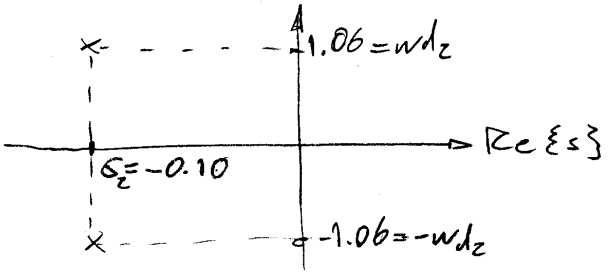
$s_1 = -0.0467 + j\,1.068$

$s_2 = -0.0467 - j\,1.068$



e) Remember $\sigma = \frac{b}{2m}$ where 'b' is damping

So if $\sigma_1 = -0.05 \frac{rad}{s}$ and $\sigma_2 = 2.\sigma_1 \Rightarrow \sigma_2 = -0.0934 \frac{rad}{s}$



$\zeta = \frac{b}{b_c}$

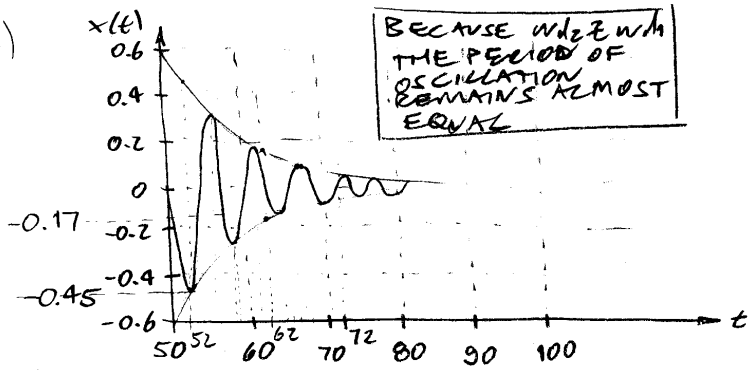
So $\zeta_1 = \frac{b_1}{b_c} \Rightarrow \zeta_2 = \frac{2b_1}{b_c}$

$\Rightarrow \zeta_2 = 2.\zeta_1 = 0.088$

$\omega_{d_2} = \omega_n\sqrt{1-\zeta^2} = 1.057 rad/s$

So $\omega_{d_2} \geq \omega_{d_1}$

f)



Because $\omega_{d_2} \geq \omega_{d_1}$ the period of oscillation remains almost equal

The decaying exponential for the original system was:

$\tau_1 = \frac{1}{\sigma_1} = 20 sec.$

And for the new system:

$\tau_2 = \frac{1}{\sigma_2} = 10 sec.$

So in 10 sec. it should be a 0.37 of its initial value and at 20 sec. at 0.13.

$-0.45 \times 0.37 = -0.17$

$-0.45 \times 0.13 = -0.06$

Figure 4.30.3:

5 Electrical Systems

5.1 Camera Flash Circuit

The solution for this problem is not available.

5.2 Batteries and Capacitors

(a) The short circuit current I_{sc} is dependent of the battery internal resistance as follows:

$$\begin{aligned} R_b &= \frac{1.5V}{I_{sc}} \\ R_b &= \frac{1.5V}{5.7A} \\ R_b &= 0.26\,\Omega \end{aligned}$$

(b) The power P dissipated will be as follows:

$$\begin{aligned} P &= VI_{sc} \\ P &= (1.5V)(5.7A) \\ P &= 8.55W \end{aligned}$$

The power is being dissipated at the battery internal resistance, heating up the battery as a consequence.

(c) In one time constant $\tau = R_l C$ the capacitor voltage should drop to 37 percent. So

$$\begin{aligned} \tau &= 10\,\text{min} \\ \tau &= 600\,\text{sec} \\ R_l &= \frac{600\,\text{sec}}{120 * 10^{-6}\,F} \\ R_l &= 5 * 10^6\,\Omega \end{aligned}$$

(d) Circuit diagram as follows:

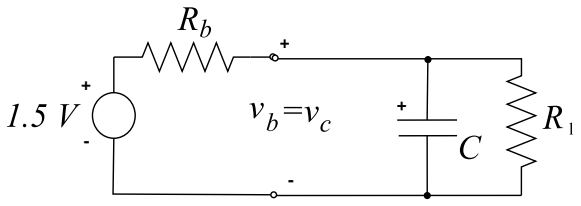


Figure 5.2.1: Circuit diagram of battery charging capacitor

(e) Making the analysis using two loop currents we have the following:

$$\begin{aligned} I_1 &= \frac{1.5V - v_c}{R_b} \\ I_2 &= \frac{v_c}{R_l} \\ I_1 - I_2 &= C\frac{dvc}{dt} \end{aligned}$$

Replacing I_1 and I_2 we have the following:

$$\begin{aligned} (R_b R_l)C\frac{dv_c}{dt} + (R_b + R_l)v_c - (1.5V)R_l &= 0 \\ 156\frac{dv_c}{dt} + 5 * 10^6 v_c - 7.5 * 10^6 &= 0 \end{aligned}$$

(f) Homogenous solution $v_c = Ae^{st}$ and $\frac{dv_c}{dt} = Ase^{st}$, replacing

$$\begin{aligned} 156Ase^{st} + 5 * 10^6 Ae^{st} &= 0 \\ s &= -32x10^3 \end{aligned}$$

Particular solution $v_c = Ae^{st} + B$ and $\frac{dv_c}{dt} = Ase^{st}$, with boundary condition $v_c(t = 0) = 0$, replacing

$$\begin{aligned} A &= -B \\ B &= 1.5V \\ A &= -1.5V \end{aligned}$$

So $v_c = 1.5(1 - e^{-32.1*10^3}t)$

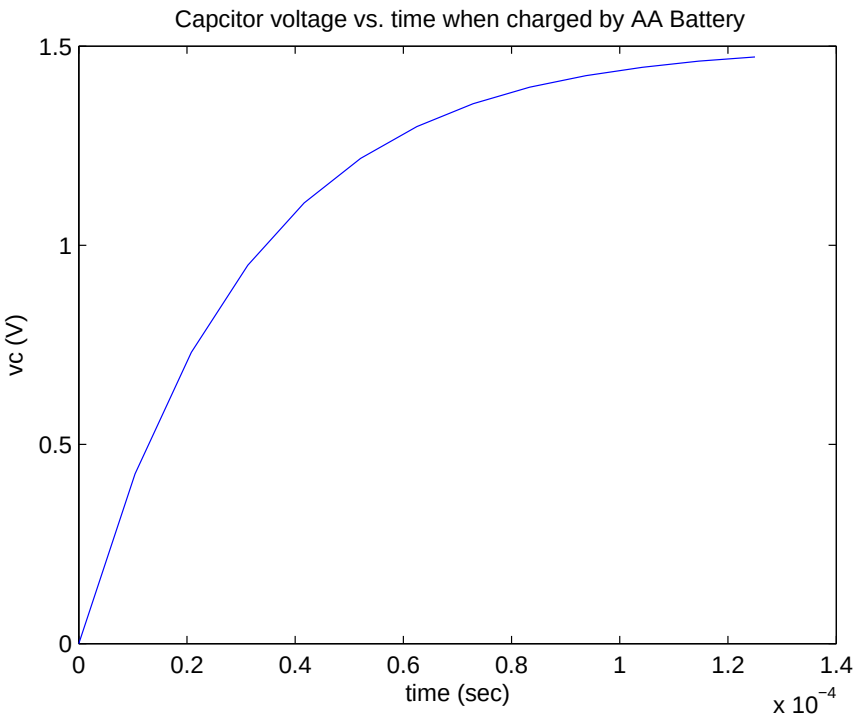


Figure 5.2.2: Charging capacitor

The system pole is located at $s = -32.1 * 10^3$, as shown on figure 5.2.3

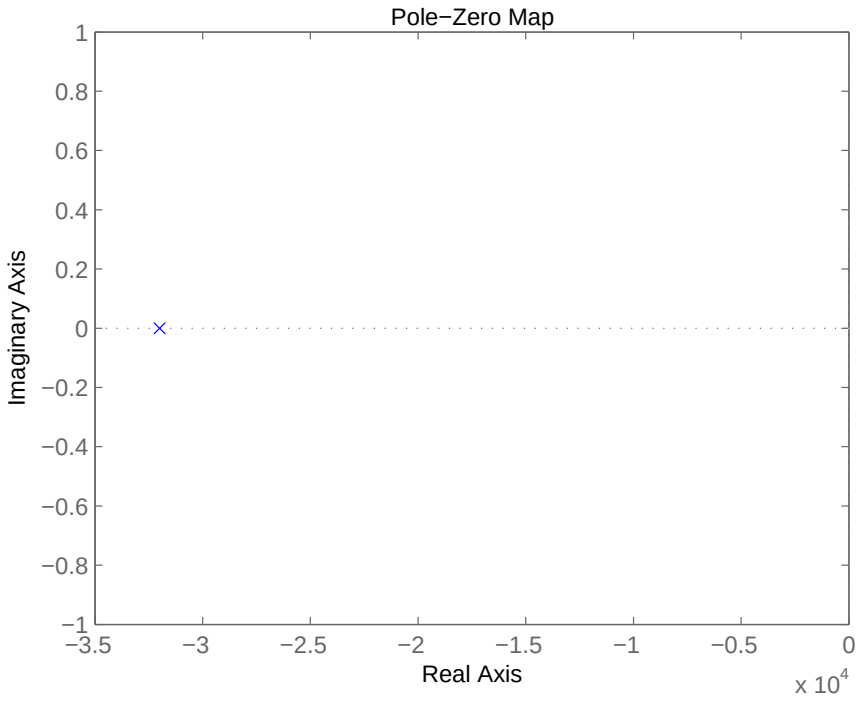


Figure 5.2.3: Pole location

(g) For the discharging capacitor we have the following

$$\begin{aligned} I &= \frac{-dv_c}{dt}C \\ I &= \frac{v_c}{R_l} \\ v_c &= Ae^{s(t-t_1)} \\ \frac{dv_c}{dt} &= Ase^{s(t-t_1)} \\ Ae^{s(t-t_1)} &= -R_lCAse^{s(t-t_1)} \\ s &= \frac{-1}{R_lC} \\ s &= \frac{-1}{600} \end{aligned}$$

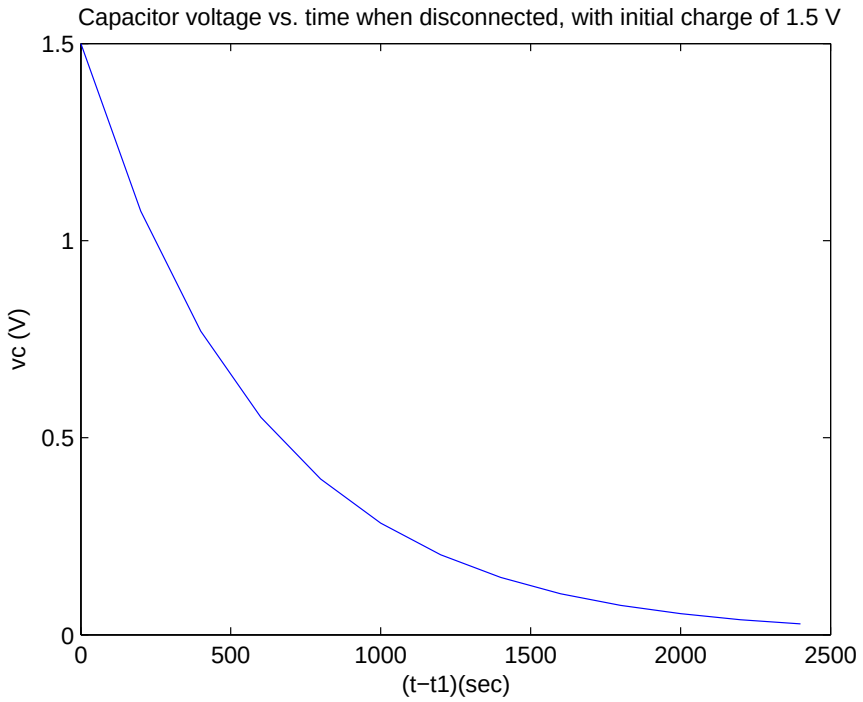


Figure 5.2.4: Discharging capacitor

The system pole is located at $s = -1.67 * 10^{-3}$, as shown on figure 5.2.5

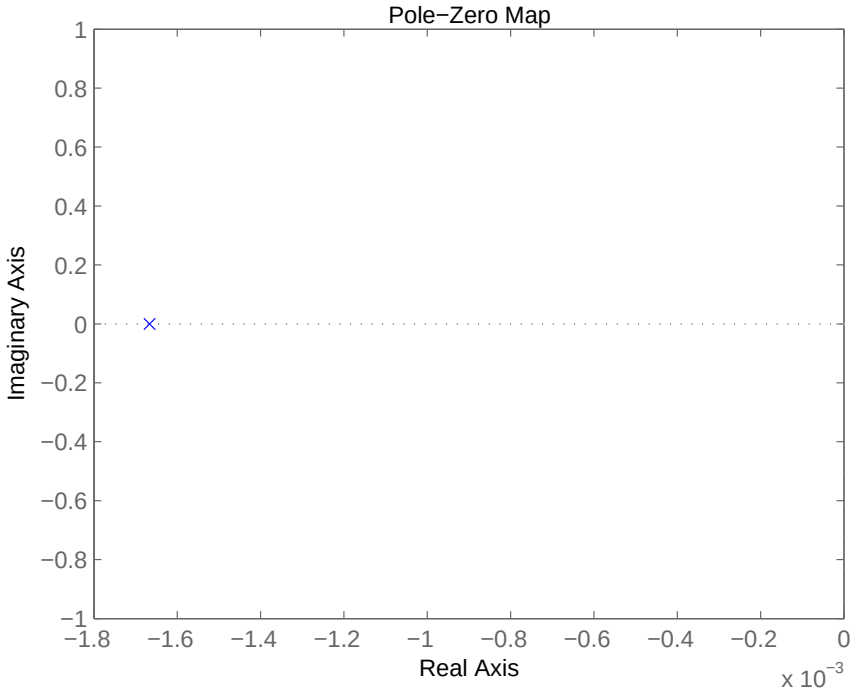


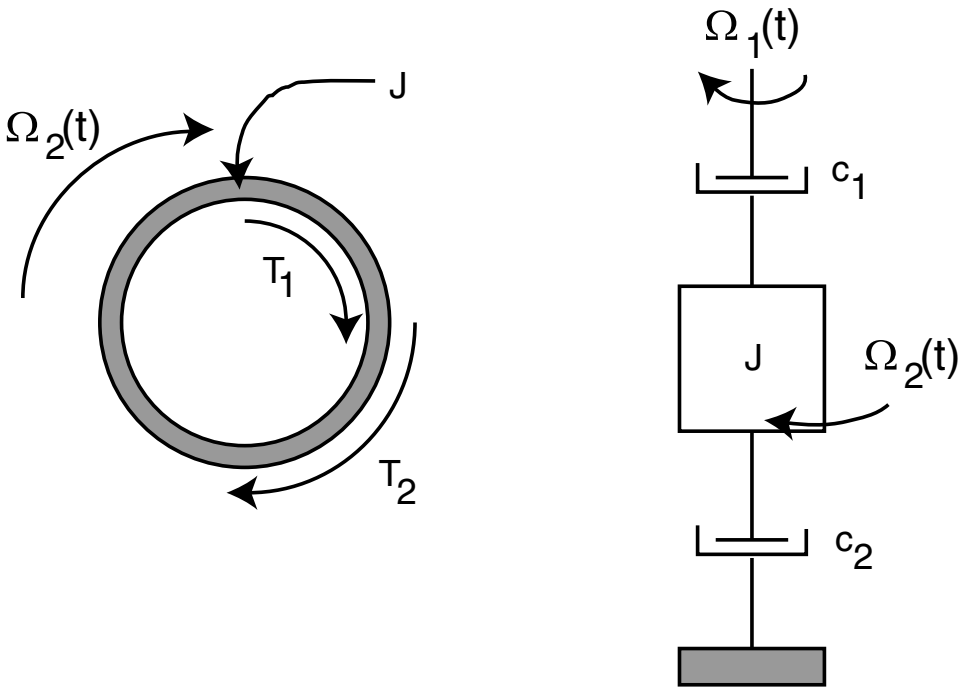
Figure 5.2.5: Pole location

5.3 Loaded Motor

The solution for this problem is not available.

6 Fluid Systems

6.1 Rotational Fluid System



a)

Figure 6.1.1: Freebody diagram of Inertia and Schematic of system.

The free body diagram for the inertia is shown in Figure 6.1.1 along with a schematic of the dynamic elements in this problem. As can be seen, the inertia has two torques acting on it: T_1 generated by the damper c_1 between the input and the inertia and T_2 generated by the damper c_2 between the inertia and ground. The schematic drawing is helpful but not required in the solution.

(b) Using the FBD from part a, we can write the following relationship:

$$\Sigma T = J\dot{\Omega}_2 = T_1 + T_2$$

This is where the schematic becomes helpful. By inspection, we can work out the following relationships

$$\begin{aligned} T_1 &= c_1(\Omega_1 - \Omega_2) \\ T_2 &= -c_2\Omega_2 \end{aligned}$$

Substituting for T_1 and T_2 yields the following

$$J\dot{\Omega}_2 = c_1(\Omega_1 - \Omega_2) - c_2\Omega_2 \Rightarrow \boxed{J\dot{\Omega}_2 + (c_1 + c_2)\Omega_2 = c_1\Omega_1}$$

(c) The input to the system has been defined as $\Omega_1(t) = u_s(t)$ thus the differential equation for this system become

$$J\dot{\Omega}_2 + (c_1 + c_2)\Omega_2 = c_1 u_s(t)$$

This differential equation (DE) matches that for a first order forced response. We know the solution to this DE is of the form

$$\begin{aligned}
\Omega_2(t) &= \Omega_h(t) + \Omega_p(t) \text{ where} \\
\Omega_p(t) &= \Omega_{ss} = \frac{c_1}{c_1 + c_2} \text{ (Particular solution)} \\
\Omega_h(t) &= Ae^{-st} \text{ (Homogeneous solution)} \\
s &= -\frac{c_1 + c_2}{J}
\end{aligned}$$

Using initial conditions

$$\Omega_2(0) = 0 = A + \frac{c_1}{c_1 + c_2} \Rightarrow A = -\frac{c_1}{c_1 + c_2}$$

$$\Omega_2(t) = \frac{c_1}{c_1 + c_2} \quad 1 - e^{-\frac{c_1 + c_2}{J}t}$$

(d) We determined earlier that

$$\Omega_{ss} = \frac{c_1}{c_1 + c_2}$$

Looking carefully at out schematic of the system, we know that the torque from the shaft acting on the inertia is T_1. Since, the torques on either side of the damper need to be equal, we know the torque acting on the shaft is equal and opposite to T_1. Since the system is in steady state, we know that the velocity of the shaft is constant which means that the sum of torques acting on the shaft equal 0.

$$\Sigma T_{shaft} = 0 = T_{in} - T_1 \Rightarrow T_{in} = T_1$$

We know that

$$\begin{aligned}
T_1 &= c_1(\Omega_1 - \Omega_2) \\
\Omega_1 &= u_s(t) = 1 \\
\Omega_2 &= \Omega_{ss} = \frac{c_1}{c_1 + c_2} \\
\text{Thus } T_1 &= c_1 \left(1 - \frac{c_1}{c_1 + c_2}\right) = \frac{c_1 c_2}{c_1 + c_2} \\
\text{More generally } T_1 &= \frac{c_1 c_2}{c_1 + c_2}\Omega_1
\end{aligned}$$

6.2 Fluid Leak

The solution for this problem is not available.

6.3 Where'd the water go?

(a) Use the parameters $R = 1.5 \cdot 10^9$ [Pa sec/m^3], and $L = 7 \cdot 10^6$ [Pa sec^2/m^3]. Begin by sketching the equivalent circuit diagram as shown below.

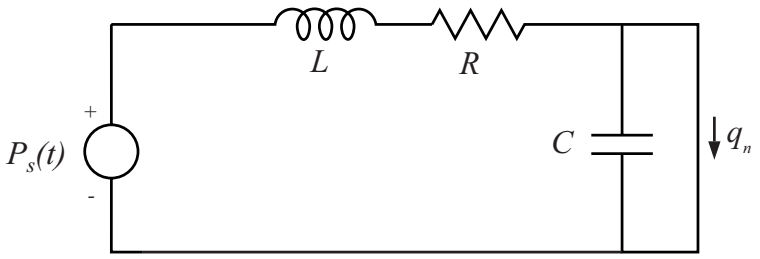


Figure 6.3.1: Shower equivalent circuit

q_n is the water flow through the shower nozzle. We need a differential equation in terms of the nozzle flow. Note that the capacitance in this circuit can be neglected as it is effectively short-circuited by the shower nozzle. Write out the loop equation for this circuit:

$$P_s(t) = P_R + P_L$$

Use the constitutive relationships for the fluidic resistor and fluid inertance:

$$P_R = q_R R$$
$$P_L = L\frac{dq_L}{dt}$$

note that $q_R = q_L = q_n$. Substitute these equations back into the loop equation:

$$P_s(t) = q_R R + L\frac{dq_L}{dt}$$
$$\frac{1}{R}P_s(t) = q_R + \frac{L}{R}\frac{dq_L}{dt}$$

the solution to this standard first order differential equation takes the form

$$q_n(t) = Ae^{-t/\tau} + B \text{ where } \tau = \frac{L}{R}$$

Our initial condition is that $P_s(0) = 3 \cdot 10^5$ [Pa]. Plugging this into our differential equation gives

$$q_n(0) = \frac{3 \cdot 10^5}{1.5 \cdot 10^9} = 2 \cdot 10^{-4} \text{ [m}^3\text{/sec]}$$

also, the final condition must be that $q_n\infty = 0$. Plugging this into the solution to the differential equation gives $A = 2 \cdot 10^{-4}$ and $B = 0$. The final solution is:

$$q_n(t) = 2 \cdot 10^{-4}e^{-t/.004667}$$

The flow will fall to below 1% of its inital value after 5τ. This is approximately .023 [s]. The sketch of the shower flow transient is shown in Figure 6.3.2.

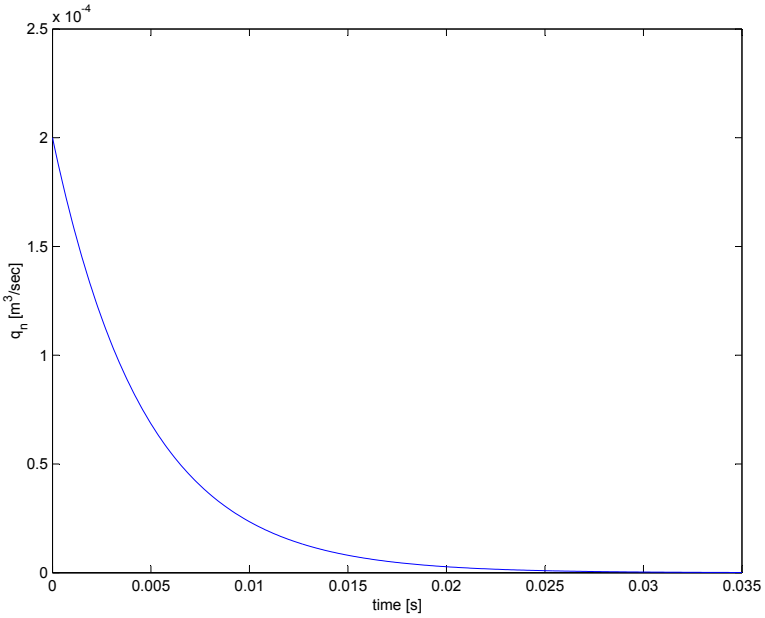


Figure 6.3.2: Shower flow transient

Because the shower nozzle is modeled as a switch, there is no pressure drop across the nozzle and therefore the pressure at the nozzle is atmospheric pressure, and this remains constant. The sketch of this plot is shown Figure 6.3.3.

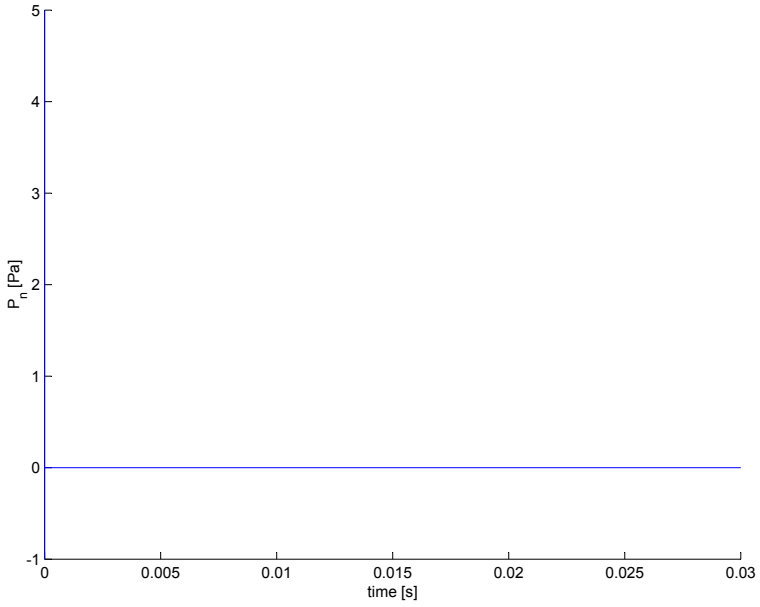


Figure 6.3.3: Shower pressure transient

(b)-(d) Don't worry, nothing to answer here.

7 Thermal Systems

7.1 Thermal Block Question

See Professor Trumper's Notes: Chapter 1 for some information on modeling thermal systems. A diagram depicting the idealized block outside of the oven is shown in Figure 7.1.1.

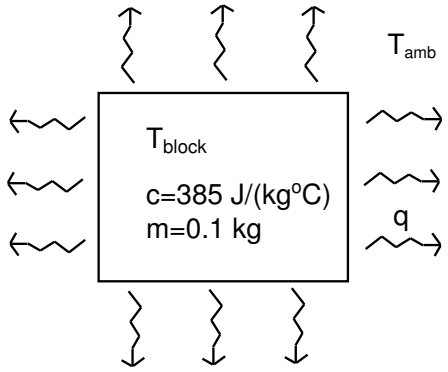


Figure 7.1.1: Diagram of copper block out of oven.

The ambient temperature surrounding the block is given as T_{amb}, the internal temperature of the block is given as T_{block}, and the heat flow from the block to the environment is given as q. The specific heat of copper (c) is given as 385 $\frac{J}{kg^oC}$, and the mass of the block is 0.1 kg.
The energy associated with the change of the block's temperature is given by

$$C\Delta T_{block} \tag{1}$$

where C is the thermal capacitance of the block given by mc. The energy associated with heat transfer for a period of time Δt is given by

$$q\Delta t. \tag{2}$$

In this idealized model, the heat loss is balanced exactly by the change in temperature of the block. Thus, the instantaneous energy/heat balance is given by

$$mcdT_{block} = -qdt. \tag{3}$$

The negative sign in front of the q is due to the fact that the heat is flowing out of the block. If you were to form a control volume around the block, heat flowing in would have a positive sign. According to the way the picture is drawn, the heat flow, q, is flowing outwards. In addition, the heat flow, q, will be modeled as linearly proportional to the difference between the block temperature and ambient temperature as

$$q = \frac{T_{block} - T_{amb}}{R} \tag{4}$$

where R is the thermal resistance between the block and the environment. By substitution we have

$$mcdT_{block} = qdt \tag{5}$$

$$= -\frac{T_{block} - T_{amb}}{R}dt \tag{6}$$

Therefore, the governing equation is

$$RC\frac{dT_{block}}{dt} + T_{block} = T_{amb} \tag{7}$$

where the time constant, τ, for the system is given by RC.

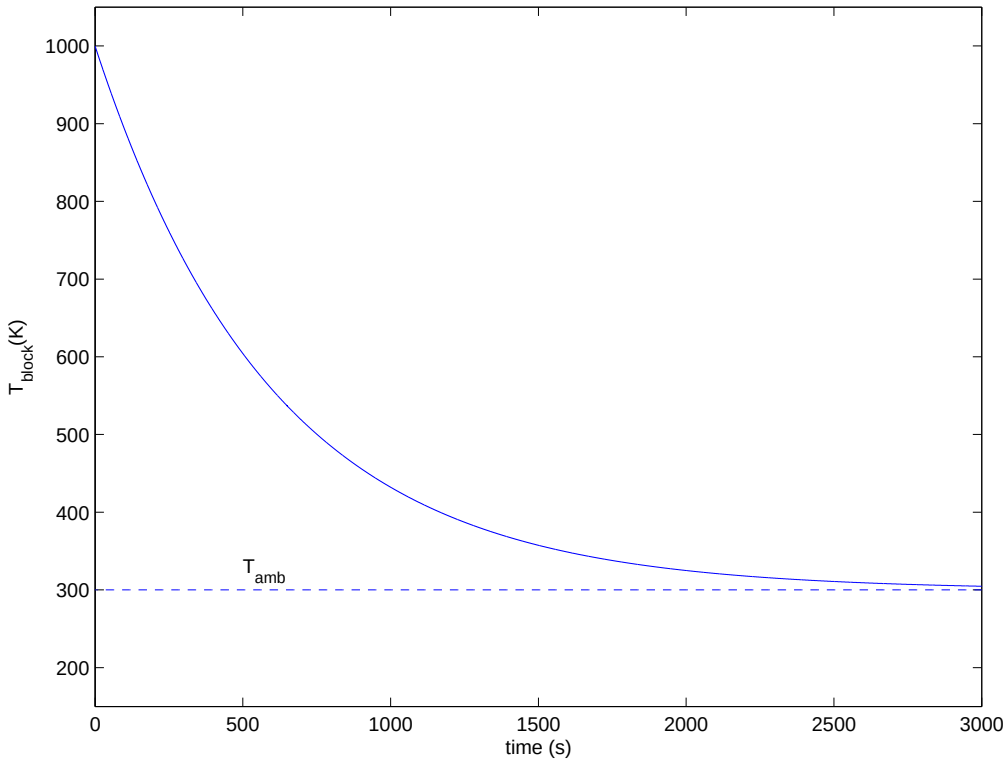


Figure 7.1.2: The block temperature (T_{block}) is plotted against time (t). The ambient temperature is given by T_{amb}.

Going back to Figure 7.1.2 we can calculate the time constant by estimating the amount of time required for the block to decay by 63% of the difference between the initial block temperature and the ambient temperature. This occurs at approximately 559 K after a time interval of 600 seconds. Therefore, the time constant for the system is 600 seconds. With $\tau = RC$ and C=38.5 $\frac{J}{^oC}$, the thermal resistance, R, equals 15.6 $\frac{^oC}{J/s}$.

7.2 The Hot Copper Block 1

(a) First, draw a diagram which shows heat flow through the system. It is useful to think about heat flowing from heat capacitances to other heat capacitances through thermal resistances. This model is shown in Figure 7.2.1.

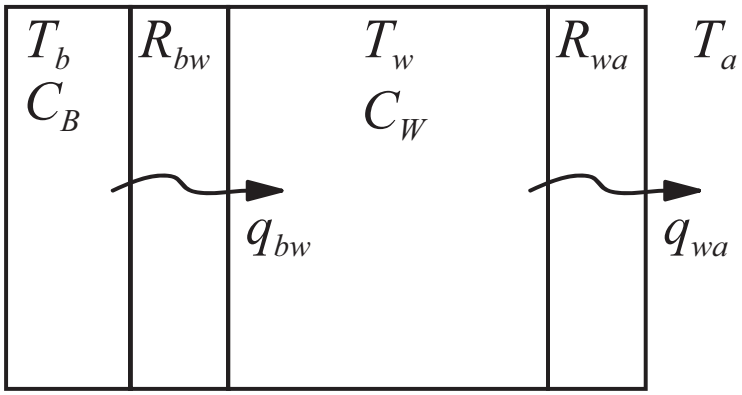


Figure 7.2.1: Thermal model

We need to use the constitutive relationships for thermal elements to write out the governing equations. Remember that

$$C\frac{dT}{dt} = q$$

where C is thermal capacitance, $\frac{dT}{dt}$ refers to the time derivative of the temperature of the capacitance and q is the heat flow into the capacitance. Also

$$T = qR$$

where T is the temperature drop across the resistance and R is the thermal resistance. An important note: We need to include the mass of the material to calculate the thermal capacitances in these equations, as the units for C are [J/°C] and you are given the specific heat of the copper and water in [J/kg °C]. Thermal capacitance is derivied by multiplying the specific heat times the mass. Using these equations and Figure 7.2.1, write out the governing equations:

$$C_B\frac{dT_b}{dt} = -q_{bw} \tag{1}$$

$$C_W\frac{dT_w}{dt} = q_{bw} - q_{wa} \tag{2}$$

$$(T_b - T_w) = q_{bw}R_{bw} \tag{3}$$

$$(T_w - T_a) = q_{wa}R_{wa} \tag{4}$$

combine equations (1) through (4) to the derivied two equations in state space form as:

$$\frac{dT_w}{dt} = \frac{1}{R_{bw}C_W}(T_b - T_w) - \frac{1}{R_{wa}C_W}(T_w - T_a) \tag{5}$$

$$\frac{dT_b}{dt} = -\frac{1}{R_{bw}C_B}(T_b - T_w) \tag{6}$$

(b) To transform equations (5) and (6) into a single 2^{nd}-order differential equation in $T_w(t)$, we need to eliminate T_b from equations (5) and (6). Start by re-arranging (5):

$$R_{bw}C_W\frac{dT_w}{dt}+\frac{R_{bw}}{R_{wa}}(T_w-T_a)+T_w=T_b \tag{7}$$

and take the time derivative

$$R_{bw}C_W\frac{d^2T_w}{dt^2}+\frac{R_{bw}}{R_{wa}}\left(\frac{dT_w}{dt}-\frac{dT_a}{dt}\right)+\frac{dT_w}{dt}=\frac{dT_b}{dt} \tag{8}$$

combine equations (7) and (8) into (6), rembering that $\frac{dT_a}{dt}=0$:

$$\begin{aligned}
R_{bw}C_W\frac{d^2T_w}{dt^2}+\left(\frac{R_{bw}}{R_{wa}}+1\right)\frac{dT_w}{dt}-\frac{R_{bw}}{R_{wa}}\frac{dT_a}{dt}&=\frac{-1}{R_{bw}C_B}\left(R_{bw}C_W\frac{dT_w}{dt}+\frac{R_{bw}}{R_{wa}}T_w-\frac{R_{bw}}{R_{wa}}T_a\right)\\
R_{bw}C_W\frac{d^2T_w}{dt^2}+\left(\frac{R_{bw}}{R_{wa}}+1+\frac{C_W}{C_B}\right)\frac{dT_w}{dt}+\frac{1}{R_{wa}C_B}T_w&=\frac{R_{bw}}{R_{wa}}\frac{dT_a}{dt}+\frac{1}{R_{wa}C_B}T_a\\
R_{bw}C_WR_{wa}C_B\frac{d^2T_w}{dt^2}+\left(R_{bw}C_B+R_{wa}C_B+R_{wa}C_W\right)\frac{dT_w}{dt}+T_w&=T_a
\end{aligned} \tag{9}$$

(c) Equation (9) is of the form

$$m\ddot{x}+b\dot{x}+kx=f(t)$$

we need to find the roots to figure out what the solution will look like.

$$\begin{aligned}
s_1,s_2&=\frac{-b\pm\sqrt{b^2-4mk}}{2m}\\
s_1&=-0.2621[\text{sec}^{-1}]\\
s_2&=-1.3878\cdot10^{-5}[\text{sec}^{-1}]
\end{aligned}$$

This means the solution takes the form

$$T_w(t)=Ae^{s_1t}+Be^{s_2t}+C$$

To find the constants A, B and C, we need three known states. We know that $T_w(0)=T_a=25°\text{C}$. The other initial condition is found by evaluating (5) at $t=0$

$$\begin{aligned}
\frac{dT_w(0)}{dt}&=\frac{1}{R_{bw}C_W}(T_b(0)-T_w(0))-\frac{1}{R_{wa}C_W}(T_w(0)-T_a(0))\\
\dot{T}_w(0)=\frac{95-25}{R_{bw}C_W}&=\frac{1}{6}[°\text{C/sec}]
\end{aligned}$$

we also know that $T_\infty=T_a$, plugging in the initial conditions yields two equations:

$$\begin{aligned}
A+B+C&=25\\
s_1A+s_2B&=\frac{1}{6}\\
C&=25
\end{aligned}$$

therefore our constants are $A = -0.6359$, $B = 0.6359$, and $C = 25$. The total solution is thus

$$T_w(t) = .6359(e^{-1.2829 \cdot 10^{-5} t} - e^{-0.0284t}) + 25$$

The system poles are shown in Figure 7.2.2

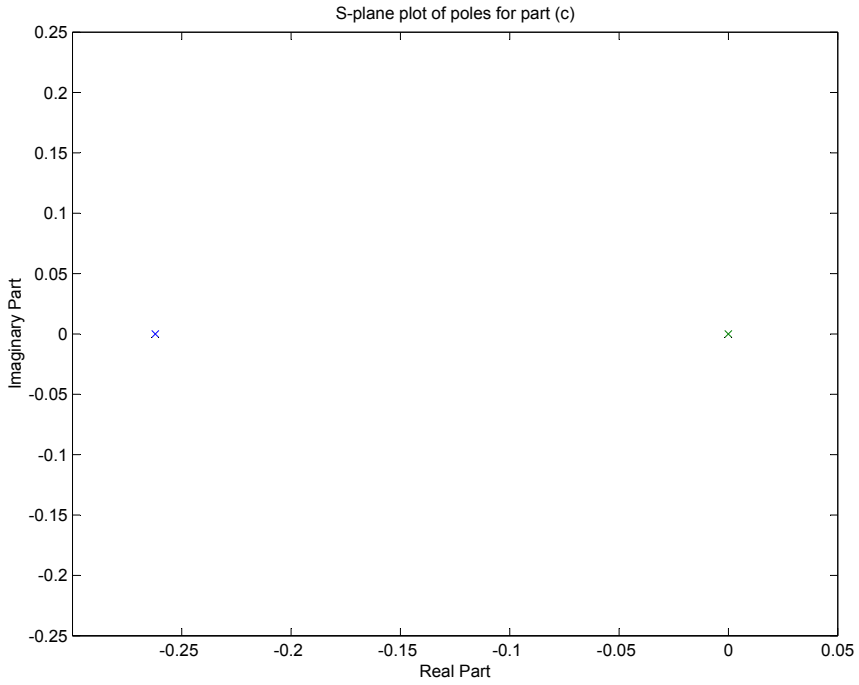


Figure 7.2.2: System Poles

(d) Figure 7.2.3 shows the plots of $T_w(t)$ over both a long and a short time scale. The plots look like this because the copper block temperature equalizes with the water temperature very quickly. Then, over a much longer time (on the order of two days; this must be a really good vacuum bottle!), the water temperature equilizes with the ambient temperature through heat losses through the walls of the vacuum bottle. You can see the effects of the time constants in the figures. The small time constant has no important effect beyond about 50 seconds, where the longer time constant dominates the response after that.

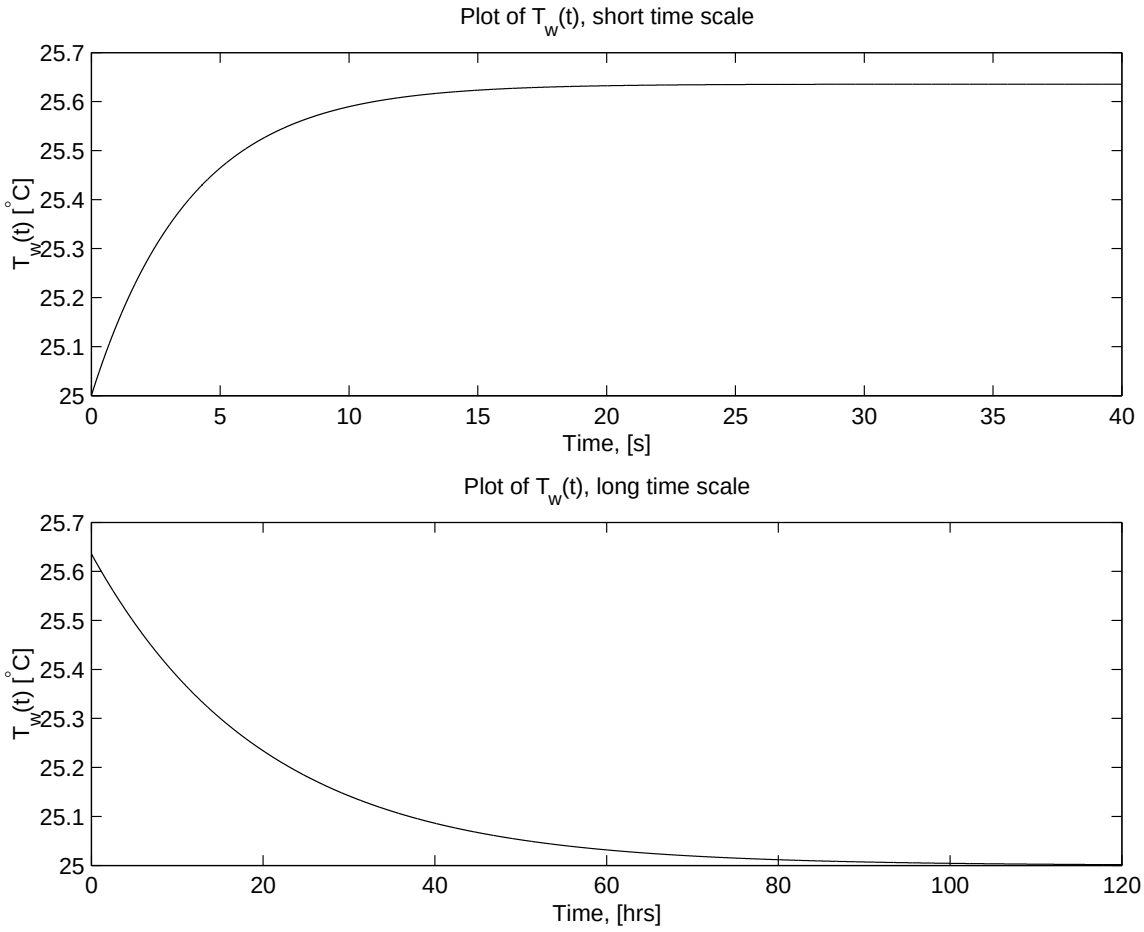


Figure 7.2.3: Plot of $T_w(t)$

(e) If the block starts out at only 60°C, we need to change the initial condition found with equation (5). It now becomes

$$\dot{T}_w(0) = \frac{60 - 25}{R_{bw} C_W} = \frac{1}{12}$$

Figure 7.2.4 show the difference from the original response. The original response is dotted. Note that since the block termperature difference to ambient has been lowered by a factor of 2, this response is one half the earlier response.

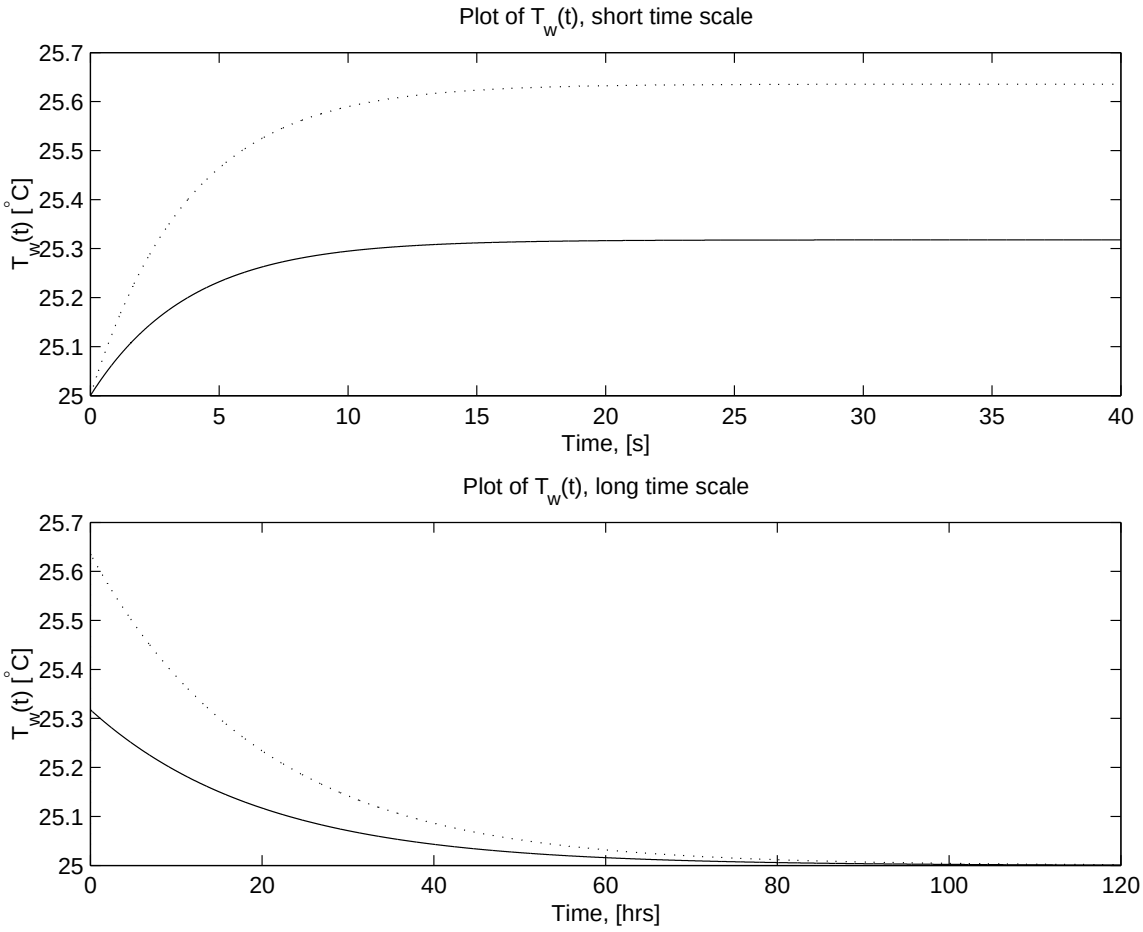


Figure 7.2.4: Plots of $T_w(t)$, $T_b(0) = 60°\text{C}$

7.3 The Hot Copper Block 2

(a) First, draw a diagram which shows heat flow through the system. It is useful to think about heat flowing from heat capacitances to other heat capacitances through thermal resistances. However, because the heater is embedded in the copper block, we model the resistance between the block and the heater to be 0. This model is shown in Figure 7.3.1.

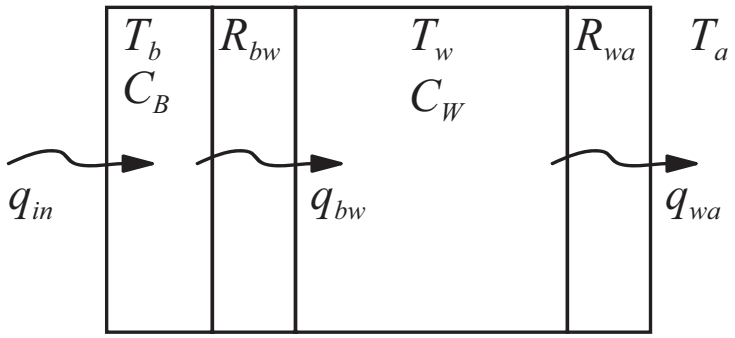


Figure 7.3.1: Thermal model

We need to use the constitutive relationships for thermal elements to write out the governing equations. Remember that

$$C\frac{dT}{dt} = q$$

where C is thermal capacitance, $\frac{dT}{dt}$ refers to the time derivative of the temperature of the capacitance and q is the heat flow into the capacitance. Also

$$T = qR$$

where T is the temperature drop across the resistance and R is the thermal resistance. An important note: We need to include the mass of the material to calculate the thermal capacitances in these equations, as the units for C are [J/°C] and you are given the specific heat of the copper and water in [J/kg °C]. Thermal capacitance is derivied by multiplying the specific heat times the mass. Using these equations and Figure 7.3.1, write out the governing equations:

$$C_B\frac{dT_b}{dt} = q_{in} - q_{bw} \tag{1}$$

$$C_W\frac{dT_w}{dt} = q_{bw} - q_{wa} \tag{2}$$

$$(T_b - T_w) = q_{bw}R_{bw} \tag{3}$$

$$(T_w - T_a) = q_{wa}R_{wa} \tag{4}$$

combine equations (1) through (4) to the derivied two equations in state space form as:

$$\frac{dT_w}{dt} = \frac{1}{R_{bw}C_W}(T_b - T_w) - \frac{1}{R_{wa}C_W}(T_w - T_a) \tag{5}$$

$$\frac{dT_b}{dt} = -\frac{1}{R_{bw}C_B}(T_b - T_w) + \frac{q_{in}}{C_B} \tag{6}$$

To transform equations (5) and (6) into a single 2^{nd}-order differential equation in $T_w(t)$, we need to eliminate T_b from equations (5) and (6). Start by re-arranging (5):

$$R_{bw}C_W\frac{dT_w}{dt}+\frac{R_{bw}}{R_{wa}}(T_w-T_a)+T_w=T_b \tag{7}$$

and take the time derivative

$$R_{bw}C_W\frac{d^2T_w}{dt^2}+\frac{R_{bw}}{R_{wa}}\left(\frac{dT_w}{dt}-\frac{dT_a}{dt}\right)+\frac{dT_w}{dt}=\frac{dT_b}{dt} \tag{8}$$

combine equations (7) and (8) into (6), rembering that $\frac{dT_a}{dt}=0$:

$$\begin{aligned}R_{bw}C_W\frac{d^2T_w}{dt^2}+\left(\frac{R_{bw}}{R_{wa}}+1\right)\frac{dT_w}{dt}-\frac{R_{bw}}{R_{wa}}\frac{dT_a}{dt}&=\\ \frac{-1}{R_{bw}C_B}\left(R_{bw}C_W\frac{dT_w}{dt}+\frac{R_{bw}}{R_{wa}}T_w-\frac{R_{bw}}{R_{wa}}T_a\right)+\frac{q_{in}}{C_B}&\\ R_{bw}C_W\frac{d^2T_w}{dt^2}+\left(\frac{R_{bw}}{R_{wa}}+1+\frac{C_W}{C_B}\right)\frac{dT_w}{dt}+\frac{1}{R_{wa}C_B}T_w=\frac{R_{bw}}{R_{wa}}\frac{dT_a}{dt}+\frac{1}{R_{wa}C_B}T_a+\frac{q_{in}}{C_B}&\\ R_{bw}C_WR_{wa}C_B\frac{d^2T_w}{dt^2}+(R_{bw}C_B+R_{wa}C_B+R_{wa}C_W)\frac{dT_w}{dt}+T_w=T_a+q_{in}R_{wa}&\end{aligned} \tag{9}$$

Equation (9) is the governing differential equation.

(b) Equation (9) is of the form

$$m\ddot{x}+b\dot{x}+kx=f(t)$$

we need to find the roots to figure out what the solution will look like.

$$\begin{aligned}s_1,s_2&=\frac{-b\pm\sqrt{b^2-4mk}}{2m}\\ s_1&=-0.2621[\text{sec}^{-1}]\\ s_2&=-1.3878\cdot10^{-5}[\text{sec}^{-1}]\end{aligned}$$

This means the solution takes the form

$$T_w(t)=Ae^{s_1t}+Be^{s_2t}+C$$

To find the constants A, B and C, we need three known states. We know that $T_w(0)=T_b(0)=T_a=25°\text{C}$. The other initial condition is found by evaluating (5) at $t=0$

$$\begin{aligned}\frac{dT_w(0)}{dt}&=\frac{1}{R_{bw}C_W}(T_b(0)-T_w(0))-\frac{1}{R_{wa}C_W}(T_w(0)-T_a(0))\\ \dot{T}_w(0)=\frac{25-25}{R_{bw}C_W}&=0[^\circ\text{C/sec}]\end{aligned}$$

we also know that $T_w(\infty)=T_a+q_{in}R_{wa}$, plugging in the initial conditions yields two equations:

$$\begin{aligned}A+B+C&=25\\ s_1A+s_2B&=0\\ C&=25+4\cdot17=93\end{aligned}$$

therefore our constants are $A = 3.6 \cdot 10^{-3}$, $B = -68.0036$, and $C = 93$. The total solution is thus

$$T_w(t) = 3.6 \cdot 10^{-3} e^{-0.2621t} - 68.0036 e^{-1.3878 \cdot 10^{-5} t} + 93$$

(c) Figure 7.3.2 shows the plots of $T_w(t)$ over both a long and a short time scale. The small time constant has no important effect beyond about 50 seconds, where the longer time constant dominates the response after that. The script `thermalblock.m` finds the system poles and plots the response on both a small and large time scale.

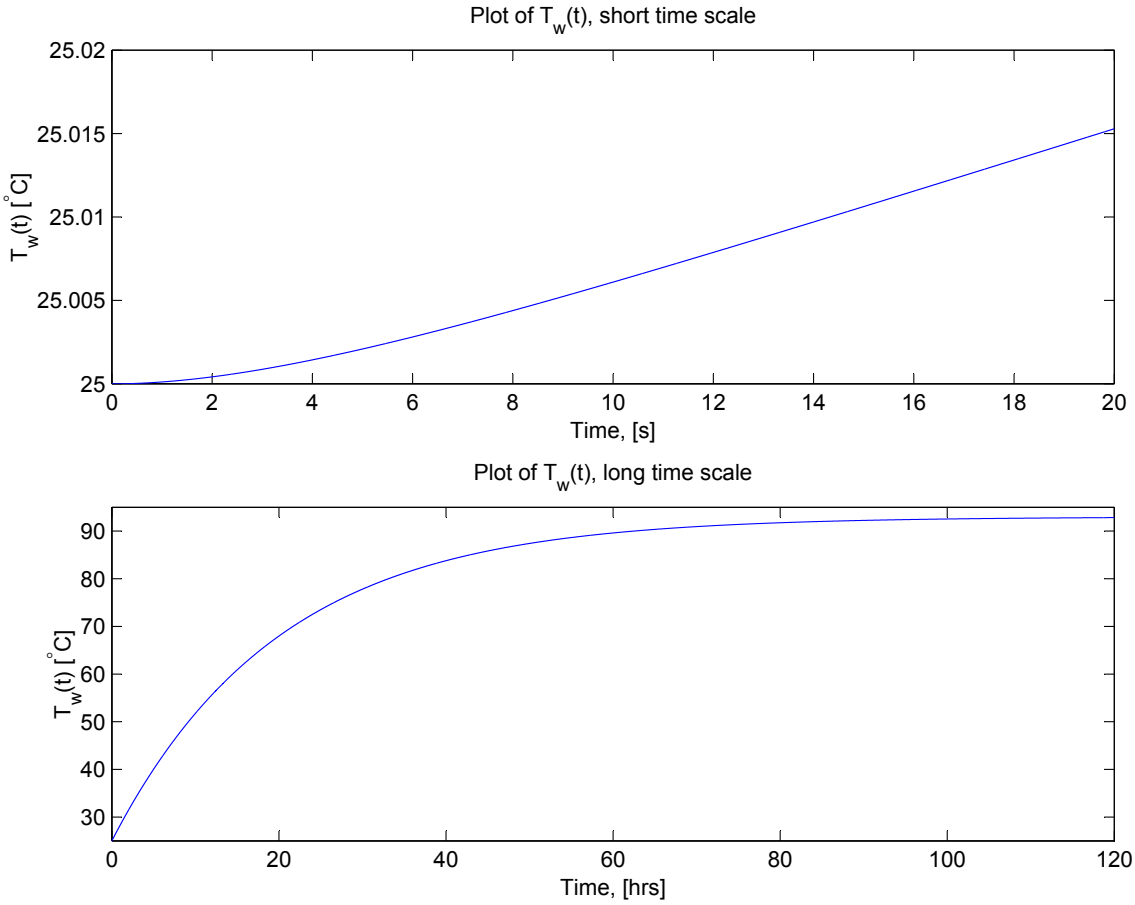


Figure 7.3.2: Plot of $T_w(t)$

7.4 Transistor on heat sink

(a) As always, first begin by drawing a figure. In this case, draw the circuit equivalent of this system. The equivalent circuit is shown in Figure 7.4.1.

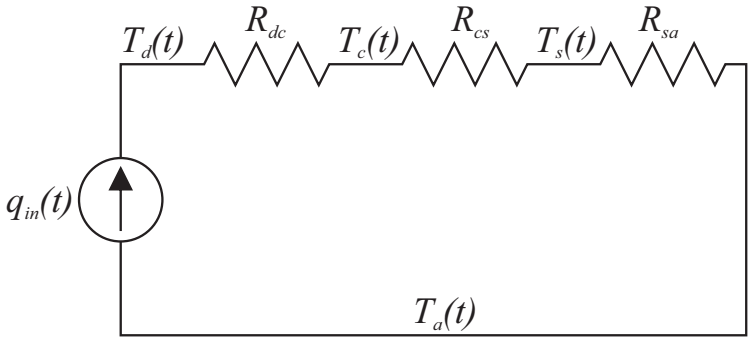


Figure 7.4.1: Transistor thermal system equivalent circuit

Because this is steady state, we do not need to worry about thermal capacitances at the moment. We need the constitutive relationships for the devices:

$$T_d - T_c = 50 \cdot R_{dc}$$
$$T_c - T_s = 50 \cdot R_{cs}$$
$$T_s - T_a = 50 \cdot R_{sa}$$

rearranging these equations and plugging in the numerical values leads to

$$T_s = T_a + 50 \cdot R_{sa} = 50°\text{C}$$
$$T_c = T_s + 50 \cdot R_{cs} = 60°\text{C}$$
$$T_d = T_c + 50 \cdot R_{dc} = 65°\text{C}$$

(b) Now draw another picture with the thermal capacitance of the heat sink added in. This new circuit equivalent is shown in Figure 7.4.2.

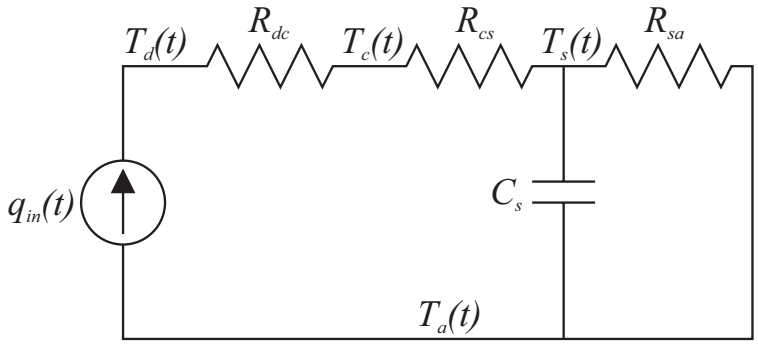


Figure 7.4.2: Transistor thermal Equivalent circuit with heatsink thermal capacitance

Why is the thermal capacitance in parallel with it's associated resistance? Imagine if they were in series. At steady state, there is no heat flow through the capacitance. If they were connected in series, there would be an open circuit with no heat flow, which would be impossible for this thermal system. The thermal capacitance is simply the specific heat times the mass: $C_s = 0.90 \cdot 500 = 450$ [J/°C] (remember that °C = °K as far as magnitude is concerned). The constitutive relation for the thermal capacitance is

$$q = C_s \frac{dT_s - dT_a}{dt} = C_s \frac{dT_s}{dt}$$

Write out the node equation at T_s

$$q_{in} = C_s \frac{dT_s}{dt} + \frac{T_s - T_a}{R_{sa}}$$

$$R_{sa} C_s \frac{dT_s}{dt} + T_s = R_{sa} q_{in} + T_a$$

This first order differential equation is the governing equation for the system. The general solution to this equation is

$$T_s(t) = Ae^{-t/\tau} + B \text{ where } \tau = R_{sa} C_s$$

First solve for the time $0 \leq t \leq 1$. During this time, $q_{in} = 500$ [W]. The initial and final conditions for this time frame are

$$T_s(0) = 50°\text{C}$$
$$T_s(\infty) = T_a + 500 \cdot R_{sa} = 275°\text{C}$$

These conditions yield $A = -225$ and $B = 275$. The solution is

$$T_s(t) = -225e^{-t/225} + 275$$

Plug this solution into the constitutive relationships for the other devices to obtain other equations:

$$T_c(t) = T_s(t) + 500 \cdot R_{cs} = -225e^{-t/225} + 375$$
$$T_d(t) = T_c(t) + 500 \cdot R_{dc} = -225e^{-t/225} + 425$$

Now, there is something to note here. These equations don't match the initial conditions of their respective steady state values. That is to say, these temperatures seem to "jump" from their steady state values to new values, and this doesn't make sense. This happens because we have approximated these thermal capacitances to be zero, and therefore their associated time constants will be zero, and their responses will be infinitely fast. In the real world, we would have to take these capacitances into effect.

Now attack the time frame $1 \leq t \leq \infty$. All that will change is the value of q_{in} to 50 [W]. The initial and final conditions will be

$$T_s(1) = -225e^{-1/225} + 275 = 51°\text{C}$$
$$T_s(\infty) = T_a + 50 \cdot R_{sa} = 50°\text{C}$$

These conditions yield $A = 1$ and $B = 50$. The solution is

$$T_s(t) = e^{-t/225} + 50$$

Plug this solution into the constitutive relationships for the other devices to obtain other equations:

$$T_c(t) = T_s(t) + 50 \cdot R_{cs} = e^{-t/225} + 60$$
$$T_d(t) = T_c(t) + 50 \cdot R_{dc} = e^{-t/225} + 65$$

The peak device temperature happens at $t = 1$:

$$T_d(1) = -225e^{-1/225} + 425 = 201°\text{C}$$

This would more than likely fry the transistor.

A bigger heatsink would not help in this case. In the limiting case as $C_s \to \infty$, $T_d(t) = 200°\text{C}$. This is because there must be a temperature difference that follows the constituent relationship $T_d - T_s = .3 \cdot 500 = 150°\text{C}$. The device will fry just as easily at 200 °C as it would 201 °C, so even the biggest heat sink won't help this chip. Smaller thermal resistances would though. The plots of the temperature transients are shown below.

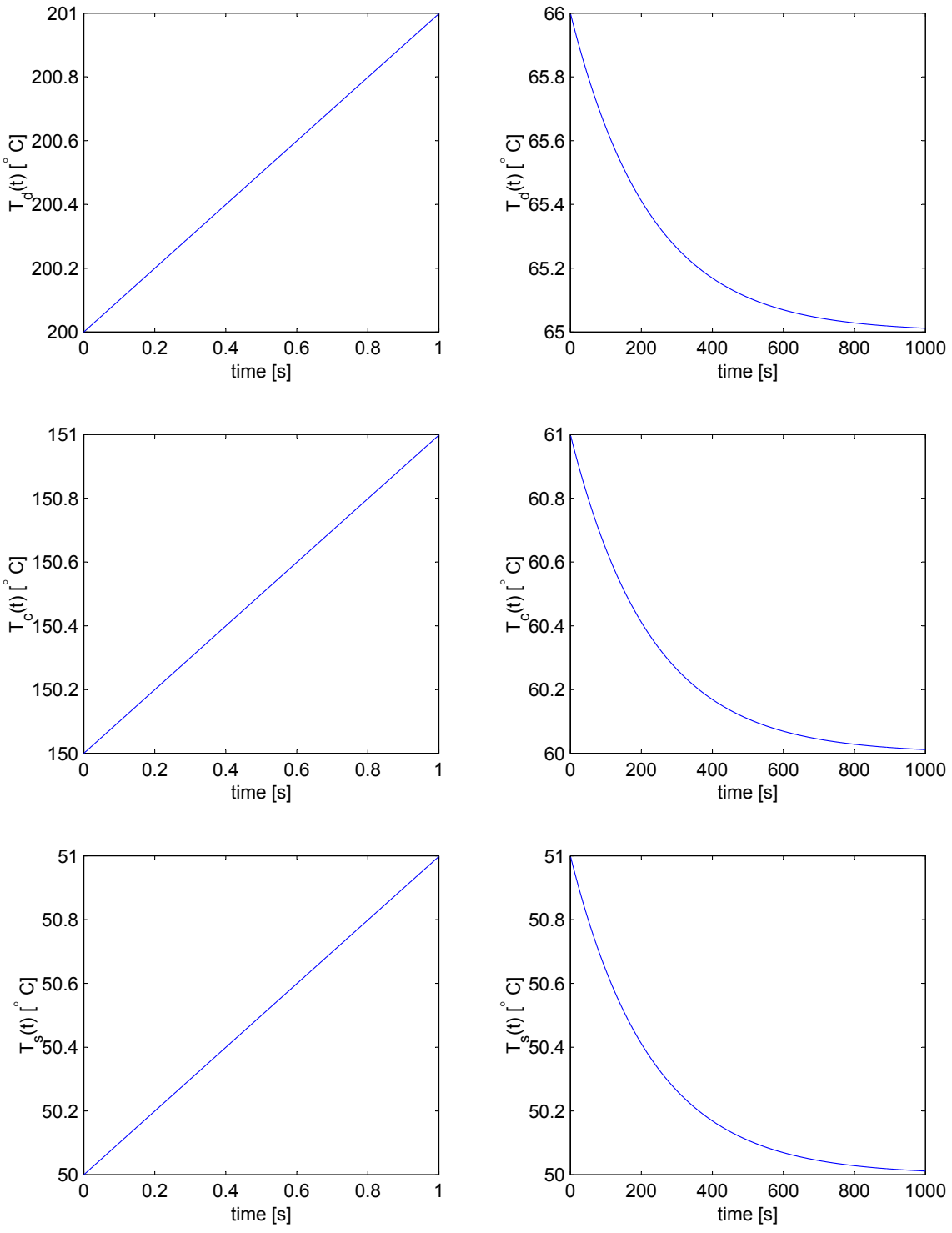


Figure 7.4.3: Plots of Temperature transients

8 Circuits

8.1 RC Transfer Function

(a) Begin by combining R_2 and C into an equivalent impedance $Z_{eq} = R_2 + \frac{1}{Cs}$. Sum the currents at the node between R_1 and R_2

$$\begin{aligned}
\frac{V_i(s) - V_o(s)}{R_1} &= \frac{V_o(s)}{R_2 + \frac{1}{Cs}} \\
\frac{V_i(s)}{R_1} &= V_o(s)\left(\frac{1}{R_1} + \frac{1}{R_2 + \frac{1}{Cs}}\right) \\
\frac{V_o(s)}{V_i(s)} &= \frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2 + \frac{1}{Cs}}} \\
\frac{V_o(s)}{V_i(s)} &= \frac{R_2Cs + 1}{(R_1 + R_2)Cs + 1}
\end{aligned}$$

This transfer function is an important result. When we find the equivalent impedance, the circuit becomes a voltage divider. In a voltage divider, the voltage drop across any resistor is simply its resistance divided by the total resistance seen at the input terminals multiplied by the input voltage. This is also true for impedances. The voltage we wanted to solve for was across the equivalent impedance $Z_{eq} = R_2 + \frac{1}{Cs}$. The total impedance seen by the input voltage source is $Z_{total} = R_1 + R_2 + \frac{1}{Cs}$. This becomes

$$\begin{aligned}
V_o(s) &= \frac{Z_{eq}}{Z_{total}} V_i(s) = \frac{R_2 + \frac{1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}} \\
\frac{V_o(s)}{V_i(s)} &= \frac{R_2Cs + 1}{(R_1 + R_2)Cs + 1}
\end{aligned}$$

This produces the same transfer function (with much less work).

(b)

$$\begin{aligned}
z_1 &= -\frac{1}{R_2C} = -8.33 \\
p_1 &= -\frac{1}{(R_1 + R_2)C} = -6.67
\end{aligned}$$

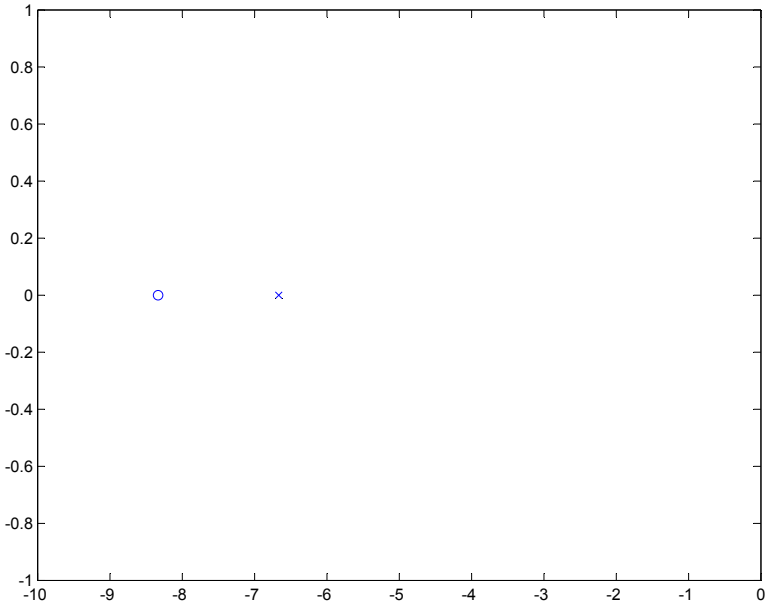


Figure 8.1.1: Pole Zero Plot

(c) This solution is not available.

8.2 RLC Transfer Function

(a) Using the voltage divider relationship:

$$V_o(s) = \frac{Z_{eq}}{Z_{total}} V_i(s) = \frac{\frac{1}{Cs}}{R + Ls + \frac{1}{Cs}} V_i(s)$$
$$\frac{V_o(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

(b)

$$C = \frac{1}{\omega_n^2 L} = 10^{-8} \text{ [F]}$$
$$R = \frac{2\zeta}{\omega_n C} = 100 \text{ [}\Omega\text{]}$$

The characteristic equation is $10^{-10}s^2 + 10^{-6}s + 1 = 0$. The roots (and therefore the poles) are

$$s_{1,2} = -5000 \pm 99875i$$

In polar form, these poles become

$$s_{1,2} = 10^5 \angle \pi \pm 1.52077$$

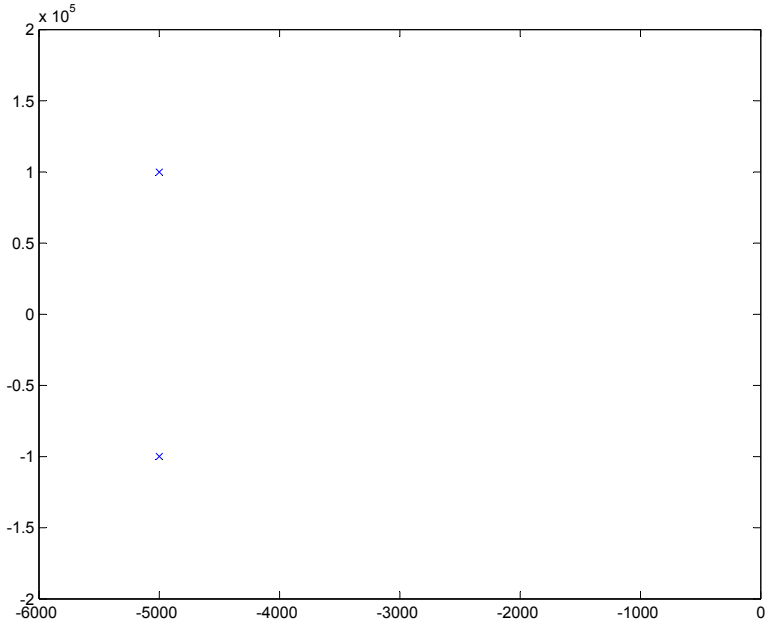


Figure 8.2.1: System Poles

(c) The step response for this system with values is shown in Figure 8.2.2.

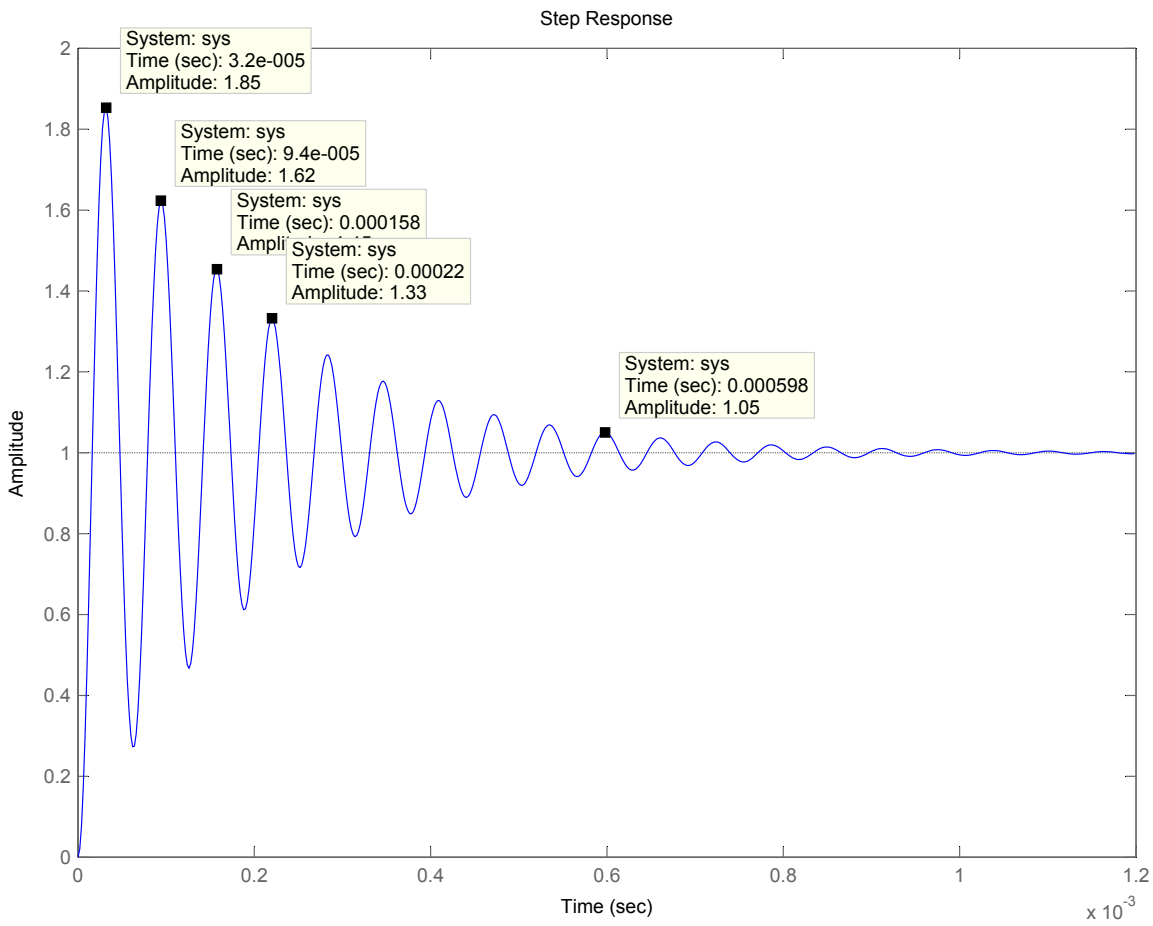


Figure 8.2.2: Step Response

(d) This solution is not available.

8.3 Bandpass Filter

(a) Using KCL, the current is $i = i_C + i_R + i_L$. The output voltage is $v_R = L\frac{di_L}{dt}$. The current in the capacitor is $i_C = C\frac{dv_r}{dt}$. The current in the resistor is $i_R = v_r/R$. The current in the inductor is $i_L = i - i_C - i_R$. Substituting for i_L into the constitutive relation for the inductor, we obtain the governing differential equation

$$RLC\frac{d^2v_R}{dt^2} + L\frac{dv_R}{dt} + Rv_R = RL\frac{di}{dt}$$

(b) Taking the Laplace Transform, we find the transfer function is the impedance

$$Z(s) = \frac{V_R(s)}{I(s)} = \frac{RLs}{RLCs^2 + Ls + R}$$

(c) To find the damping ratio and natural frequency, we compare the characteristic equation ($RLCs^2 + Ls + R = 0$) to the canonical form ($s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$). Thus, we find

$$\omega_n = \frac{1}{\sqrt{LC}} \text{ and } \zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

The zero is located at $s_z = 0$. The poles are located at

$$s_p = \frac{-1}{2RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}$$

(d) Using the given values ($R = 50\text{k}\Omega$, $C = 10\text{pF}$, and $L = 3.14815 \times 10^{-7}\text{H}$, we find $\omega_n = 563.6 \times 10^6$ rad/sec $= 89.7\text{MHz}$ and $\zeta = 0.18$.

(e) The Bode plot is shown in Figure 8.3.1 The circuit is called a bandpass filter because it passes signals with a frequency in a band centered at 89.7MHz. Frequency content away from this band is attenuated. The frequency 89.7MHz is frequency of the Boston radio station WGBH-FM.

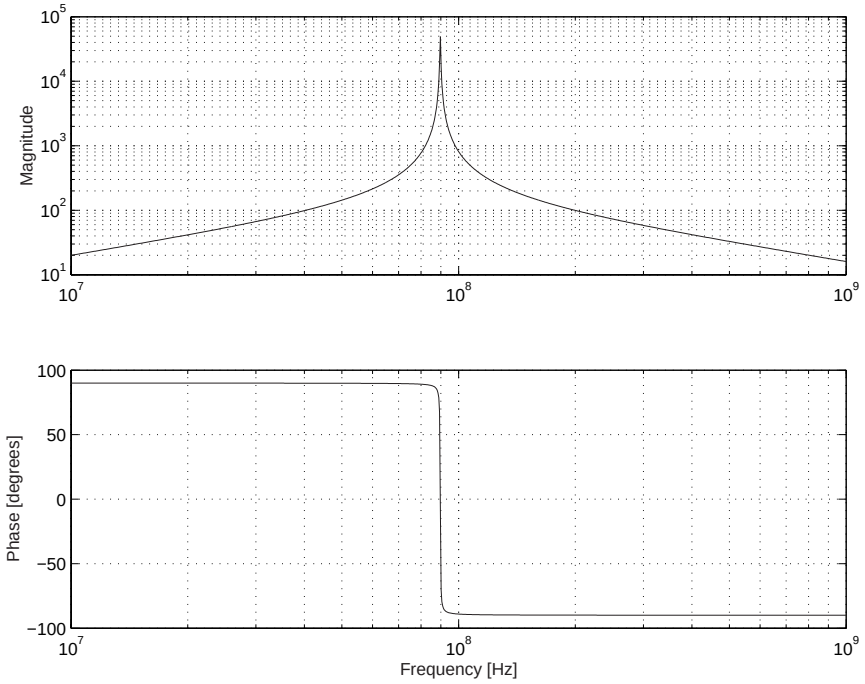


Figure 8.3.1: Bode Plot

(f) If radio stations are separated by at least 400kHz, we must examine the attenuation at 89.3MHz and 90.1MHz. The magnitude of the transfer function is

$$|H(j\omega)| = \sqrt{\frac{(RL\omega)^2}{(R(1-LC\omega^2)+1)^2 + (L\omega)^2}}$$

The peak magnitude (f = 89.7MHz) is 5×10^4. The magnitude at 89.3MHz is 1.85×10^4. The magnitude at 90.1MHz is 1.85×10^4. The attenuation of the neighboring radio stations is shown in Figure 8.3.2.

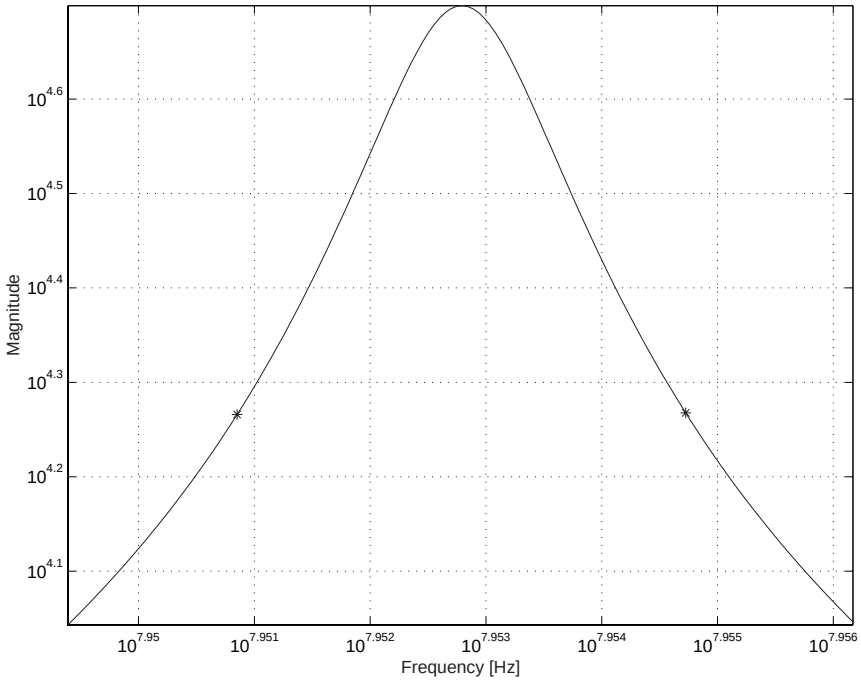


Figure 8.3.2: Attenuation of neighboring frequencies

8.4 LC Circuit Differential Equations

(a) Using KCL at the node between the inductor and capacitor with the assumed currents both positive into the node gives the following:

$$i_L + i_C = 0 \tag{1}$$

$$i_L = \frac{1}{L}\int v_L dt$$

$$i_C = C\frac{dv_c}{dt}$$

Equation (1) must be differentiated before substituting for the currents and from the direction of our assumed currents, $v_L = v_i - v_o$ and $v_C = 0 - v_o$. The governing differential equation is then

$$\frac{d^2v_o}{dt^2} + \frac{v_o}{LC} = \frac{v_i}{LC} \tag{2}$$

(b) The Laplace transform of Equation (2), with $\omega_n = 1/\sqrt{LC}$, initial rest and some rearranging, is

$$v_o(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} v_i(s)$$

With the input voltage a unit step and after performing partial fraction decomposition

$$v_o(s) = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

The inverse Laplace transform reveals the output voltage as a function of time after substituting $\omega = 1/\sqrt{LC}$

$$v_o(t) = 1 - cos\frac{t}{\sqrt{LC}}$$

Note in the period of the output voltage in Figure 8.4.1

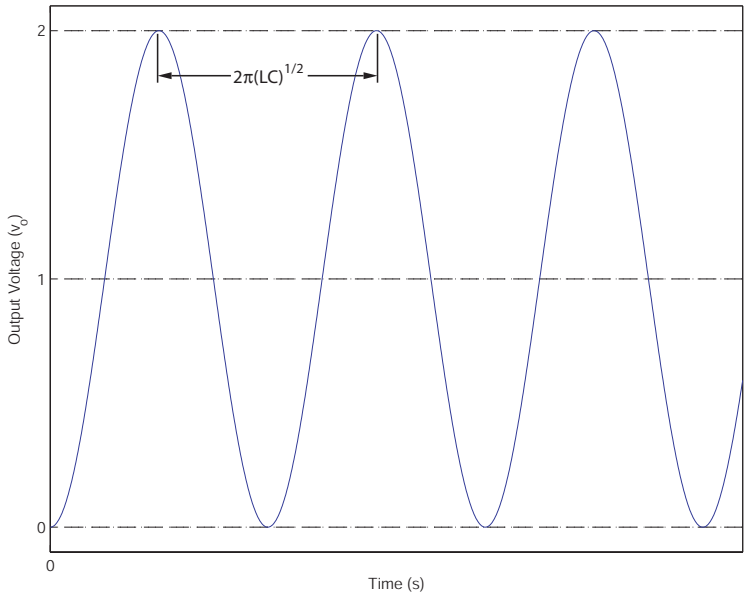


Figure 8.4.1: Labeled plot of the output voltage

8.5 Equivalent Impedance

The equivalent impedance of the circuit is

$$Z_{eq} = Z_L + (Z_{R1} + Z_{R2})||(Z_{C1}||Z_{C2}) \tag{1}$$

But realize the series resistive impedance, $Z_{R1} + Z_{R2}$, can be simplified into an equivalent resistive impedance by $R_{12} = R_1 + R_2$ and $Z_{R12} = Z_{R1} + Z_{R2}$. Also, the parallel capacitors can be replaced by an equivalent capacitance $C_{12} = C_1 + C_2$. The impedance of the equivalent capacitor is then $Z_{C12} = Z_{C1}||Z_{C2}$. Equation (1) is now simplified to

$$Z_{eq} = Z_L + (Z_{R12}||Z_{C12})$$

Substitute $Z_L = Ls$, $Z_{R12} = R_{12}$, and $Z_{C12} = 1/C_{12}s$ gives

$$Z_{eq} = Ls + \frac{R_{12}(1/C_{12}s)}{R_{12} + 1/C_{12}s}$$

Now expand the R_{12} and C_{12} substitutions, then simplify and

$$Z_{eq} = \frac{L\,(R_1 + R_2)\,(C_1 + C_2)\,s^2 + Ls + (R_1 + R_2)}{(R_1 + R_2)\,(C1 + C2)\,s + 1}$$

8.6 Circuit Response

From Prof. Trumper's handouts and lecture, we know that the solution to any differential equation is the sum of the homogenous and particular solutions.

(a) The differential equation for i_c is

$$\begin{aligned} RC\frac{di_c}{dt} + i_c &= C\frac{dv_{in}}{dt} \\ v_{in} &= u(t) \\ \frac{dv_{in}}{dt} &= \delta(t) \\ \text{thus for } t > 0\ RC\frac{di_c}{dt} + i_c &= 0 \\ \text{thus for } t > 0\ i_c(t) &= Ae^{-\frac{t}{RC}} \\ \text{where } A &= i(0^+) \\ \text{and } i(0^+) &= \frac{v_{in}(0) - v_c(0)}{R} \end{aligned}$$

$$\begin{aligned} \text{i) } i(t) &= \frac{1}{R}e^{-\frac{t}{RC}} \\ \text{ii) } i(t) &= -\frac{9}{R}e^{-\frac{t}{RC}} \\ \text{iii) } i(t) &= 0 \end{aligned}$$

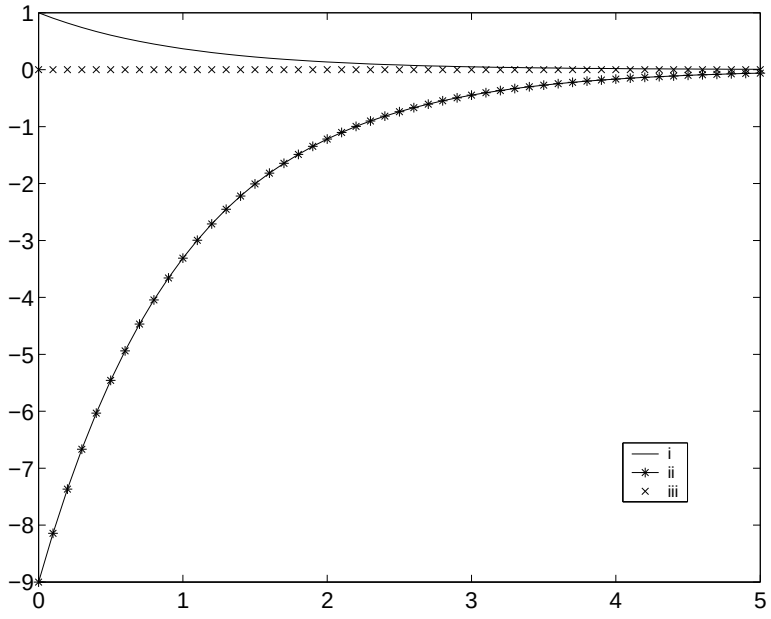


Figure 8.6.1: Step Response for Part (a)

(b)

$$
\begin{aligned}
L\frac{dv_L}{dt} + Rv_L &= L\frac{dv_{in}}{dt} \\
v_{in} &= u(t) \\
\frac{dv_{in}}{dt} &= \delta(t) \\
\text{thus for } t > 0 \; L\frac{dv_L}{dt} + Rv_L &= 0 \\
\text{thus for } t > 0 \; v_L(t) &= Ae^{-\frac{Rt}{L}} \\
\text{where } A &= v_L(0^+) \\
\text{and } v_L(0^+) &= v_{in}(0) - Ri_L(0)
\end{aligned}
$$

$$
\begin{aligned}
\text{i) } v(t) &= e^{-\frac{Rt}{L}} \\
\text{ii) } v(t) &= -9e^{-\frac{Rt}{L}} \\
\text{iii) } v(t) &= 0
\end{aligned}
$$

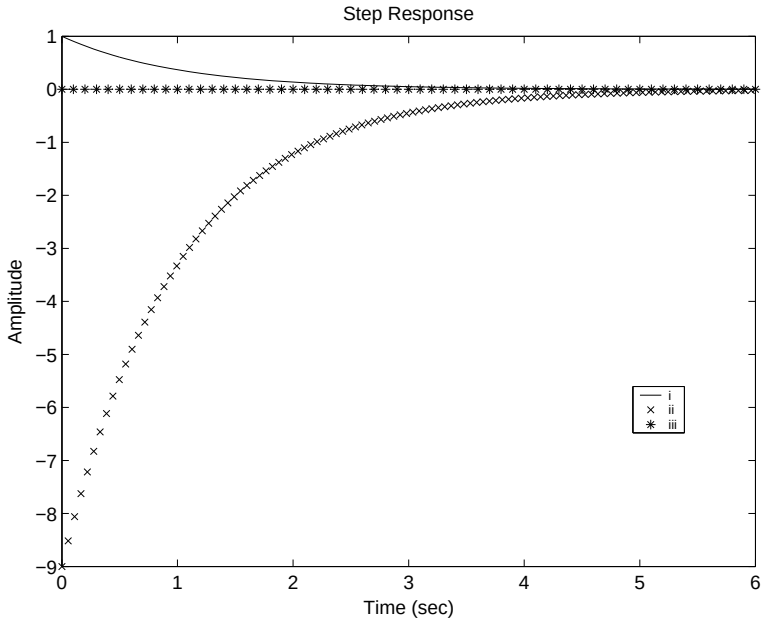


Figure 8.6.2: Step Response for Part (b)

(c)

$$\begin{aligned}
\frac{L}{R}\frac{dv_L}{dt} + v_L &= L\frac{di_{in}}{dt} \\
\text{for } t > 0 \; \frac{L}{R}\frac{dv_L}{dt} + v_L &= 0 \\
\text{for } t > 0 \; v_L(t) &= Ae^{-\frac{Rt}{L}} \\
i_{in}(0) &= i_R(0) + i_L(0) = i_R(0) + 0 \\
v_R(0) &= v_L(0) = A = i_{in}R \\
v_L(t) &= Re^{-\frac{Rt}{L}}
\end{aligned}$$

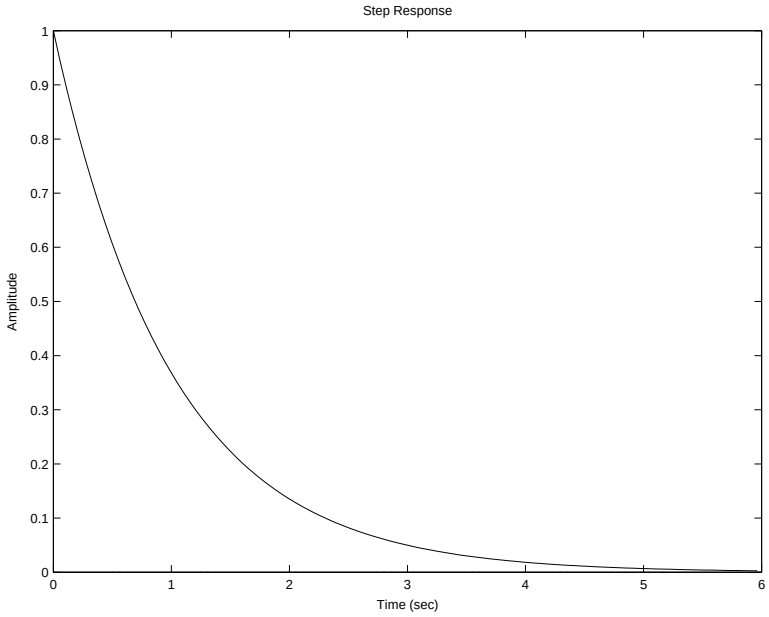


Figure 8.6.3: Step Response for Part (c)

(d)

$$\begin{aligned}
RC\frac{dv_c}{dt} + v_c &= Ri_{in} \\
v_c(t) &= Ri_{in} + Ae^{-\frac{t}{RC}} \\
v_c(0) &= 0 \rightarrow A = -R \\
v_c(t) &= \left(1 - e^{-\frac{t}{RC}}\right) R
\end{aligned}$$

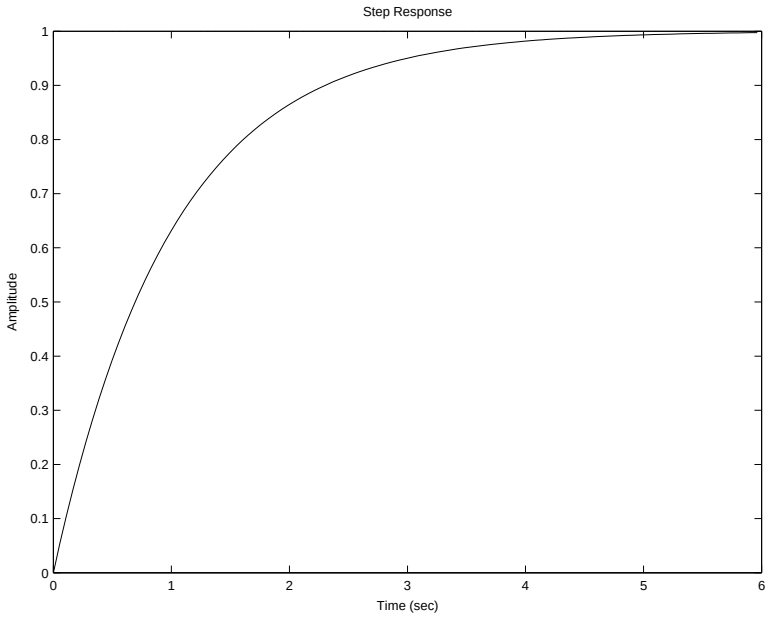


Figure 8.6.4: Step Response for Part (d)

(e) (a)

$$\frac{V_c}{V_{in}} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{1}{RCs + 1}$$
$$\frac{I_c}{V_{in}} = \frac{CsV_c}{V_{in}} = \frac{Cs}{RCs + 1}$$

(b)

$$\frac{V_L}{V_{in}} = \frac{Ls}{R + Ls}$$

(c)

$$I_{in} = \frac{V_L}{R} + \frac{V_L}{Ls}$$
$$\frac{V_L}{I_{in}} = \frac{LRs}{R + Ls}$$

(d)

$$I_{in} = \frac{V_c}{R} + \frac{V_c}{\frac{1}{Cs}}$$
$$\frac{V_c}{I_{in}} = \frac{R}{RCs + 1}$$

8.7 LRC Circuit 1

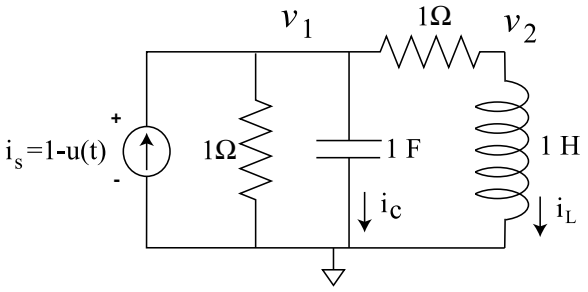


Figure 8.7.1: LRC Circuit

(a) $v_1 = 0.5$ V, $v_2 = 0$ V, $i_l = 0.5$ A, $i_c = 0$

(b)

$$\begin{aligned}
v_1(0^+) &= v_c(0^-) = 0.5V \\
\frac{dv_1}{dt}(0^+) &= dv_1 dt(0^+) = \frac{i_c(0^+)}{C} = \frac{-(i_L(0^+) + i_R(0^+)}{C} \\
i_R(0^+) &= \frac{v_c(0^+)}{R} = 0.5A \\
i_L(0^+) &= i_L(0^-) = 0.5A \\
\frac{dv_1}{dt}(0^+) &= -1
\end{aligned}$$

(c)

$$\begin{aligned}
v_1 &= v_c = v_c(0) - \frac{i_c}{sC} \Rightarrow C\frac{dv_1}{dt} = -i_c \\
i_c &= i_R + i_L = \frac{v_c}{R} + \frac{v_c}{R + Ls} \\
C\frac{dv_1}{dt} &= -\frac{v_c}{R} - \frac{v_c}{R + Ls} \Rightarrow RLC\frac{d^2v_1}{dt^2} + (R^2C + L)\frac{dv_1}{dt} + 2Rv_1 = 0 \\
\frac{d^2v_1}{dt^2} + 2\frac{dv_1}{dt} + 2v_1 &= 0 \\
v_1(t) &= 0.707e^{-t}\sin(t + 3\pi/4)
\end{aligned}$$

8.8 LRC Circuit 2

(a) The node equations are

$$0 = i_s - \frac{v_1}{1\Omega} - i_c - \frac{v_1 - v_2}{1\Omega} \tag{1}$$

$$0 = \frac{v_1 - v_2}{1\Omega} - i_L \tag{2}$$

for nodes v_1 and v_2 respectively. The constituent relationships for the capacitor and inductor are

$$i_c = C\frac{dv_c}{dt} = \frac{dv_1}{dt} \tag{3}$$

$$v_L = L\frac{di_L}{dt} = v_2 \tag{4}$$

These are the 4 main equations that we need to solve this problem. We need to find v_2 in terms of v_1. Begin by differentiating (2) and plugging in equation (4).

$$\frac{dv_1}{dt} - \frac{dv_2}{dt} = v_2 \tag{5}$$

Substitute equation (3) into equation (1) and differentiate

$$0 = i_s - v_1 - \frac{dv_1}{dt} - v_1 + v_2 \tag{6}$$

$$\frac{di_s}{dt} = \frac{d^2v_1}{dt^2} + 2\frac{dv_1}{dt} - \frac{dv_2}{dt} \tag{7}$$

plug equation (5) into (7)

$$\begin{aligned} \frac{di_s}{dt} &= \frac{d^2v_1}{dt^2} + \frac{dv_1}{dt} + v_2 \\ v_2 &= \frac{di_s}{dt} - \frac{d^2v_1}{dt^2} - \frac{dv_1}{dt} \end{aligned} \tag{8}$$

Finally, substitute (8) into (6) to obtain the final differential equation.

$$\frac{di_s}{dt} + i_s = \frac{d^2v_1}{dt^2} + 2\frac{dv_1}{dt} + 2v_1$$

(b) $i_s = 1$ A, $v_1 = 0.5$ V, $v_2 = 0$ V, $i_l = 0.5$ A, $i_c = 0$ A.

(c)

$$\begin{aligned} \frac{di_s}{dt} + i_s &= \frac{d^2v_1}{dt^2} + 2\frac{dv_1}{dt} + 2v_1 \\ I(s)(s+1) - i_s(0^-) &= V_1(s)\left(s^2 + 2s + 2\right) - sv_1(0^-) - \dot{v}_1(0^-) - 2v_1(0^-) \\ V_1(s) &= \frac{I(s) - i_s(0^-) + v_1(0^-)(s+2) + \dot{v}_1(0^-)}{s^2 + 2s + 2} \end{aligned}$$

Plug in the initial conditions $i_s(0^-) = 1$, $v_1(0^-) = .5$, and $\dot{v}_1(0^-) = 0$. Also, $I(s) = 0$.

$$\begin{aligned} V_1(s) &= \frac{-1 + .5(s+2)}{s^2 + 2s + 2} \\ &= \frac{.5s}{s^2 + 2s + 2} \end{aligned} \tag{9}$$

Equation (9) does not appear in our Laplace transform table. We must manipulate it until it matches a standard form.

$$\begin{aligned} V_1(s) &= \frac{.5s}{s^2 + 2s + 2} \\ &= \frac{.5s}{(s+1)^2 + 1^2} \\ &= .5\frac{s+1}{(s+1)^2 + 1^2} - .5\frac{1}{(s+1)^2 + 1^2} \end{aligned}$$

Now we have the transform in a form available on the table. The final equation is then:

$$v_1(t) = \frac{e^{-t}}{2}(\cos t - \sin t)$$

(d) The plot is shown in Figure 8.8.1.

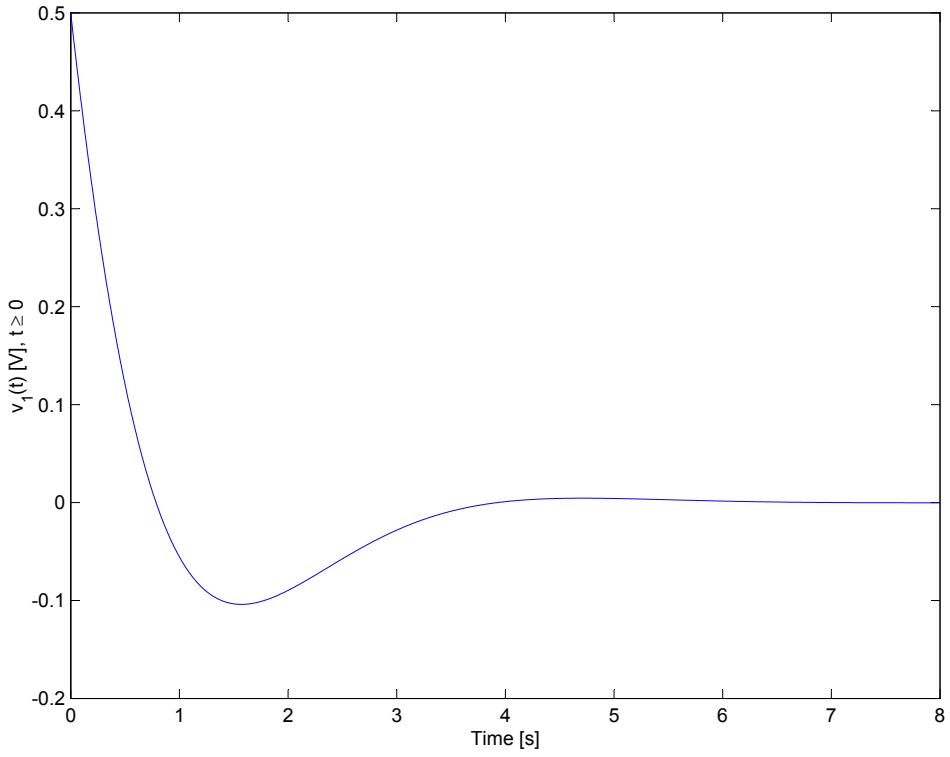


Figure 8.8.1: Plot of $v_1(t)$

(e) Begin by finding the transfer function $\frac{V_1(s)}{I_s(s)}$ by using a current divider relationship after combining the inductor and resistor into a single impedance.

$$I_s(s) = V_1 \left(\frac{1}{R+Ls} + \frac{1}{R} + \frac{1}{\frac{1}{Cs}} \right)$$
$$\frac{V_1}{I_s(s)} = \frac{1+s}{s^2+2s+2}$$

To find $\frac{I_L(s)}{I_s(s)}$, recognize that

$$\frac{I_L(s)}{I_s(s)} = \frac{I_L(s)}{V_1(s)} \frac{V_1(s)}{I_s(s)}$$

Thus, all we need is a relationship between V_1 and I_L to obtain the correct transfer function. Impedance provide that when we combine the resistor and the inductor into a single impedance.

$$I_L(s)(R+Ls) = V_1(s)$$
$$\frac{I_L(s)}{V_1(s)} = \frac{1}{R+Ls}$$

This works out because the resistor and inductor are in series, and thus the current through them are equal. Therefore

$$\frac{I_L(s)}{I_s(s)} = \frac{I_L(s)}{V_1(s)} \frac{V_1(s)}{I_s(s)} = \frac{1}{s^2+2s+2}$$

8.9 LR Circuit Step

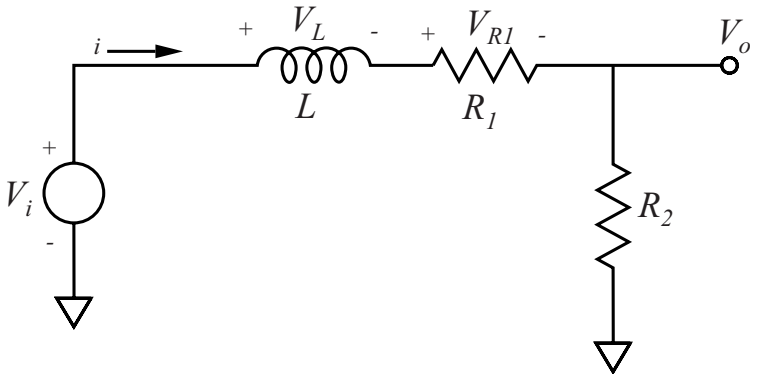


Figure 8.9.1: Circuit Diagram

(a) First, write out constituent relationships for each element:

$$V_L = L\frac{di_L}{dt} \tag{1}$$

$$V_{R1} = i_{R1}R_1 \tag{2}$$

$$V_{R2} = V_o = i_{R2}R_2 \tag{3}$$

From the circuit diagram in Figure 8.9.1, notice that

$$i = i_L = i_{R1} = i_{R2} \tag{4}$$

However, we want the differential equation to be in terms of V_i and V_o. Apply KVL to the entire circuit loop to obtain

$$V_i - V_L - V_{R1} - V_o = 0 \tag{5}$$

To get an expression for V_L, differentiate equation (3) with respect to time and substitute into equation (1), taking into account (4):

$$V_L = \frac{L}{R_2}\frac{dV_o}{dt} \tag{6}$$

To get an expression for V_{R1}, substitute equation (3) into equation (2):

$$V_{R1} = \frac{R_1}{R_2}V_o \tag{7}$$

Substitute equations (6) and (7) back into (5) to obtain the differential equation

$$V_i - \frac{L}{R_2}\frac{dV_o}{dt} - \frac{R_1}{R_2}V_o - V_o = 0$$

$$\frac{L}{R_2}\frac{dV_o}{dt} + \left(1 + \frac{R_1}{R_2}\right)V_o = V_i$$

To find the transfer, substitue $sV(s)$ for each derivative to obtain:

$$V_o(s) \quad \frac{L}{R_2}s + \quad 1 + \frac{R_1}{R_2} \qquad = V_i(s)$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{\frac{L}{R_2}s + \quad 1 + \frac{R_1}{R_2}}$$

$$H(s) = \frac{R_2}{R_1 + R_2} \frac{1}{\frac{L}{R_1+R_2}s + 1}$$

The final form of $H(s)$ is the same as the form above it. However, it is in standard form for easier manipulation later. The pole zero plot is shown below.

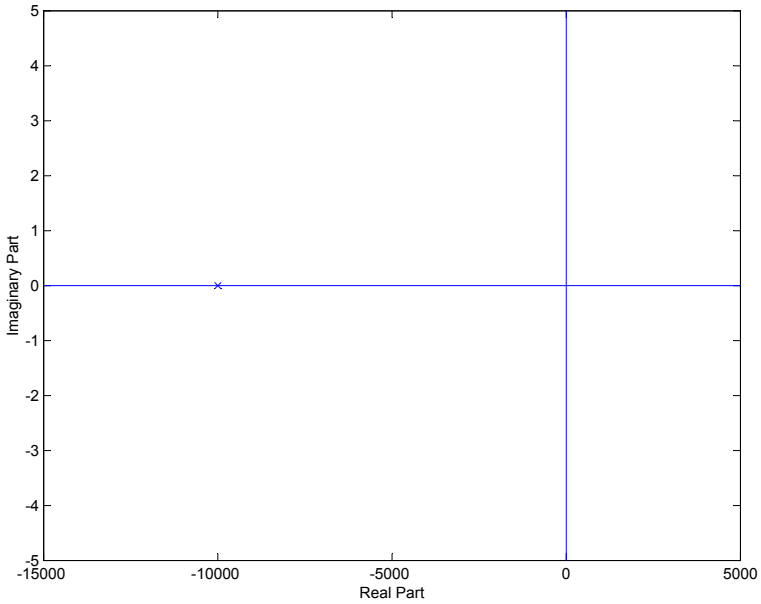


Figure 8.9.2: Pole Zero plot of circuit

(b) To find R_1, apply the Final Value Theorem to the transfer function, using the final value .1 as obtained from the graph. Remember that the Laplace Transform of the unit step function is $\frac{1}{s}$.

$$V_o(\infty) = .1 = \lim_{s\to 0} s \cdot \frac{R_2}{R_1 + R_2} \frac{1}{\frac{L}{R_1+R_2}s + 1} \cdot \frac{1}{s}$$

$$.1 = \frac{R_2}{R_1 + R_2}$$

$$R_1 = 9000[\Omega]$$

To find L, recognize that $\tau = \frac{L}{R_1+R_2}$ from the standard form of a transfer function for an exponential decay $\frac{1}{\tau s+1}$. τ is found from the graph to be $1 \cdot 10^{-4}$ seconds.

$$1 \cdot 10^{-4} = \frac{L}{R_1 + R_2}$$

$$L = 1 \text{ [H]}$$

(c) The value M is the magnitude of the transfer function evaluated at $s = j\omega$, and the vaule ϕ is the phase of the transfer function evaluated at $s = j\omega$. The magnitude of the transfer function is

$$\begin{aligned} |H(j\omega)| = \frac{V_o(j\omega)}{V_i(j\omega)} &= \frac{1}{\sqrt{\left(\frac{R_1}{R_2}+1\right)^2 + \left(\frac{L}{R_2}\omega\right)^2}} \\ &= \frac{1}{\sqrt{100 + \frac{1}{10^6}\omega^2}} \end{aligned}$$

The phase of the transfer function is

$$\begin{aligned} \angle H(j\omega) &= \tan^{-1}\left[\frac{\Im H(j\omega)}{\Re H(j\omega)}\right] \\ &= -\tan^{-1}\left(\frac{\frac{L}{R_2}\omega}{\frac{R_1}{R_2}+1}\right) \\ &= -\tan^{-1}\left(\frac{\omega}{10000}\right) \end{aligned}$$

(d) The bode plot is shown in Figure 8.9.3. The DC gain is the magnitude of the transfer function as $\omega \to 0$, which is 0.1. The high-frequency asypmtotic gain is the magnitude of the transfer function as $\omega \to \infty$, which is 0.

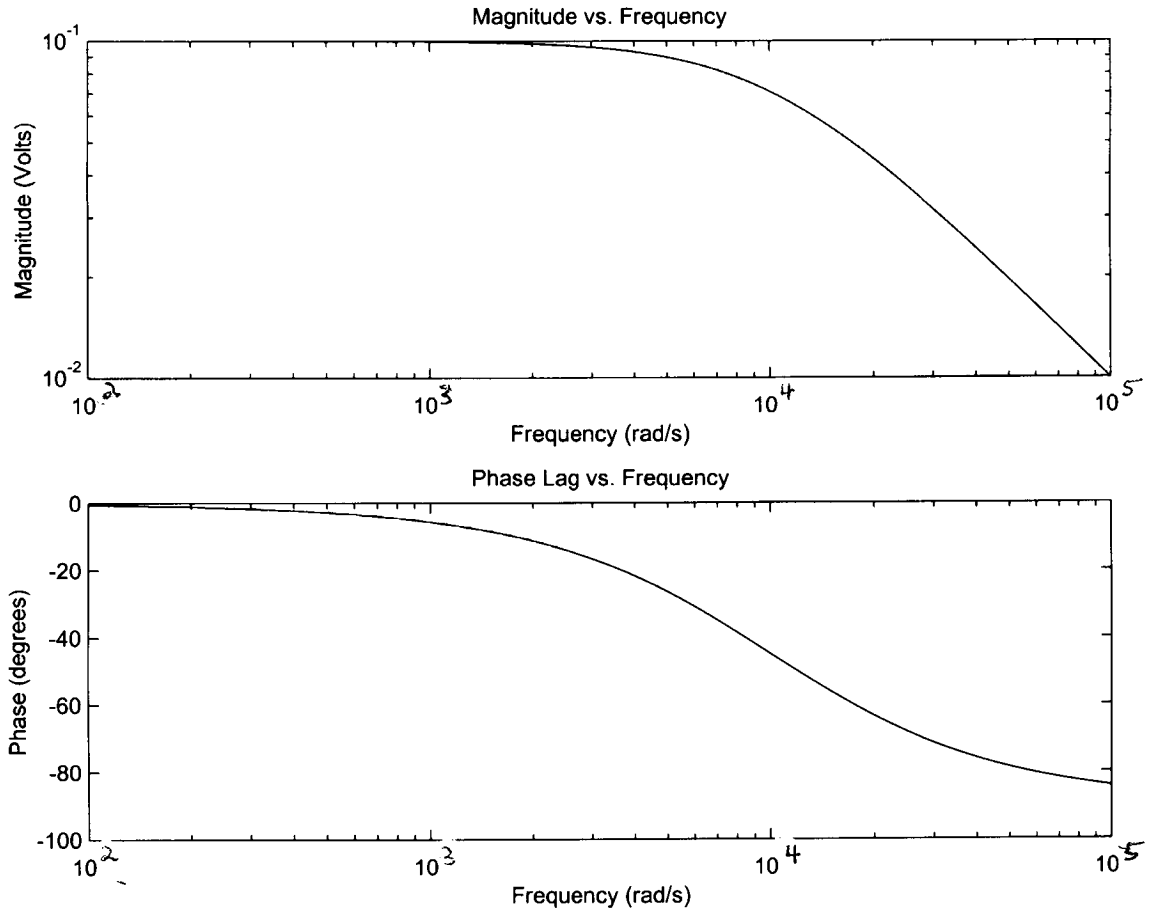


Figure 8.9.3: Bode Plot, M and ϕ

9 Op Amps

9.1 Opamp Arithmetic

The solution for this problem is not available.

9.2 Inverting Op-Amp

The solution for this problem is not available.

9.3 Opamp Transfer Function

The solution for this problem is not available.

9.4 Op-Amp Differentiator 1

In this problem, we examine a practical differentiator. Notice that the second op-amp circuit is simply a unity gain inverter. The transfer function of the circuit is

$$\frac{V_o(s)}{V_i(s)} = \frac{RCs}{R_1Cs+1}$$

(a) The zero of the transfer function occurs at $s_z = 0$. The pole occurs at $s_p = -1/R_1C = -0.5$rad/s.

(b) The magnitude and phase of the transfer function are found by evaluating the transfer function at $s = j\omega$. The magnitude is

$$M = \frac{RC\omega}{\sqrt{(R_1C\omega)^2+1}} = \frac{0.2\omega}{\sqrt{(2\omega)^2+1}}$$

The phase is

$$\phi = \frac{\pi}{2} - \arctan R_1C\omega = \frac{\pi}{2} - \arctan(2\omega)$$

(c) The step response is given in Figure 9.4.1. An ideal differentiator has a transfer function of $\frac{1}{s}$. This transfer function is not ideal because of the resistance R_1. An ideal differentiator has no pole, only a zero. The step response of an ideal differentiator is an impulse.

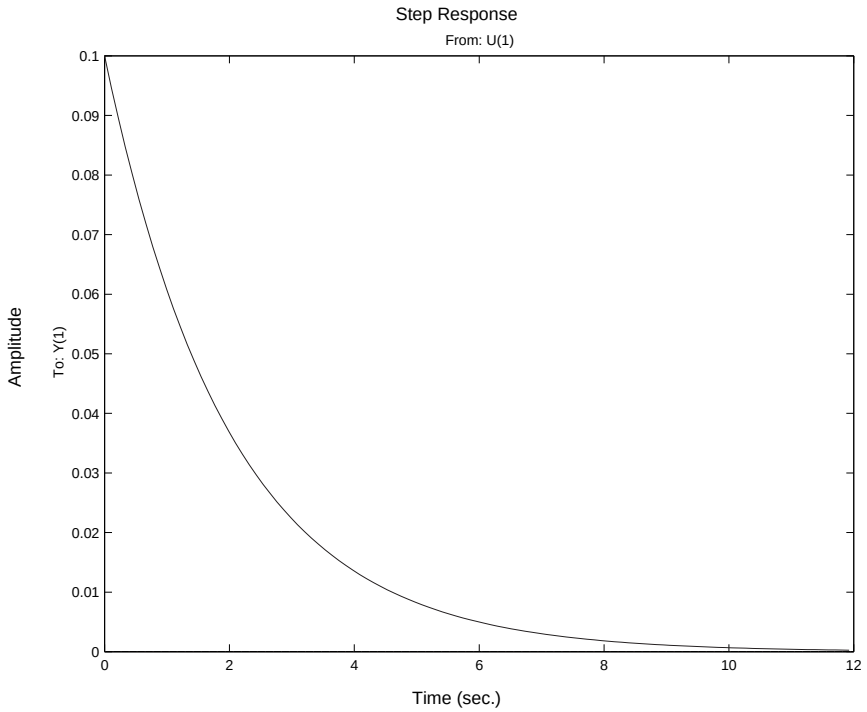


Figure 9.4.1: Step Response

(d) The Bode plot is given in Figure 9.4.2. The breakpoint frequency, also known as the corner frequency, is identical to the pole location. The corner frequency is used for sketching the

asymptotes of the magnitude portion of the Bode plot. At the corner frequency, the transfer function has a value of $10\frac{\sqrt{2}}{2}$. At the corner frequency, actual plot is 3 decibels below the asymptote ($20 \log \frac{\sqrt{2}}{2} \approx -3$).

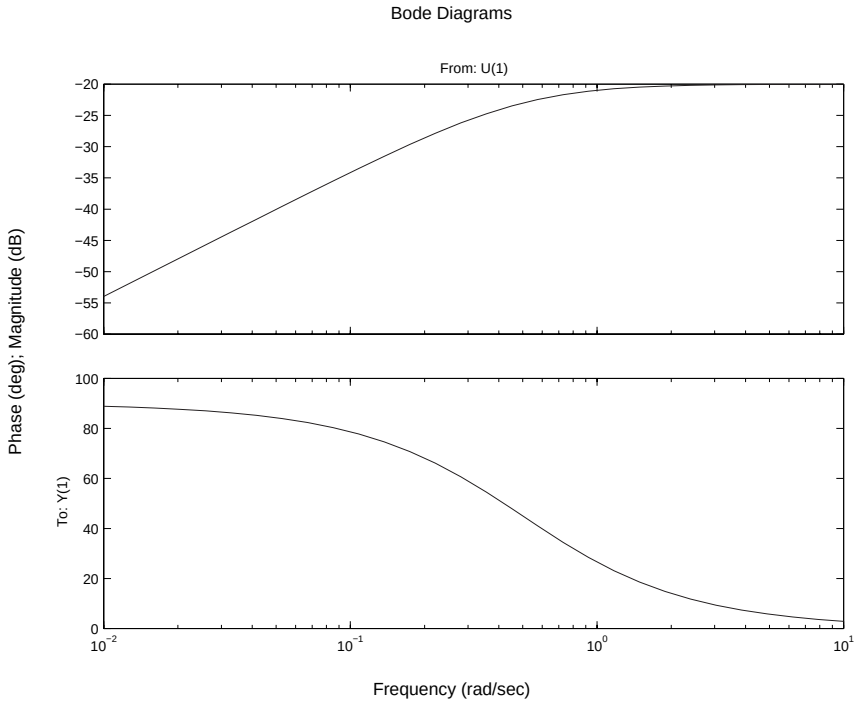


Figure 9.4.2: Bode Plot

9.5 Op-Amp Proportional plus Integral Controller

We are asked to analyze a Proportional plus Integral (PI) circuit. The transfer function for this circuit is

$$\frac{V_o(s)}{V_i(s)} = -\frac{R_f C s + 1}{R_i C s}$$

(a) The pole occurs at $s_p = 0$. The zero is located at $s_z = -1/R_f C = -1$.

(b) The magnitude and phase of the transfer function are found by evaluating the transfer function at $s = j\omega$. The magnitude is

$$M = \frac{\sqrt{(R_f C\omega)^2 + 1}}{R_i C\omega} = \frac{\sqrt{\omega^2 + 1}}{0.1\omega}$$

The phase is

$$\phi = \arctan R_f C\omega - \frac{3\pi}{2} = \arctan \omega - \frac{3\pi}{2}$$

(c) The step response is given in Figure 9.5.1. The transfer function for a PI circuit may be written as

$$H_{PI}(s) = K_p + \frac{K_I}{s}$$

The transfer function may be re-written as

$$\frac{V_o(s)}{V_i(s)} = -\frac{R_f}{R_i} - \frac{1}{R_i C s}$$

The step response will have a non-zero initial value and then grow linearly with time.

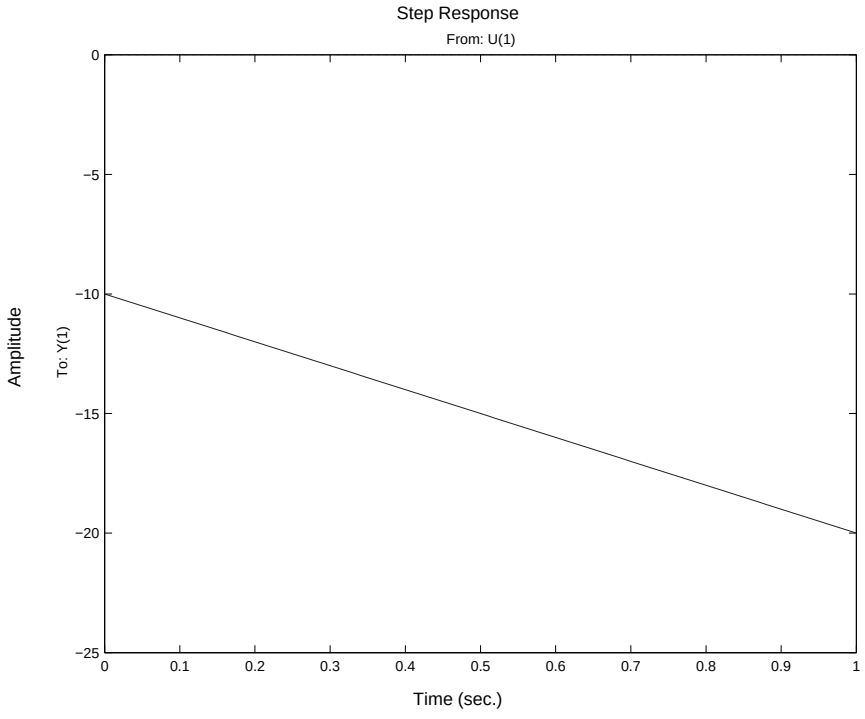


Figure 9.5.1: Step Response

(d) The Bode plot is given in Figure 9.5.2. The breakpoint frequency, also known as the corner frequency, is identical to the zero location.

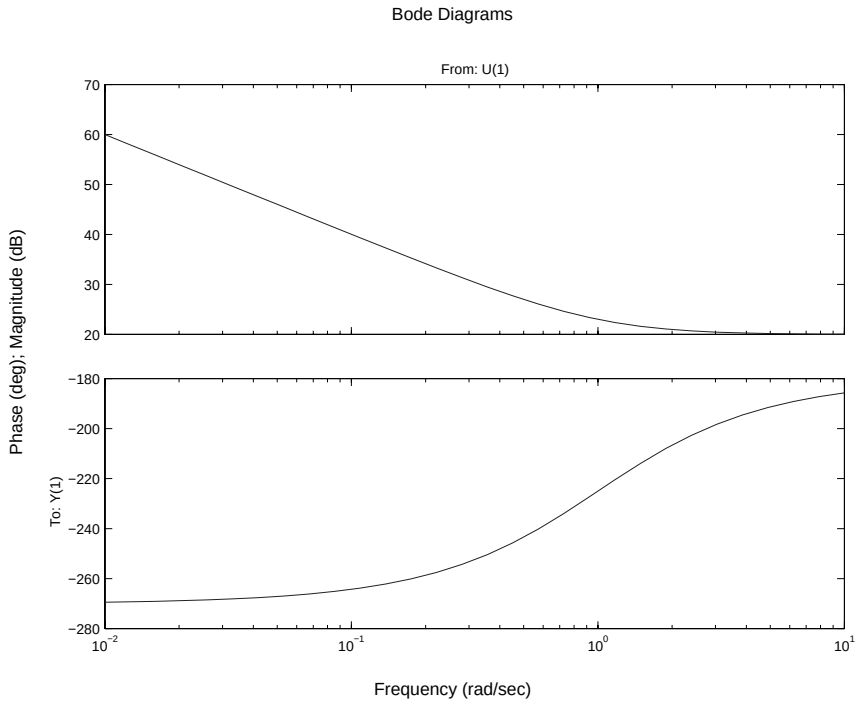


Figure 9.5.2: Bode Plot

9.6 Op-Amp Circuit Design 1

We want to design an non-inverting summing circuit whose output is

$$V_o = 5V_1 + 3V_2 - 10V_3$$

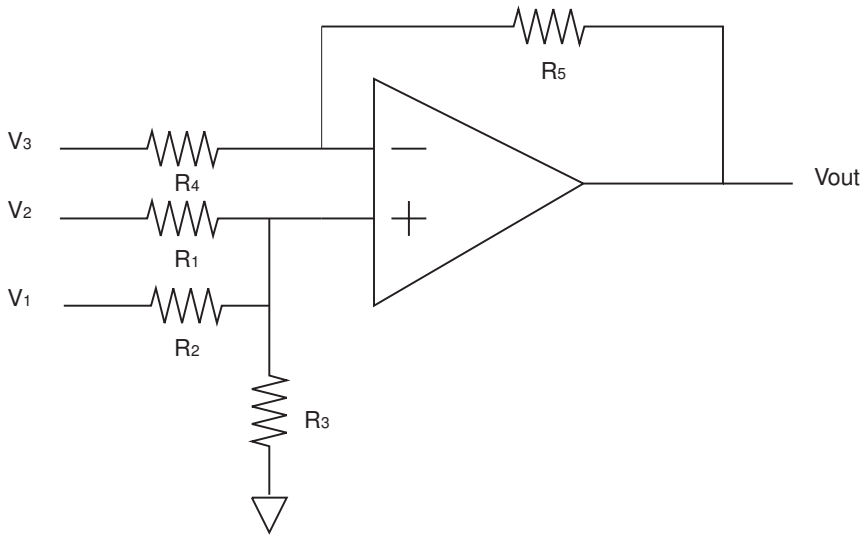


Figure 9.6.1: Summing Amplifier

We should use the principle of superposition, and we must realize that for adding voltages, we should have the inputs go to the non-inverting terminal and subtracting a voltage inputs should go to the inverting terminal. The non-inverting configuration gives the addition of V_1 and V_2, while the inverting configuration gives the subtraction of V_3. Together, they can realize the given equation. The circuit in Figure 9.6.1 gives the desired output for appropriately chosen resistors.

As always, we start with the equation $V_o = A(s)[V_+ - V_-]$ and find V_+ and V_-. We then plug them back into the equation. Using superposition (i.e. setting $V_o = 0$ and solving for V_-, then setting $V_3 = 0$ and solving for V_-, and summing the two) gives

$$V_- = \frac{R_4}{R_4 + R_5}V_o + \frac{R_5}{R_4 + R_5}V_3$$

Using the voltage divider and superposition, we find

$$\begin{aligned} V_+ &= \frac{R_2//R_3}{R_2//R_3 + R_1}V_2 + \frac{R_1//R_3}{R_1//R_3 + R_2}V_1 \\ &= \frac{R_2R_3}{R_2R_3 + R_1(R_2 + R_3)}V_2 + \frac{R_1R_3}{R_1R_3 + R_2(R_1 + R_3)}V_1 \end{aligned}$$

Plugging these values back into the equation and solving for V_o, with $A(s)$ having infinite gain, gives

$$V_o = \frac{R_4 + R_5}{R_4}\left[\frac{R_1R_3}{R_1R_3 + R_2(R_1 + R_3)}V_1 + \frac{R_2R_3}{R_2R_3 + R_1(R_2 + R_3)}V_2 - \frac{R_5}{R_4 + R_5}V_3\right]$$

The V_3 coefficient gives us the constraint that $R_5/R_4 = 10$. We arbitrarily choose $R_4 = 1k\Omega$ and this then sets $R_5 = 10k\Omega$. We then choose $R_3 = 1k$, and this leaves 2 equations and 2 unknowns for the coefficients of V_1 and V_2. Solving the system of equations gives $R_1 = 1k\Omega$ and $R_2 = 600\Omega$. The final solution has the following form:

$$V_o = K\left[C_1V_1 + C_2V_2 - C_3V_3\right]$$

where $K = 11$, $C_1 = 0.4545$, $C_2 = 0.2727$, and $C_3 = 0.9091$.

9.7 Op-Amp Block Diagram

The solution for this problem is not available.

9.8 Op-Amp Circuit Design 2

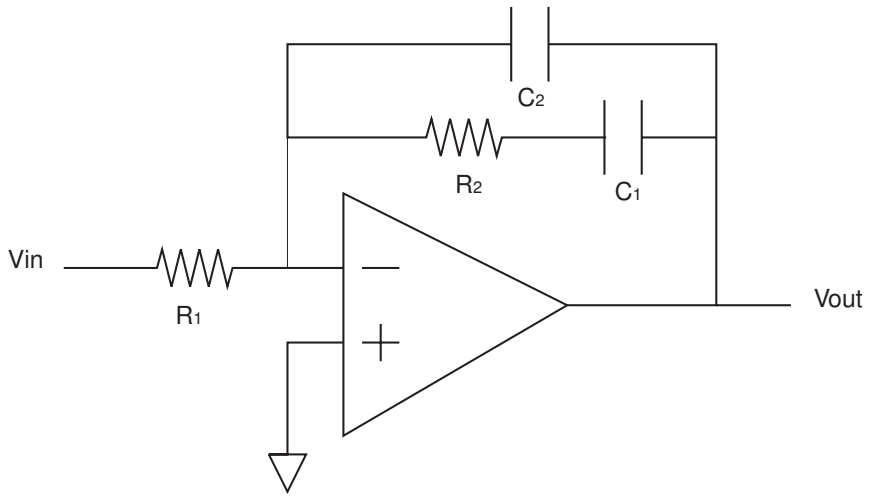


Figure 9.8.1: Circuit implementation of first stage

The circuit in Figure 9.8.1 gives one implementation for the desired output for appropriately chosen resistors and capacitors. There is more than one possible design.

As presented in tutorials, the circuit without the C_2 capacitor gives a pole at the origin and a zero. This leaves us to figure out how to get an additional pole. The impedance of a capacitor is $1/Cs$ so that may be a good starting point to achieving a pole. We add a capacitor in parallel with the capacitor and resistor in series and this does the trick.

The impedance across the negative and output terminals is

$$Z = \frac{R_2Cs + 1}{s(R_2C_1s + C_1 + C_2)}$$

We know that for an inverting op using the infinite gain model, $V_{out}/V_{in} = Z/R_1$. The transfer function then reduces to

$$\frac{V_o}{V_{in}} = -\frac{1}{R_1(C_1 + C_2)} \frac{R_2Cs + 1}{\frac{R_2C_1C_2}{C_1+C_2}s + 1}$$

We choose $C_1 = 0.01\mu$F and that sets $R_2 = 100k\Omega$. Solving the rest of the equations, we get $C_2 = 1.11n$F and $R_1 = 900k\Omega$.

Finally, we add a negative unity gain buffer to invert the signal and make it positive overall. This circuit is cascaded onto the first and it is shown in Figure 9.8.2.

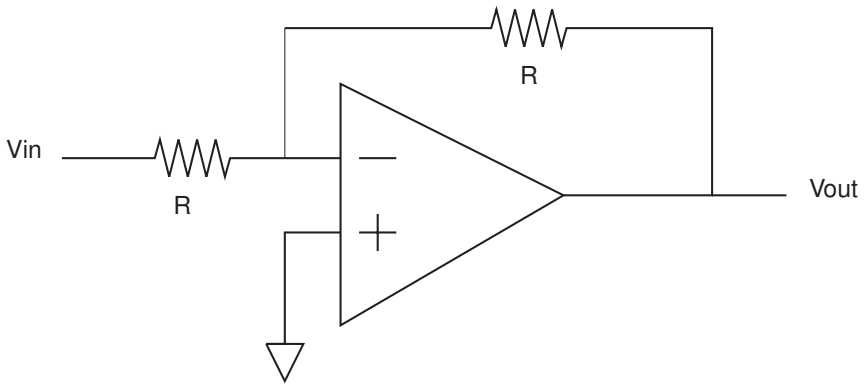


Figure 9.8.2: Negative unity gain buffer

9.9 Op-Amp Differentiator 2

The op-amp output is $V_{out} = A(s)(V_+ - V_-)$.

(a) Because the op-amps have a low output impedance, we may model the system as two blocks cascaded (see Figure 9.9.1).

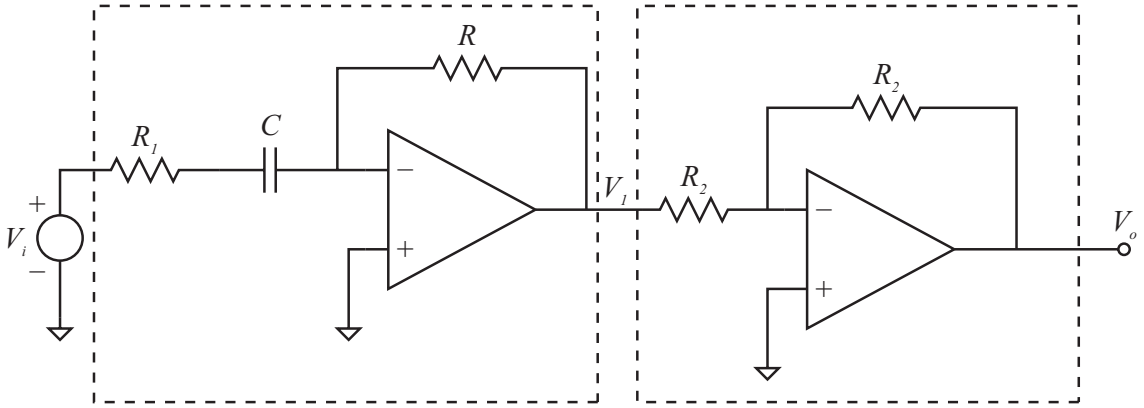


Figure 9.9.1: Cascaded Block Circuit

First, consider the differentiator block. Using the voltage divider relation and the principle of superposition, the voltage at the inverting terminal is

$$V_- = \frac{R_1 + \frac{1}{Cs}}{R_1 + R + \frac{1}{Cs}} V_1 + \frac{R}{R_1 + R + \frac{1}{Cs}} V_i$$

The voltage at the non-inverting terminal is zero. The amplifier outputs $V_1 = A(s)(-V_-)$. The block diagram for this part of the circuit is given in Figure 9.9.2.

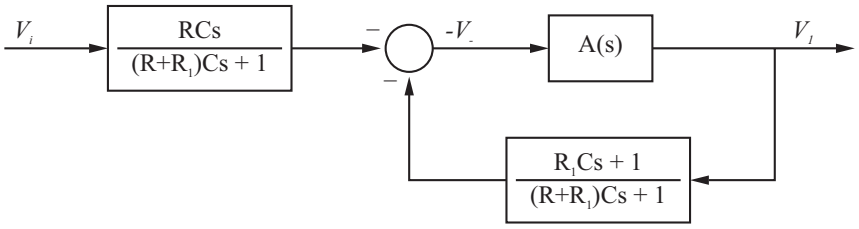


Figure 9.9.2: Block Diagram

Next, consider the inverting amplifier part of the circuit. The voltage input is V_1. The voltage at the non-inverting terminal is zero. The voltage at the inverting terminal is

$$V_- = \frac{V_o + V_1}{2}$$

The output voltage of the op-amp is $V_o = -A(s)V_-$. The block diagram is shown in Figure 9.9.3.

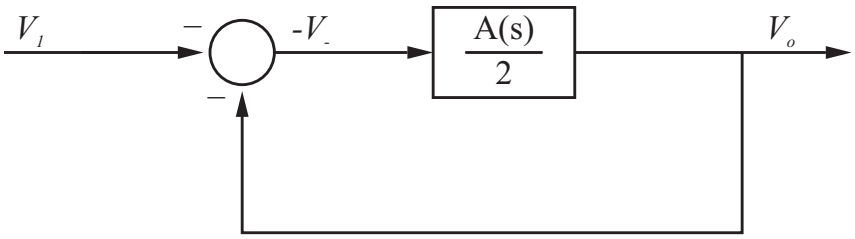


Figure 9.9.3: Block Diagram for Inverting Amplifier

To obtain the total block diagram, we simply cascade the two blocks. In general, we cannot simply cascade two systems because there will interaction between the two systems. With two cascaded blocks, the second block has no effect on the first. However, because the op-amps have a very small output impedance, we may cascade the two blocks. The total block diagram is shown in Figure 9.9.4.

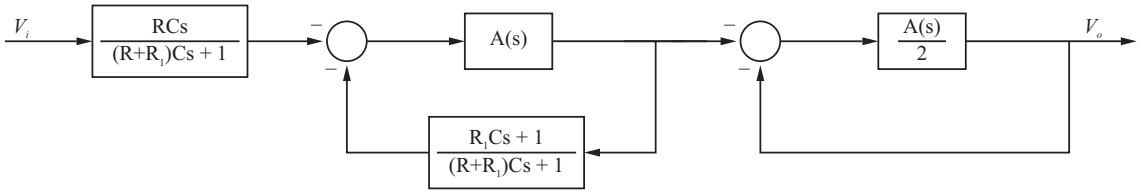


Figure 9.9.4: Block Diagram of Cascaded System

(b) Assuming $A(s) = g/s$, the transfer function for the first block (practical differentiator) is

$$H_1(s) = \frac{-A(s)RCs}{\big((R+R_1)Cs+1\big) + A(s)(R_1Cs+1)}$$

$$H_2(s) = \frac{\frac{-A(s)}{2}}{1+\frac{A(s)}{2}}$$

Substituting $A(s) = g/s$, we obtain

$$H_1(s) = \frac{-gRCs}{s\big((R+R_1)Cs+1\big) + g(R_1Cs+1)}$$

and

$$H_2(s) = \frac{-1}{\frac{2}{g}s+1}$$

The total transfer function is

$$\frac{V_o(s)}{V_i(s)} = \frac{1.257\times10^7 s}{7.003\times10^{-8}s^3 + 6.2s^2 + 1.257\times10^8 s + 6.283\times10^7}$$

(c) The pole-zero plot is shown in Figure 9.9.5.

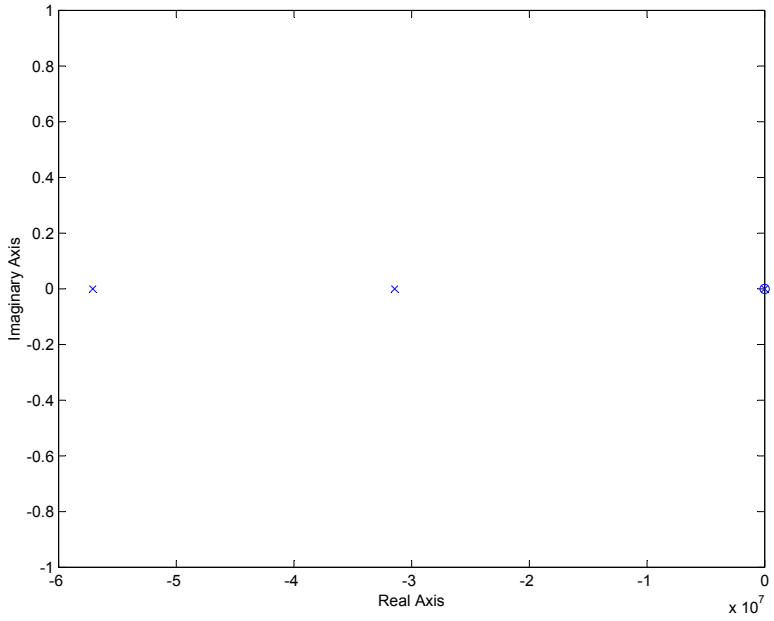


Figure 9.9.5: Pole Zero Plot

The zero is located at $s_z = 0$. The poles are located at $s_p = -0.5, -3.1437\times10^7, -5.7096\times10^7$. The dominant pole is $s = -0.5$. The transfer function for the dominant roots is of the form

$$H(s) = \frac{s}{s + 0.5}$$

In other words, the approximate transfer function is a non-ideal differentiator. The step response for the approximate system is shown in Figure 9.9.6.

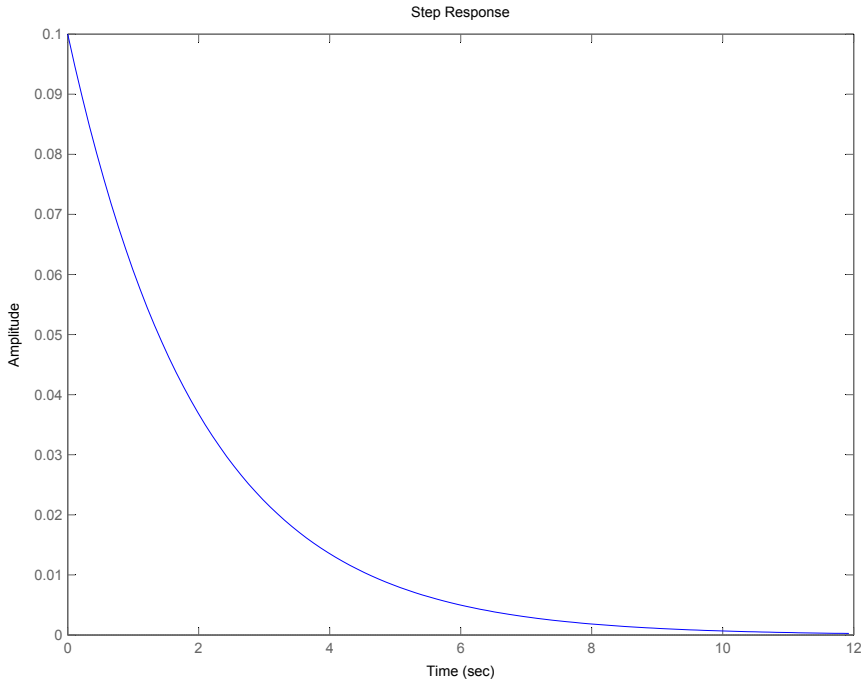


Figure 9.9.6: Step response ignoring higher-order dynamics

(d) The step response of the exact transfer function is shown in Figure 9.9.7. We see that step response is the same as for the system including only the dominant dynamics except for the first .02 seconds. The reason that the actual system takes a finite amount of time to approach the ideal behavior is because of the high frequency dynamics. As seen from the Bode plot (see Figure 9.9.8), the system has a very small response at high frequencies (above 1MHz). The step response excites all frequencies, but the contribution to the response is very small when the dynamics of the op-amps are considered.

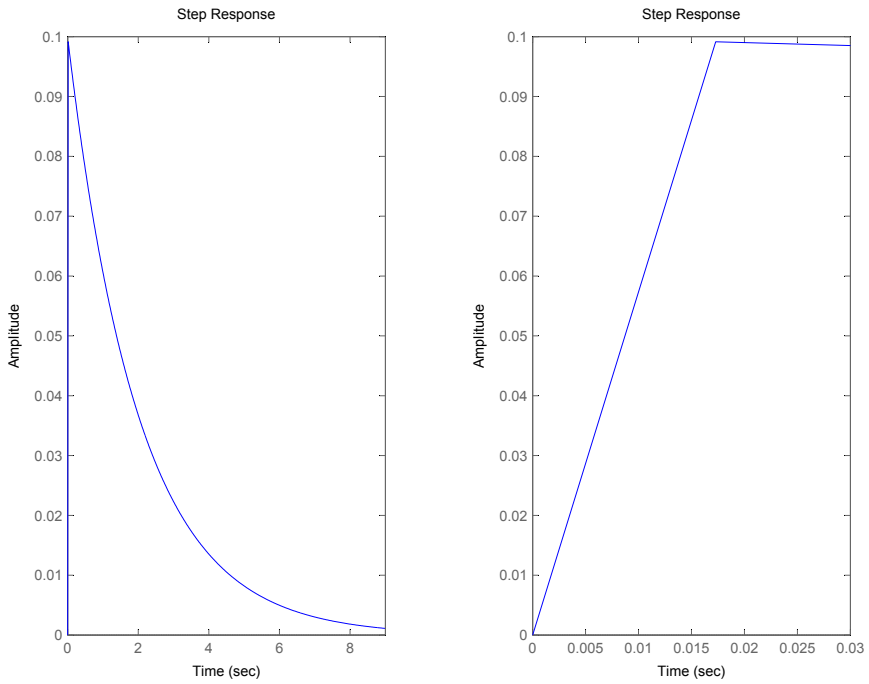


Figure 9.9.7: Step response of exact transfer function

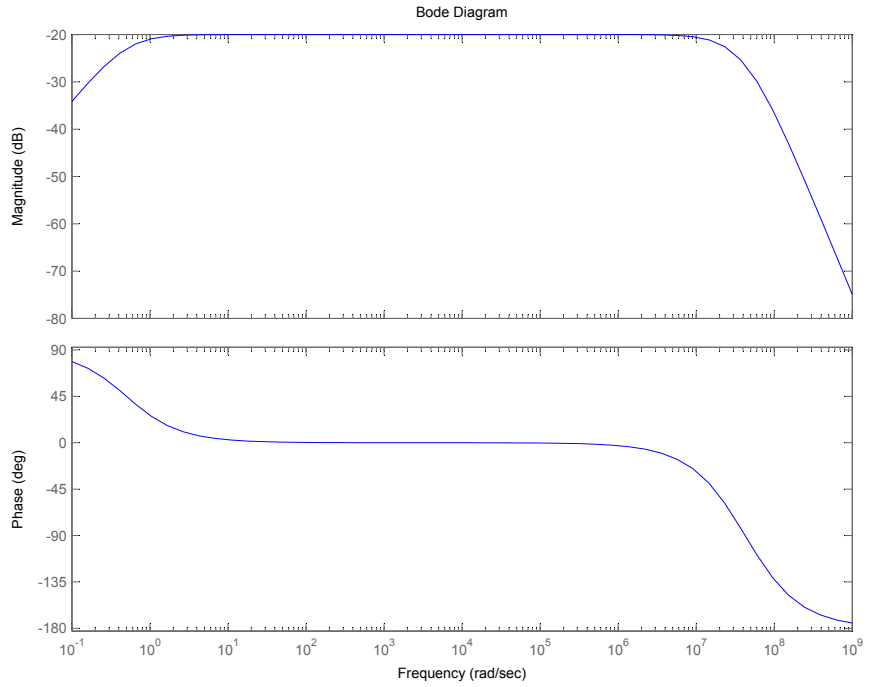


Figure 9.9.8: Bode Plot of full Transfer Function

9.10 RC Op-Amp Frequency Response

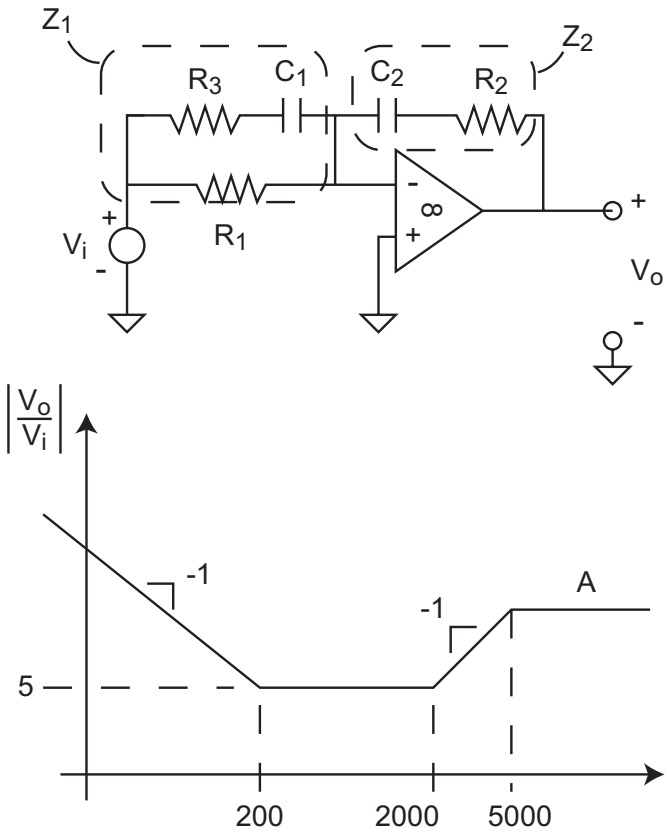


Figure 9.10.1: RC Op-Amp Circuit

(a) Figure 9.10.1 shows the circuit for this problem. I solved this using impedance methods. Both Z_1 and Z_2 share the same current i. With $a(s) = \infty$, we can make the assumption that $V^- = 0$.

$$\begin{aligned}
Z_1 &= \left(\frac{1}{R_1} + \frac{C_1 s}{R_3 C_1 s + 1}\right)^{-1} = \frac{R_1 R_3 C_1 s + R_1}{C_1(R_1 + R_3)s + 1} \\
Z_2 &= R_2 + \frac{1}{C_2 s} = \frac{R_2 C_2 s + 1}{C_2 s} \\
I &= \frac{C_1(R_1 + R_3)s + 1}{R_1 R_3 C_1 s + R_1} V_i \\
I &= -\frac{C_2 s}{R_2 C_2 s + 1} V_o \\
\frac{V_o}{V_i} &= -\frac{(R_2 C_2 s + 1)(C_1(R_1 + R_3)s + 1)}{C_2 s(R_1 R_3 C_1 s + R_1)}
\end{aligned}$$

(b) Our transfer function has two poles and two zeros. We need to associate these poles and zeros with the breakpoints on the frequency response. We have a pole at the origin which is associated with the initial negative slope. We then have a zero at 200 r/s (slope goes to

zero). We then have a 2nd zero at 2000 r/s and the 2nd pole at 5000 r/s. To start with we note that the low frequency response is dominated by the pole at the origin. We can express the low frequency magnitude as

$$M(j\omega) = \frac{1}{C_2 R_1 \omega}$$

Since we are given R_1 if we know the magnitude at some low frequency we can find C_2. Well we know the magnitude at 200 is 5 and thus using just the low frequency asymptote

$$5 = \frac{1}{R_1 C_2 200} \rightarrow C_2 = \frac{1}{5 * 10000 * 200} = 0.1\mu F$$

Now we can tie the $(R_2 C_2 s + 1)$ zero or $(C_1(R_1 + R_3)s + 1)$ with the breakpoint at 200. I chose the following:

$$200 = \frac{1}{R_2 C_2} \rightarrow R_2 = \frac{1}{200 * 0.1E - 6} = 50\ k\Omega.$$

Now all that is left is the zero at 2000 and the pole at 5000, thus:

$$\begin{aligned} 2000 &= \frac{1}{C_1(R_1 + R_3)} \\ 5000 &= \frac{1}{R_3 C_1} \\ 2000 C_1(R_1 + R_3) &= 5000 R_3 C_1 \rightarrow 3000 R_3 = 2000 R_1 \\ R_3 &= \frac{2}{3} R_1 = 6667\ \Omega \\ C_1 &= \frac{1}{5000 * R_3} = 0.03\mu F \end{aligned}$$

Now we need to figure out A. Well we know that the slope of magnitude between 2000 and 5000 is +1. Thus

$$\begin{aligned} A &= 10^{\log(5) + \Delta x} \\ \Delta x &= \log(5000) - \log(2000) = 3.699 - 3.301 = 0.395 \\ A &= 10^{(0.7 + 0.395)} = 12.5 \end{aligned}$$

Alternately, you could use the fact that at high frequencies

$$M(\omega) = \frac{R_2(R_1 + R_3)}{R_1 R_3} = \frac{50000(16667)}{10000 * 6667} = 12.5$$

Figure 9.10.2 shows the actual frequency response of this circuit with the asymptotes.
If you choose to set the first zero using $(C_1(R_1 + R_2)s + 1)$ you would get the following component values:

$$\begin{aligned} R_2 &= 5000\ \Omega \\ R_3 &= 417\ \Omega \\ C_1 &= 0.48\ \mu F \end{aligned}$$

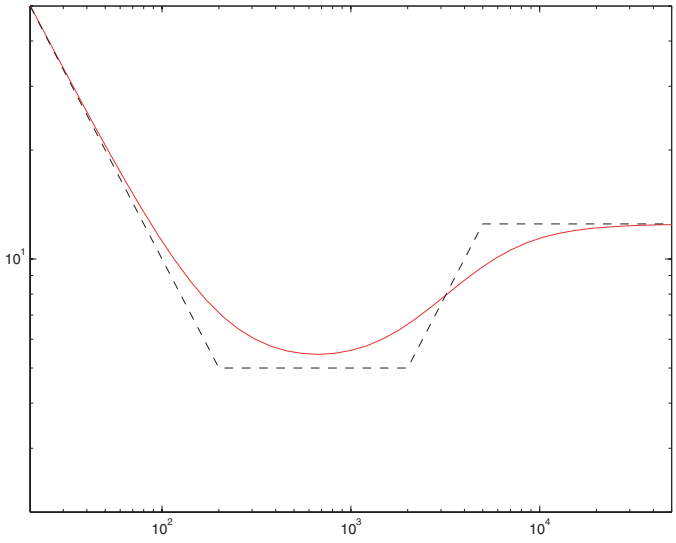


Figure 9.10.2: Frequency Response

10 Differential Equations

10.1 Differential Equations 1

(a) $500\dot{y} + 1000y = 0$

$$y_h(t) = c_1 e^{s_1 t} \text{and} \frac{dy_h}{dt} = c_1 s_1 e^{s_1 t}$$
$$500 c_1 s_1 e^{s_1 t} 1000 c_1 e^{s_1 t} = 0$$
$$(500 s_1 + 1000) c_1 e^{s_1 t} = 0$$

therefore $s_1 = -2$

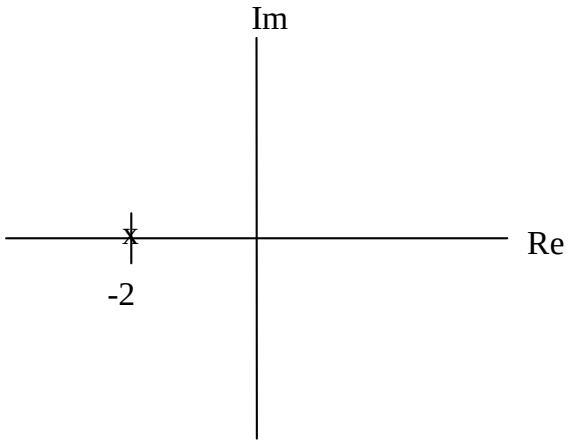


Figure 10.1.1: Plot of poles for part (a)

(b) $y(-3) = 10$

$$y_h(t) = c_1 e^{s_1 t}$$
$$c_1 e^6 = 10$$
$$c_1 = \frac{10}{e^6}$$
$$y_h(t) = \frac{10}{e^6} e^{-2t}$$

(c) $y_h(t) < 10^{-6}$

$$\frac{10}{e^6} e^{-2t} < 10^{-6}$$
$$e^{-2t} < 10^{-7} e^6$$
$$-2t < \ln(10^{-7} e^6)$$
$$t > 5.06$$

(d) Using units of meters for y, we have the following plot shown in Figure 10.1.2, made with the script 'diff2sol.m' found at the end of the solution.

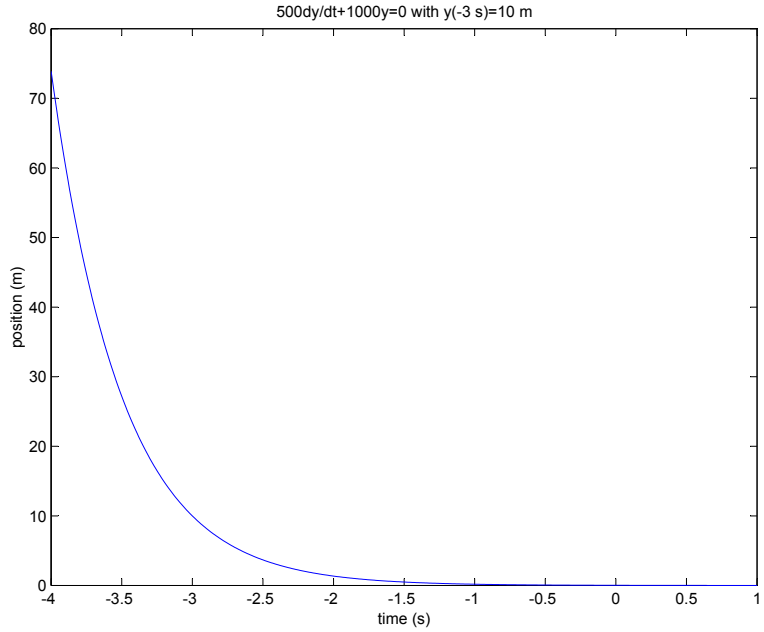


Figure 10.1.2: Plot for Part (d)

(e) One physical system would be a spring-dashpot system like the one modeled in Lab 1. The variable y would represent the displacement in units of meters (m). The dashpot would have a viscous damping constant (b) of 500 Ns/m, and the spring would have a spring constant (k) of 1000 N/m. First order systems are not just limited to springs and dashpots but describe electrical systems, fluid flow, and many other natural behaviors. Figure 10.1.3 is taken from the ActivLab website.

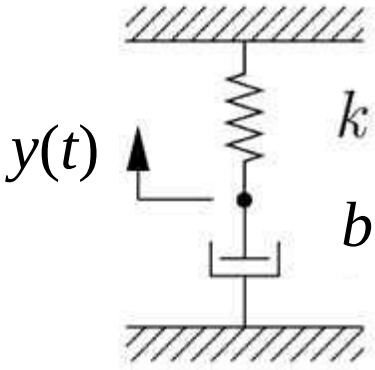


Figure 10.1.3: Physical System

10.2 Differential Equations 2

(a) $\ddot{y} + 10\dot{y} + 10000y = 0$

$$\omega_n = \sqrt{\frac{k}{m}} \text{ and } \zeta = \frac{b}{2\sqrt{km}} \text{ and } \omega_d = \omega_n\sqrt{1-\zeta^2}$$

so

$$\begin{aligned} \omega_n &= 100\text{rad/s} \\ \zeta &= 0.05 \\ \omega_d &= 99.9\text{rad/s} \end{aligned}$$

(b) $y(0.1) = -3$ and $\dot{y}(0.1) = 0$

From lecture for an underdamped system:

$$y_h(t) = c_1 e^{-\sigma t}\cos\omega_d t + c_2 e^{-\sigma t}\sin\omega_d t$$

so

$$\dot{y}_h(t) = -c_1\sigma e^{-\sigma t}\cos\omega_d t - c_1\omega_d e^{-\sigma t}\sin\omega_d t - c_2\sigma e^{-\sigma t}\sin\omega_d t + c_2\omega_d e^{-\sigma t}\cos\omega_d t$$

From part (a), we know ω_n, ζ, and ω_d. We also know $\sigma = \zeta\omega_n$. Therefore we can solve for c_1 and c_2 based on the initial conditions.

$$\begin{aligned} c_1 &= 4.051 \\ c_2 &= 2.848 \end{aligned}$$

(c)

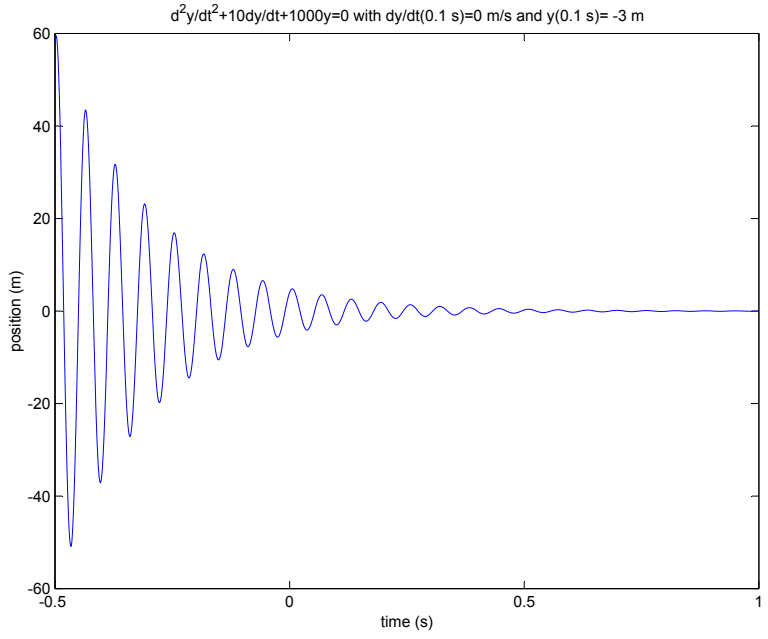


Figure 10.2.1: Plot of Position vs. time

We can zoom in on the plot shown in Figure 10.2.1 near the given initial conditions to look at $\dot{y}$ and y at $t = .01$s. This zoomed plot is shown in Figure 10.2.2.

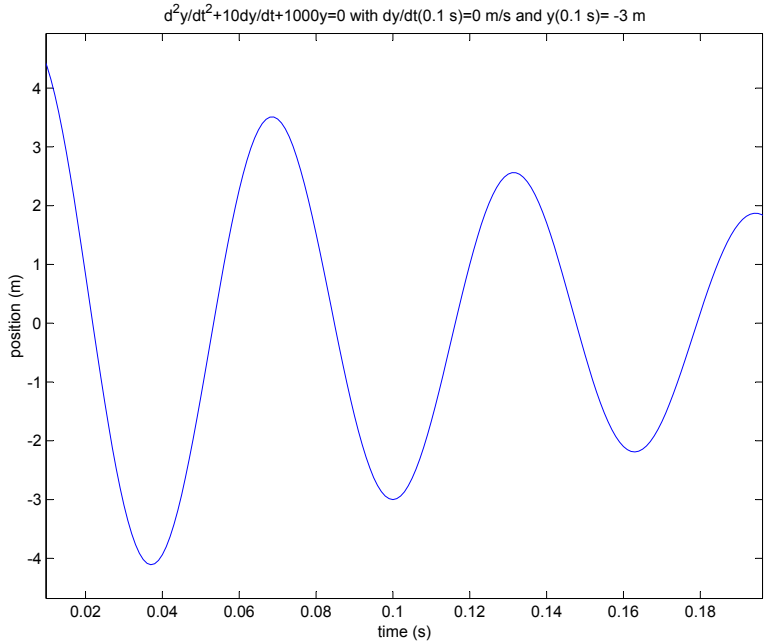


Figure 10.2.2: Zoomed Plot of Position vs. time

If we look at Figure 10.2.2 at $t = 0.1$s, we can see that the position is approximately -3 m. Because the position curve is at a peak, we can also conclude that the velocity $\dot{y}$ is 0.

(d) One physical system would be a mass-spring-dashpot system like the one shown below. The variable y would represent the displacement in units of meters (m). The dashpot would have a viscous damping constant (b) of 10 Ns/m, the spring would have a spring constant (k) of 10000 N/m, and the mass (m) would be 1 kg. Second order systems are not just limited to masses, springs, and dashpots but describe electrical systems, fluid flow, and many other natural behaviors. Figure 10.2.3 below is taken from the ActivLab website.

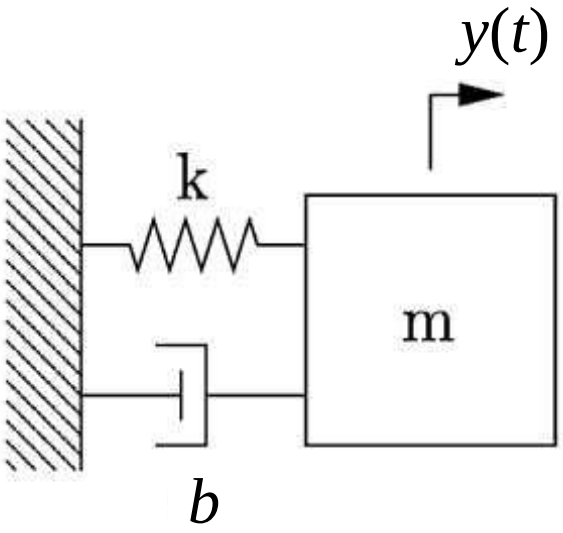


Figure 10.2.3: Physical System

10.3 Differential Equations 3

(a) $\ddot{y} + 200\dot{y} + 10000y = 0$

$$\omega_n = \sqrt{\frac{k}{m}} \text{ and } \zeta = \frac{b}{2\sqrt{km}} \text{ and } \omega_d = \omega_n\sqrt{1-\zeta^2}$$

so

$$\begin{aligned} \omega_n &= 100 \text{ rad/s} \\ \zeta &= 1 \\ \omega_d &= 0 \text{ rad/s} \end{aligned}$$

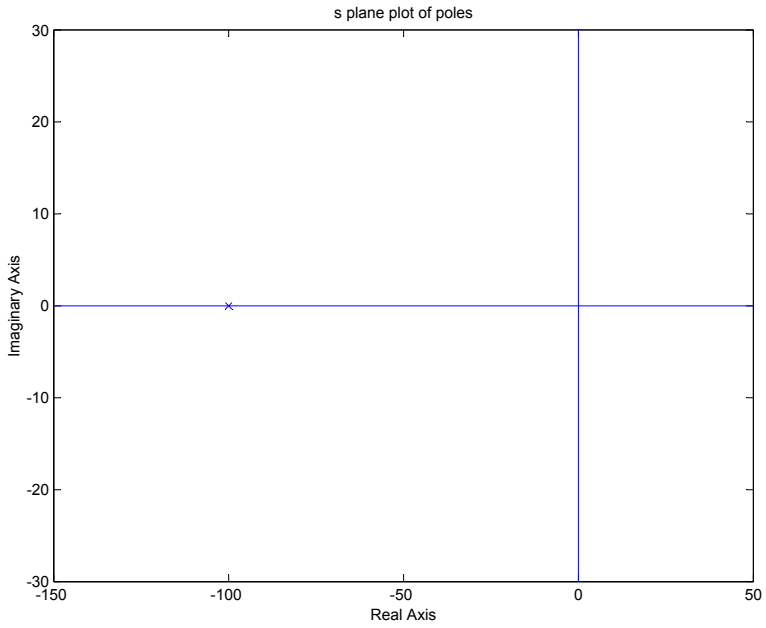


Figure 10.3.1: Poles for Part (a)

(b) $y(0) = 1$ and $\dot{y}(0) = 1$

For the critically damped case:

$$\begin{aligned} y_h(t) &= c_1e^{-\sigma t} + c_2te^{-\sigma t} \\ \dot{y}_h(t) &= -\sigma c_1e^{-\sigma t} + c_2e^{-\sigma t} - \sigma c_2te^{-\sigma t} \\ \sigma &= \zeta\omega_n \end{aligned}$$

so

$$c_1 = 1 \text{ given } y(0) = 1$$
$$c_2 = 101 \text{ given } \dot{y}(0) = 1$$

therefore

$$y_h(t) = e^{-100t} + 101te^{-100t}$$

Looking at the last plot of Figure 10.3.2, we can see that both $\dot{y}(0)$ and $y(0)$ are 1 at $t = 0$.

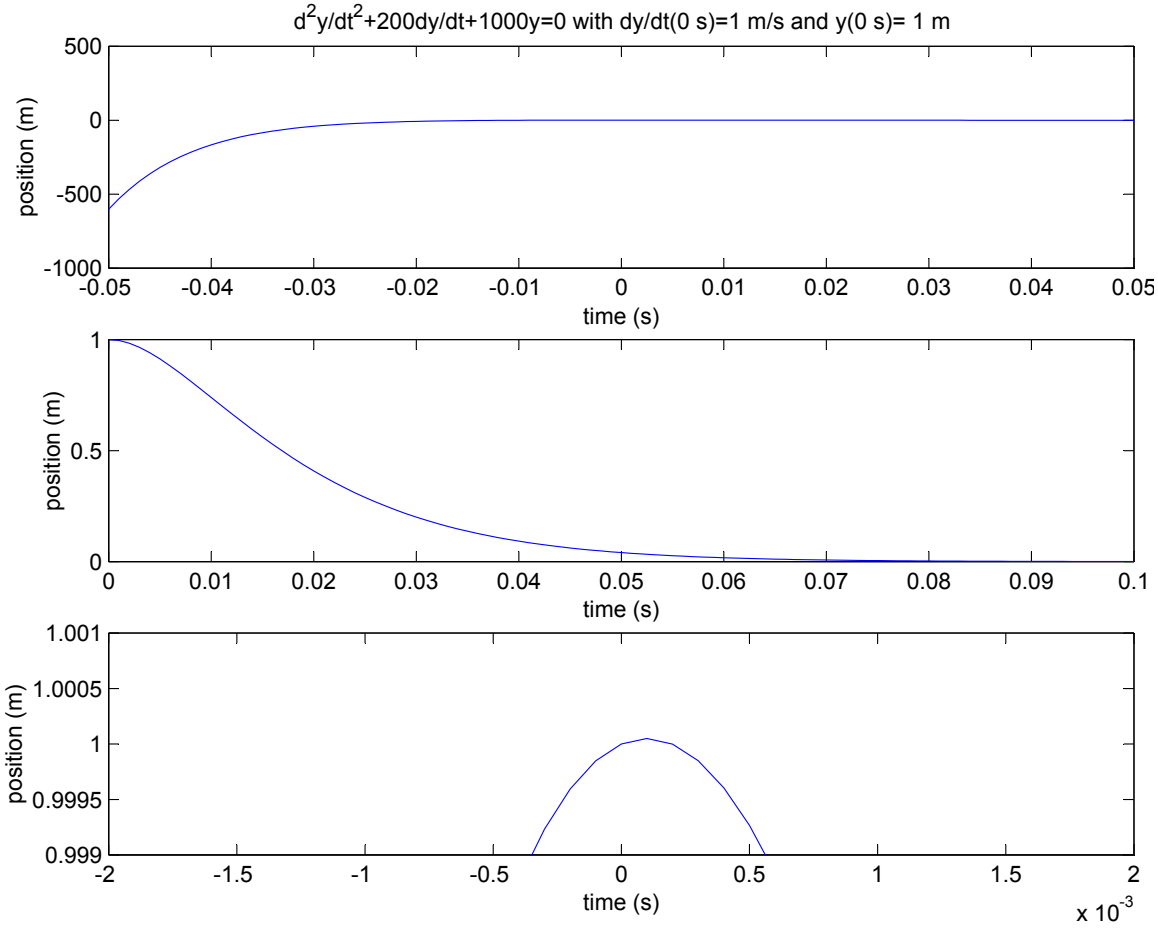


Figure 10.3.2: Plots for Part (b)

10.4 Euler's Theorem 1

Three Forms of Eulers Theorem:

$$\begin{aligned} e^{j\phi} &= \cos\phi + j\sin\phi \\ \cos\phi &= \left(\tfrac{e^{j\omega t}+e^{-j\omega t}}{2}\right) \\ \sin\phi &= \left(\tfrac{e^{j\omega t}-e^{-j\omega t}}{2j}\right) \end{aligned}$$

(a)

$$\begin{aligned} ce^{j\omega t} + c^* e^{-j\omega t} &= A\cos\omega t + B\sin\omega t \text{ where } c = \alpha + j\beta \text{ and } c^* = \alpha - j\beta \\ ce^{j\omega t} + c^* e^{-j\omega t} &= c(\cos\omega t + j\sin\omega t) \\ &= (\alpha + j\beta)(\cos\omega t + j\sin\omega t) + (\alpha - j\beta)(\cos\omega t - j\sin\omega t) \\ &= \alpha\cos\omega t + j\alpha\sin\omega t + j\beta\cos\omega t - \beta\sin\omega t + \alpha\cos\omega t - j\alpha\sin\omega t - j\beta\cos\omega t - \beta\sin\omega t \\ &= 2\alpha\cos\omega t - 2\beta\sin\omega t \\ &= A\cos\omega t + B\sin\omega t \end{aligned}$$

so

$$\begin{aligned} \alpha = \frac{A}{2} \quad \text{and} \quad \beta = \frac{-B}{2} \\ c = \frac{A}{2} - j\frac{B}{2} \quad \text{and} \quad c^* = \frac{A}{2} + j\frac{B}{2} \end{aligned}$$

Or, if you don't like that method:

$$\begin{aligned} A\cos\omega t + B\sin\omega t &= A\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) + B\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) \\ &= \frac{A}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) + \frac{B}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \\ &= \left(\frac{A}{2} + \frac{B}{2j}\right)e^{j\omega t} + \left(\frac{A}{2} + \frac{B}{2j}\right)e^{-j\omega t} \end{aligned}$$

therefore

$$\begin{aligned} c &= \frac{A}{2} + \frac{B}{2j} \\ &= \frac{A}{2} + \frac{Bj}{2jj} \\ &= \frac{A}{2} - j\frac{B}{2} \end{aligned} \qquad \begin{aligned} c^* &= \frac{A}{2} - \frac{B}{2j} \\ &= \frac{A}{2} - \frac{Bj}{2jj} \\ &= \frac{A}{2} + j\frac{B}{2} \end{aligned}$$

where c^* is the complex conjugate of c.

(b) $c = \frac{7}{2} - j$ and $c^* = \frac{7}{2} + j$

(c) Using units of meters for A and B and rad/s for ω, we get the plots seen in Figure 10.4.1. The following MATLAB script creates this plot.

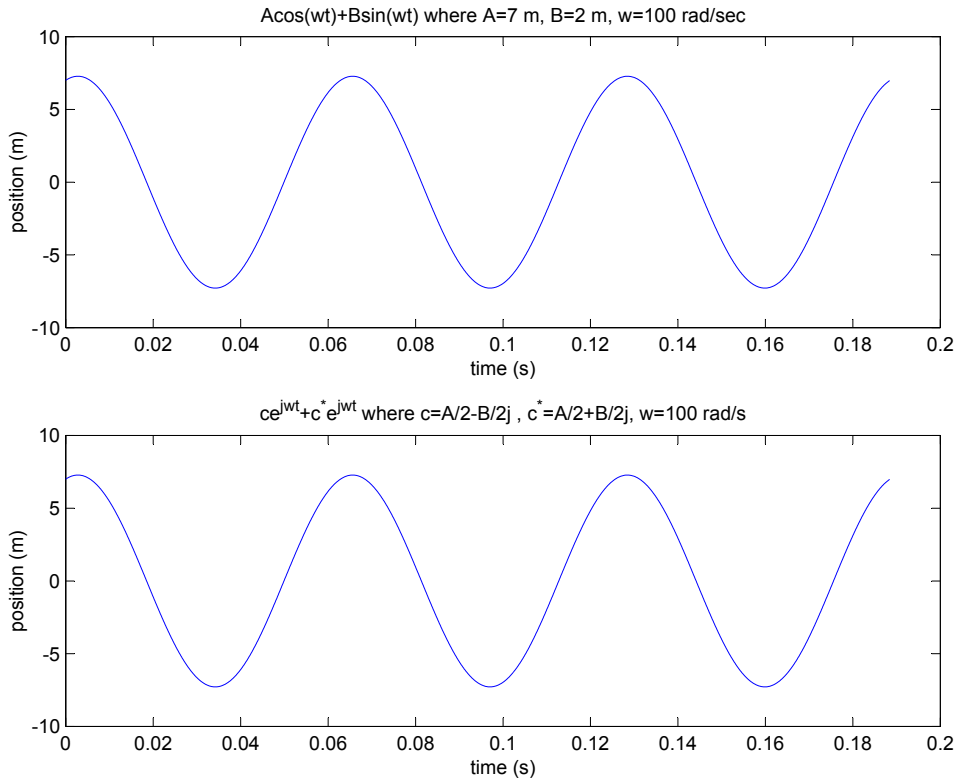


Figure 10.4.1: Plots for Part (c)

10.5 Euler's Theorem 2

(a)

$$\begin{aligned} M\cos\omega t+\phi &= M\left(\frac{e^{j(\omega t+\phi)}+e^{-j(\omega t+\phi)}}{2}\right) \\ &= \frac{M}{2}\left(e^{j\omega t}e^{j\phi}+e^{-j\omega t}e^{-j\phi}\right) \\ &= \frac{M}{2}e^{j\omega t}e^{j\phi}+\frac{M}{2}e^{-j\omega t}e^{-j\phi} \end{aligned}$$

From Problem 10.4:

$$A\cos\omega t+B\sin\omega t = A\left(\frac{e^{j\omega t}+e^{-j\omega t}}{2}\right)+B\left(\frac{e^{j\omega t}-e^{-j\omega t}}{2j}\right)$$

so

$$\left(\frac{A}{2}+\frac{B}{2j}\right)e^{j\omega t}=\frac{M}{2}e^{j\phi}e^{j\omega t} \qquad \text{and} \qquad \left(\frac{A}{2}-\frac{B}{2j}\right)e^{-j\omega t}=\frac{M}{2}e^{-j\phi}e^{-j\omega t}$$

$$\left(\frac{A}{2}+\frac{B}{2j}\right)=\frac{M}{2}e^{j\phi} \qquad \left(\frac{A}{2}-\frac{B}{2j}\right)=\frac{M}{2}e^{-j\phi}$$

therefore

$$\begin{aligned} \frac{M}{2}\left(\cos\phi+j\sin\phi\right) &= \frac{A}{2}-\frac{B}{2}j \\ M\cos\phi &= A \\ M\sin\phi &= B \\ M^2\ \ \cos\phi^2+\sin\phi^2\ \ &= A^2+B^2 \\ \tan\phi &= \frac{-B}{A} \end{aligned}$$

(b)

$$M=4(\Re\{c\})^2+4(\Im\{c\})^2 \text{and} \tan\phi=\frac{\Im\{c\}}{\Re\{c\}}$$

M is the amplitude of the waveform in Problem 10.4.

10.6 Force Steps

In this problem, we examine the behavior of the three basic mechanical elements when a step force is applied. Assuming zero initial position and velocity of the elements, for the mass

$$m\ddot{x}_m = u_s(t) \tag{1}$$

$$\dot{x}_m = v_m = \int_{-\infty}^{t} \frac{1}{m} u_s(t)dt = \frac{1}{m}t \tag{2}$$

$$x_m = \int_{-\infty}^{t} \frac{1}{m} tdt = \frac{1}{2m}t^2 \tag{3}$$

For the spring,

$$u_s(t) = f_k(t) = kx_k(t) \tag{4}$$

$$x_k(t) = \frac{1}{k}u_s(t) \tag{5}$$

$$v_k(t) = \frac{d}{dt}x_k(t) = \frac{1}{k}\delta(t) \tag{6}$$

For the dashpot,

$$f_c(t) = c\dot{x}_c = u_s(t) \tag{7}$$

$$\dot{x}_c = v_c = \frac{1}{c}u_s(t) \tag{8}$$

$$x_c = \int_{-\infty}^{t} \frac{1}{c} u_s(t)dt = \frac{t}{c} \tag{9}$$

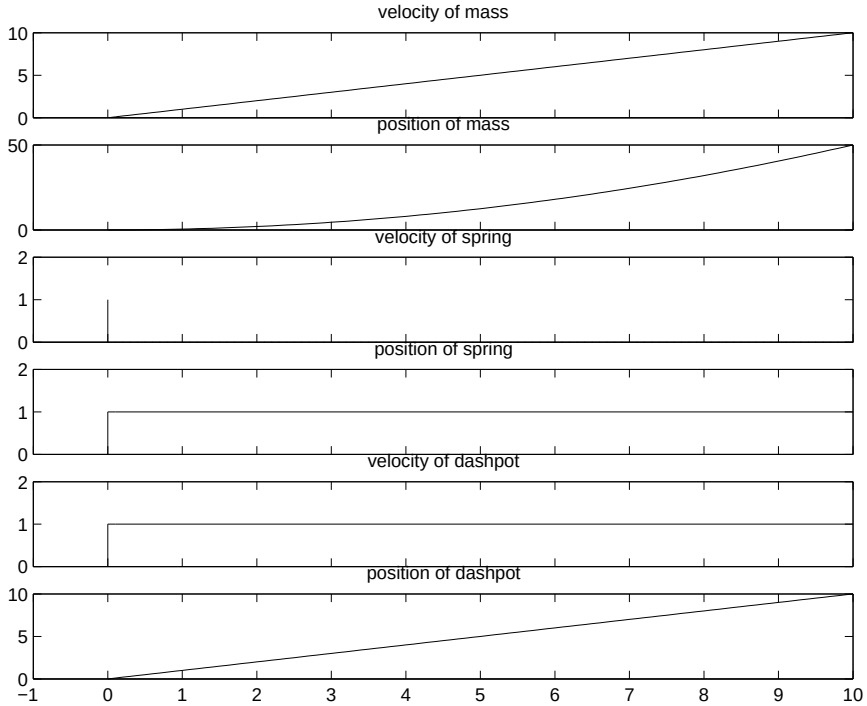


Figure 10.6.1: Step Response of Mass, Spring and Dashpot

10.7 Input for Zero Output

The solution for this problem is not available.

10.8 Zero Input Response

In this problem we would like to solve for the Zero Input Response (ZIR) of the systems given. First, we start with the rotational system. The governing equation is

$$\frac{d\omega}{dt} + \frac{c_1 + c_2}{J}\omega = 0$$

Solving this equation, we obtain

$$\omega(t) = Ae^{-t/\tau}$$

where $\tau = J/(c_1 + c_2)$ and A is determined from initial conditions. Using the initial condition $\dot{\theta}(0) = \omega(0) = \omega_0$ we find $A = \omega_0$. To solve for the position $\theta(t)$, we integrate $\omega(t)$ to find

$$\theta(t) = -\frac{\omega_0}{\tau}e^{-t/\tau} + C$$

Using the initial condition $\theta(0) = \theta_0$, we find $C = \frac{\omega_0}{\tau} + \theta_0$. Thus, the total response is

$$\theta(t) = \theta_0 + \frac{\omega_0}{\tau}\left(1 - e^{-t/\tau}\right)$$

For the translational system, we begin with the governing equation

$$M\ddot{x} + (c_1 + c_2)\dot{x} + (k_1 + k_2)x = 0$$

with the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$. Dividing the equation by M, we find the natural frequency to be

$$\omega_n = \sqrt{\frac{k_1 + k_2}{M}}$$

The damping ratio is

$$\zeta = \frac{c_1 + c_2}{2\sqrt{M(k_1 + k_2)}}$$

Assuming the system is underdamped ($\zeta < 1$), we find the damped natural frequency to be

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = \sqrt{\frac{k_1 + k_2}{M} - \frac{(c_1 + c_2)^2}{4M^2}}$$

To solve for $x(t)$, we assume a harmonic solution $x(t) = Ae^{st}$. The governing equation simplifies to an algebraic equation

$$ms^2 + (c_1 + c_2)s + (k_1 + k_2) = 0$$

or equivalently,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Solving this quadratic equation, we obtain

$$\omega = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} = -\frac{c_1 + c_2}{2M} \pm \sqrt{\frac{(c_1 + c_2)^2}{4M^2} - \frac{k_1 + k_2}{M}}$$

Assuming the underdamped solution, the response $x(t)$ is

$$x(t) = A_1e^{s_1 t} + A_2e^{s_2 t}$$

At $t = 0$, we have

$$x(0) = x_0 = A_1 + A_2 \qquad \text{and} \qquad \dot{x}(0) = v_0 = A_1 s_1 + A_2 s_2$$

After some algebra, we arrive at the solution

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos(\sqrt{1-\zeta^2}\omega_n t) + \frac{v_0 + \zeta\omega_n x_0}{\omega_n\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\omega_n t)\right)$$

Other convenient forms exist. For example, we could combine the *cos* and *sin* terms into a single *cos* term with a phase.

$$x(t) = X_0 e^{-\zeta\omega_n t} \cos\left(\sqrt{1-\zeta^2}\omega_n t - \phi\right)$$

where

$$X_0 = \sqrt{x_0^2 + \left(\frac{v_0 + \zeta\omega_n x_0}{\omega_n \sqrt{1-\zeta^2}}\right)^2} \qquad \text{and} \qquad \phi = \tan^{-1}\left(\frac{x_0}{\frac{v_0+\zeta\omega_n x_0}{\omega_n\sqrt{1-\zeta^2}}}\right)$$

10.9 Zero Step Response 1

The forced-response with zero initial conditions is composed of the particular solution and the homogeneous solution.

First, we solve the rotational system. The governing equation is

$$\frac{d\omega}{dt} + \frac{c_1 + c_2}{J}\omega = \frac{u_s(t)}{J}$$

The particular solution is given by $\omega_p = \frac{1}{c_1+c_2}$. The homogeneous solution is $\omega_h = Ae^{-t/\tau}$. The total solution is

$$\omega(t) = Ae^{-t/\tau} + \frac{1}{c_1 + c_2}$$

Using the initial condition that $\omega(0) = 0$, we obtain

$$\omega(t) = \frac{1}{c_1 + c_2}\left(1 - e^{-t/\tau}\right)$$

Integrating this equation, and using the initial condition that $\theta(0) = 0$, we obtain

$$\theta(t) = \frac{t}{c_1 + c_2} + \frac{\tau}{c_1 + c_2}\left(e^{-t/\tau} - 1\right)$$

where $\tau = J/(c_1 + c_2)$.

For the translational system, the particular solution is $x_p = F/(k_1 + k_2)$. The total solution is

$$x(t) = \frac{F}{k_1 + k_2} + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Solving using the initial conditions, we obtain

$$A_1 = \frac{F}{2(k_1 + k_2)}\left(\frac{-j\zeta}{\sqrt{1 - \zeta^2}} - 1\right)$$

$$A_2 = \frac{F}{2(k_1 + k_2)}\left(\frac{j\zeta}{\sqrt{1 - \zeta^2}} - 1\right)$$

After some algebra, we obtain

$$x(t) = \frac{F}{k_1 + k_2}\left(1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta \sin\omega_d t}{\sqrt{1 - \zeta^2}}\right)\right)$$

where $\zeta = \frac{c_1+c_2}{2M\omega_n}$, $\omega_n = \sqrt{(k_1 + k_2)/M}$, and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$. Again, various equivalent forms may be found using trigonometric identities.

10.10 Zero Step Response 2

(a) The unit step response for the rotational system is given in Figure 10.10.1. To aid in the sketching the response, find the difference between $\omega_p t$ and $x(t)$ as $t \to \infty$. This quantity represents the vertical distance between $\omega_p t$ and $x(t)$ in the response. The initial slope is zero because the initial velocity is zero.

The unit step response for the translational system is given in Figure 10.10.2. To aid in sketching, draw the envelope defined by the real part of s and use the damped natural frequency to determine how fast the response oscillates inside of the envelope. We also know from the initial conditions that the initial slope is zero because the initial velocity is zero.

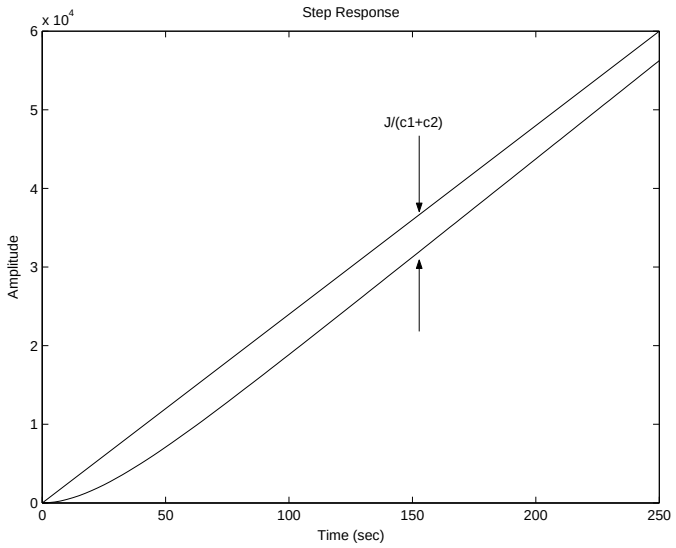


Figure 10.10.1: Sketch of the step response for the rotational system

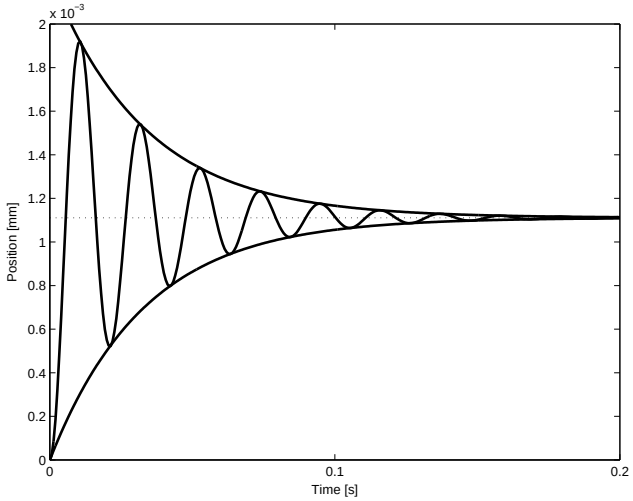


Figure 10.10.2: Sketch of the step response for the translational system

(b) The unit step response for the rotational system is given in Figure 10.10.3. The step response for the translational system is given in Figure 10.10.4. Both responses were made with the Matlab script 'zsrsol.m' found at the end of this solution.

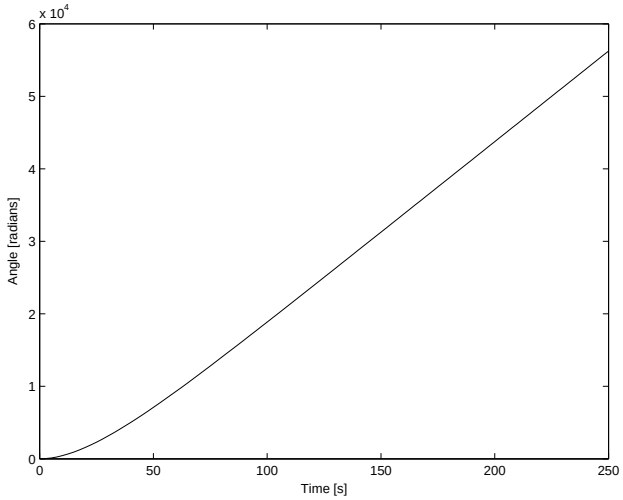


Figure 10.10.3: Unit step response $\theta(t)$ for the rotational system

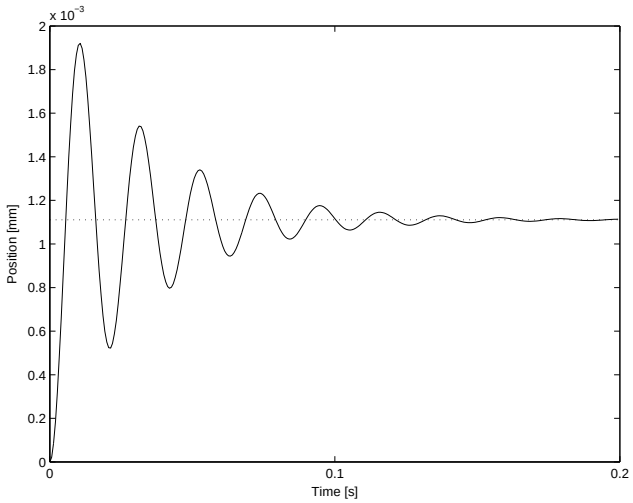


Figure 10.10.4: Unit step response $x(t)$ for the translational system

10.11 Nonlinear String System

The solution for this problem is not available.

10.12 Homogenous and Particular Solutions

(a) Figure 10.12.1 shows the model for inertia and damper system. The arrows in the figure show the reference directions for the physical variables. Separating the model into parts for each element, we can get the free body diagram as in Figure 10.12.2. Note the two T_d have equal values but opposite directions, because they are a pair of acting force and reacting force.

According to Newton's second law, we can derive the differential equation for the inertia element:

$$I\dot{\omega} = \sum T = T_{in} - T_d. \tag{1}$$

For the damping, the constitute relation is:

$$T_d = C\omega \tag{2}$$

Combining equation 1 and 2, we can get

$$I\dot{\omega} + C\omega = T_{in} \tag{3}$$

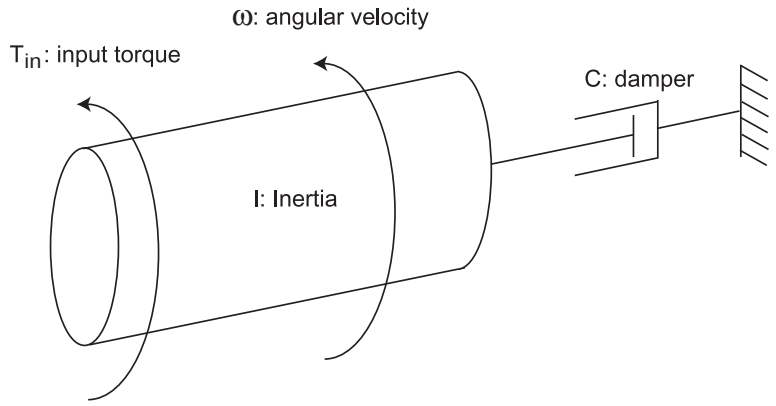
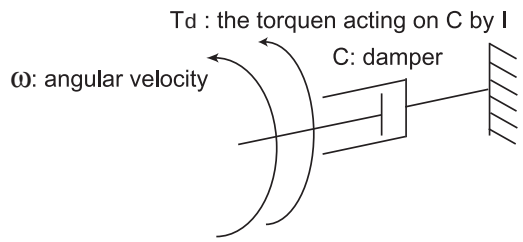
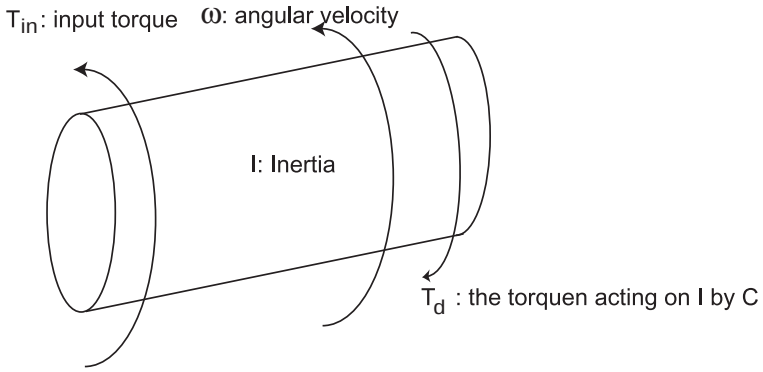


Figure 10.12.1: System Model



(a) Free Body Diagram for damper



(b) Free Body Diagram for intertia

Figure 10.12.2: Free Body Diagram

(b) The characteristic equation is

$$Is + C = 0 \tag{4}$$

The solution is $s = -\frac{C}{I}$, so the homogenous solution is $\omega_h = Ae^{st} = Ae^{-\frac{C}{I}t}$. The parameter A will be determined by the initial condition. To get the particular solution for input $T_{in} = Ku_s(t)$, we guess that the solution has the form $\omega_p = B$. Plugging it into equation 3, we get $B * C = K$. So $B = K/C$, and the particular solution is $\omega_p = K/C$. The complete solution is

$$\omega = \frac{K}{C} + Ae^{-\frac{C}{I}t} \tag{5}$$

With the initial condition, we have

$$\omega_0 = \frac{K}{C} + A. \tag{6}$$

$$A = \omega_0 - \frac{K}{C}. \tag{7}$$

The complete solution is

$$\omega = (\omega_0 - \frac{K}{C})e^{-\frac{C}{I}t} + \frac{K}{C}. \tag{8}$$

When $\omega_0 = \frac{K}{C}$, there will be no transient.

(c) For $K = 10^{-2}Nm$, $I = 0.000317kg - m^2$ and $c = 0.0011N - m - sec/rad$,
(i) when $\omega_o = 0rad/sec$, $\omega = -9.0909e^{-3.4700t} + 9.0909$,
(ii) when $\omega_o = 40rad/sec$, $\omega = 30.9091e^{-3.4700t} + 9.0909$.
The result is plotted in Figure 10.12.3.

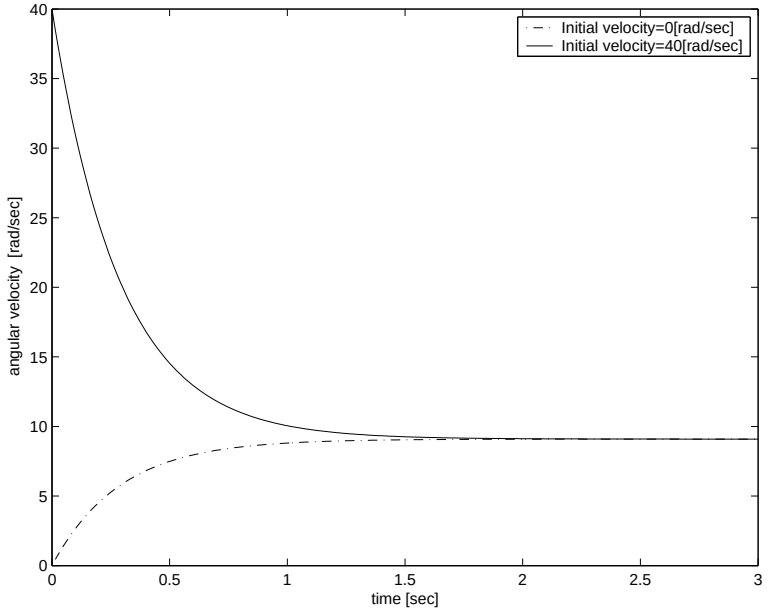


Figure 10.12.3: Step Response

(d) (i) For cosine input, we guess that the particular solution takes the form of $\omega_p = B_1 cos(\Omega t) + B_2 sin(\Omega t)$. Plugging it back into the differential equation 3,

$$I * (-\Omega B_1 sin(\Omega t) + \Omega B_2 cos(\Omega t)) + C * (B_1 cos(\Omega t) + B_2 sin(\Omega t)) = K cos(\Omega t) \quad (9)$$

$$-\Omega I B_1 + C B_2 = 0 \quad \Omega I B_2 + C B_1 = K \quad (10)$$

$$B_1 = \frac{KC}{C^2 + I^2\Omega^2} \quad (11)$$

$$B_2 = \frac{KI\Omega}{C^2 + I^2\Omega^2} \quad (12)$$

The complete solution is

$$\omega_p = \frac{KC}{C^2 + I^2\Omega^2} cos(\Omega t) + \frac{KI\Omega}{C^2 + I^2\Omega^2} sin(\Omega t) \quad (13)$$

(ii) In complex fomat, the cosine input can be expressed as $T_{in} = K cos\Omega t = Re\{Ke^{j\Omega t}\}$, we guess that the particular solution takes the form of $\omega_p = Re\{Be^{j\Omega t}\}$. Plugging it back into the differential equation 3,

$$Re\{(BI(j\Omega) + CB)e^{j\Omega t}\} = Re\{Ke^{j\Omega t}\} \quad (14)$$

$$B = \frac{K}{C + j\Omega I} \quad (15)$$

The complete solution is

$$\omega_p = Re\{\frac{K}{C + j\Omega I} e^{j\Omega t}\} \tag{16}$$

$$= \frac{KC}{C^2 + I^2\Omega^2} cos(\Omega t) + \frac{KI\Omega}{C^2 + I^2\Omega^2} sin(\Omega t) \tag{17}$$

(e) The complete solution is

$$\omega = \omega_h + \omega_p \tag{18}$$

$$= Ae^{-\frac{C}{I}t} + \frac{KC}{C^2 + I^2\Omega^2} cos(\Omega t) + \frac{KI\Omega}{C^2 + I^2\Omega^2} sin(\Omega t) \tag{19}$$

For zero initial condition, we can get $A = -\frac{KC}{C^2+I^2\Omega^2}$,

$$\omega = \frac{KC}{C^2 + I^2\Omega^2}(cos(\Omega t) - e^{-\frac{C}{I}t}) + \frac{KI\Omega}{C^2 + I^2\Omega^2} sin(\Omega t) \tag{20}$$

$$\tag{21}$$

The total solution is plotted in Figure 10.12.4. From the figure we can see that the output amplitude at $\Omega = 0.5 rad/sec$ is much larger than the output at $\Omega = 20 rad/sec$. For $\Omega = 0.5 rad/sec$, the homegeous solution have significant effect in the first cycle. For $\Omega = 20 rad/sec$, the effect of the homegeous solution is negligible. From the total solution expression, at high frequency the system significantly attenuate the output sinusoidal magnitude , "sin" component will dominate the output and thus the phase shift will be 90 degree.

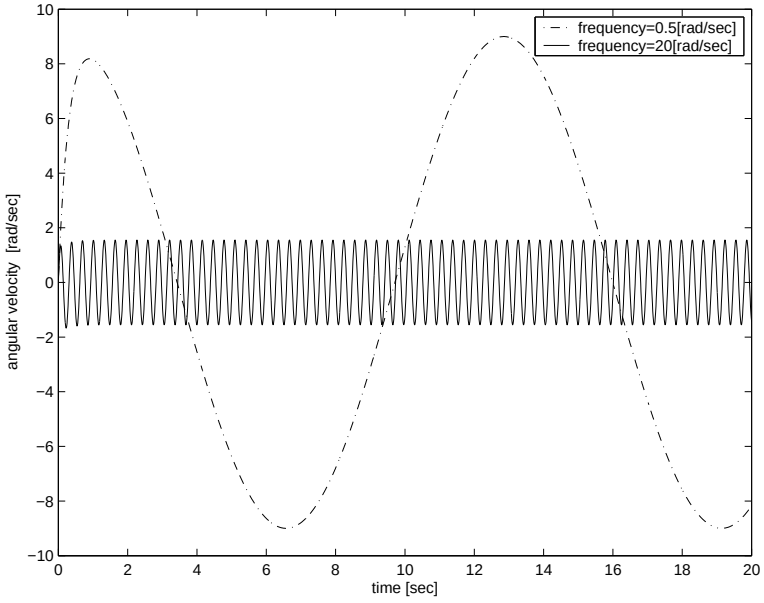


Figure 10.12.4: Sinusoidal Response

11 Frequency Response

11.1 Circuit Bode Plots

The solution for this problem is not available.

11.2 Sketch Bode Plots

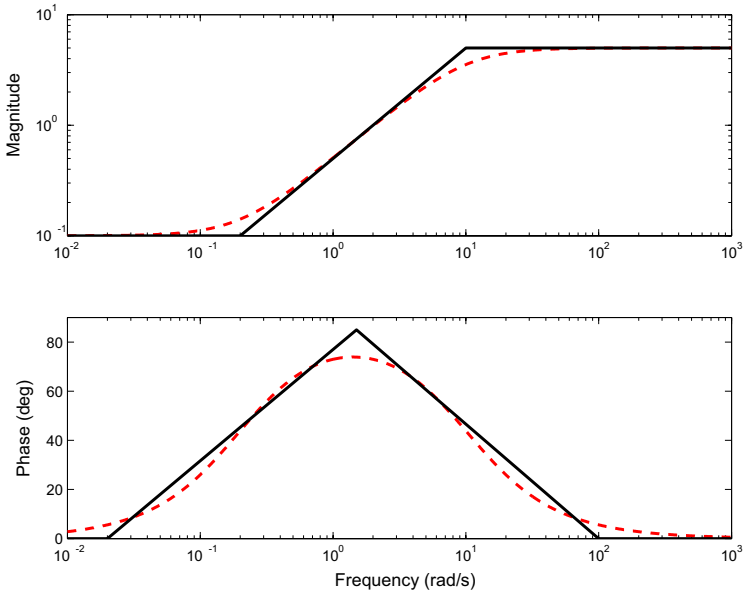


Figure 2: Frequency response for $G(s) = \frac{5s+1}{s+10}$

$$
\begin{aligned}
x_2(t) &= A\left|\frac{X_2}{U}(j\omega)\right|\sin\left(\omega t + \angle\frac{X_2}{U}(j\omega)\right) \\
\left|\frac{X_2}{U}(j\omega)\right| &= \frac{k_2}{\sqrt{(k_1k_2 - b_1b_2\omega^2)^2 + ([k_1b_2 + b_2(k_1+k_2)]\omega)^2}} \\
\angle\frac{X_2}{U}(j\omega) &= -\tan^{-1}\left(\frac{[k_1b_2 + b_1(k_1+k_2)]\omega}{k_1k_2 - b_1b_2\omega^2}\right)
\end{aligned}
$$

Problem 2

Figures 2-6 show the frequency response plots for this problem. The solid black line on the plots is the asymptotic frequency response, while the dashed red line is the actual frequency response. In order to sketch, the frequency response you need to determine the frequencies of the poles and zeros of the transfer function. You also need to find the magnitude of at least one frequency often either at $\omega = 0$ or ∞.
A) $G(s) = \frac{5s+1}{s+10} = 5\frac{s+0.2}{s+10}$, pole @ -10, zero @ -0.2. $|G(0)| = 0.1$, $|G(\infty)| = 5$.
B) $G(s) = \frac{5s+1}{s^2+3s+1} = 5\frac{s+0.2}{(s+0.38)(s+2.62)}$, poles @ -.38 and -2.62, zeros @ -0.2, $|G(0)| = 1$, $G(\infty) = 0$
C) $G(s) = \frac{s+10}{(s+2)(s^2+10s+100)}$, poles @ -2 and -5$\pm$8.66$j$, $\omega_{n1} = 10$, zeros @ -10, $|G(0)| = 1/20$, $G(\infty) = 0$

Figure 11.2.1:

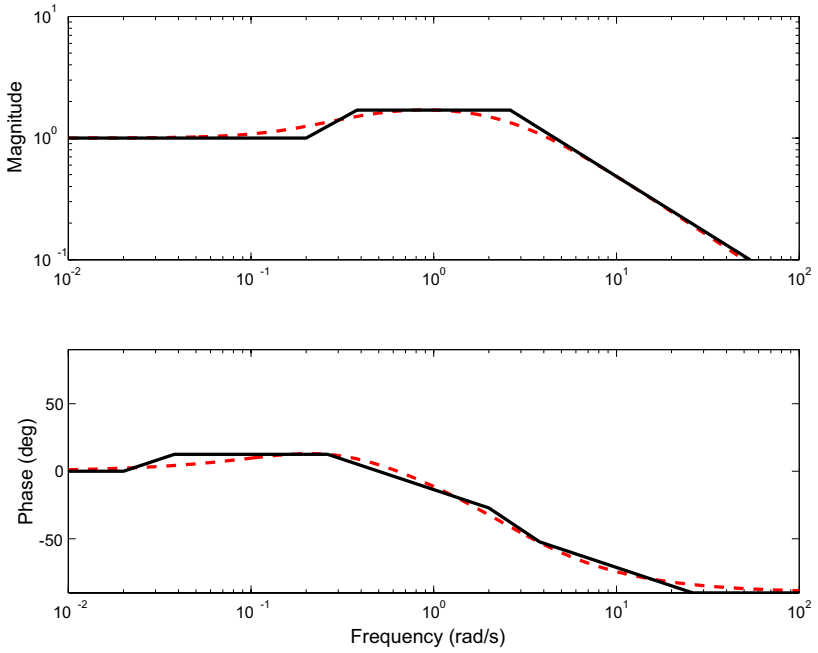


Figure 3: Frequency response for $G(s) = \frac{5s+1}{s^2+3s+1}$

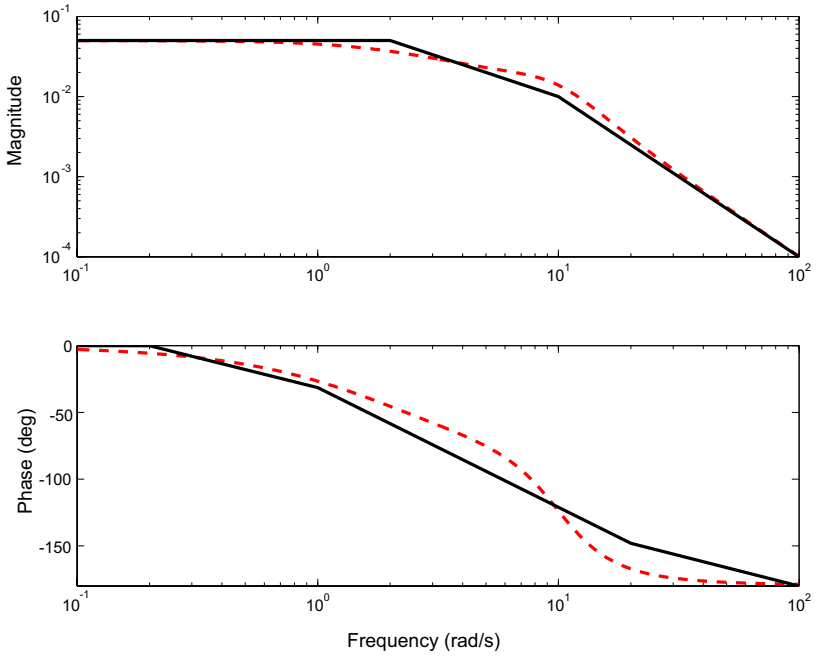


Figure 4: Frequency response for $G(s) = \frac{s+10}{(s+2)(s^2+10s+100)}$

Figure 11.2.2:

11.3 LC Circuit Frequency Response

(a) It is easiest to use impedances for this problem. Combine the capacitor and inductor into an equivalent impedance Z.

$$Z = \frac{\frac{Ls}{Cs}}{Ls + \frac{1}{Cs}} = \frac{Ls}{LCs^2 + 1}$$

The constitutive relationship for this equivalent impedance is $V_z(s) = ZI_Z(s)$. By definition, $V_Z(s) = V_c(s)$ and $I_Z(s) = I_s(s)$. Combine these equations and take the inverse Laplace transform to derive the differential equation.

$$\begin{aligned} V_c(s) &= \frac{Ls}{LCs^2 + 1} I_s(s) \\ (LCs^2 + 1)V_c(s) &= LsI_s(s) \\ LC\ddot{v}_c + v_c &= L\dot{I}_s \end{aligned}$$

The characteristic equation for this differential equation is $LCs^2 + 1$. This is of the standard form $\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n} + 1$. This yields

$$\begin{aligned} \omega_n &= \sqrt{\frac{1}{LC}} \text{ [rad/sec]} \\ \zeta &= 0 \\ \omega_d &= \sqrt{\frac{1}{LC}}\sqrt{1 - \zeta^2} = \sqrt{\frac{1}{LC}} \text{ [rad/sec]} \\ \sigma &= \zeta\omega_n = 0 \text{ [1/sec]} \end{aligned}$$

The pole zero plot is shown in Figure 11.3.1

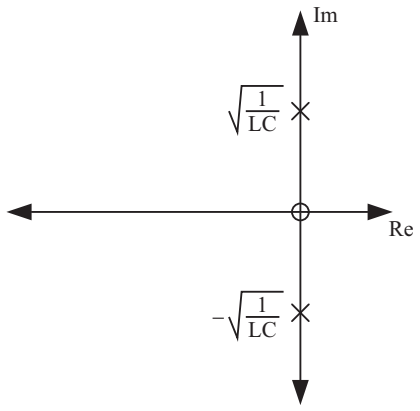


Figure 11.3.1: Pole Zero Plot

(b) Start with the differential equation and take the Laplace transform, including initial condi-

tions.

$$\begin{aligned} LC\ddot{v}_c + v_c &= L\dot{I}_s \\ V_c(s)(LCs^2+1) - LCv_c(0^-) - LC\dot{v}_c(0^-) &= LsI(s) - I(0^-) \\ V_c(s) &= \frac{LsI(s) + LC\dot{v}_c(0^-)}{(LCs^2+1)} \end{aligned}$$

We need an expression for $\dot{v}_c(0^-)$. We know that $i_L(0^-) = -1$ [A]. Summing currents at the top node with $I_s = 0$, we find that $i_c = -i_L = 1$ [A]. By using the constitutive relationship for a capacitor we find that $\dot{v}_c(0^-) = 1/C$. Therefore

$$V_c(s) = \frac{LsI_s(s) + L}{(LCs^2+1)}$$

Substitute $I_s(s) = 1/s$ and take the inverse Laplace transform to find an expression for $v_c(t)$.

$$\begin{aligned} V_c(s) &= \frac{2L}{(LCs^2+1)} \\ &= \frac{\frac{2}{C}}{s^2 + \frac{1}{LC}} \\ &= 2\sqrt{\frac{L}{C}}\frac{\sqrt{\frac{1}{LC}}}{s^2 + \frac{1}{LC}} \\ v_c(t) &= 2\sqrt{\frac{L}{C}}\sin\sqrt{\frac{1}{LC}}t \end{aligned}$$

This function is plotted in Figure 11.3.2

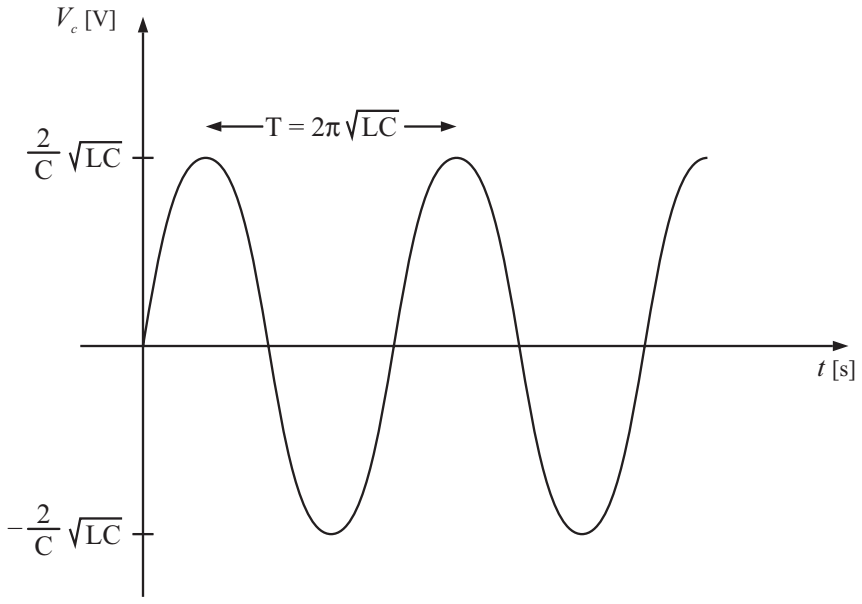


Figure 11.3.2: Step Response with $v_c(0) = 0$ [V] and $i_L(0) = -1$ [A]

(c) M is simply the magnitude of the transfer function evaluated at $s = j\omega$.

$$M = \left| \frac{V_c(s)}{I_s(s)} \right| = \frac{L\omega}{|1 - LC\omega^2|}$$

The phase is

$$\begin{aligned} \phi &= \tan^{-1} \frac{L\omega}{0} - \tan^{-1} \frac{0}{1 - LC\omega^2} \\ &= 90^\circ, \omega^2 < \frac{1}{LC} \text{ and } -90^\circ, \omega^2 > \frac{1}{LC} \end{aligned}$$

(d) The plot is shown in Figure 11.3.3

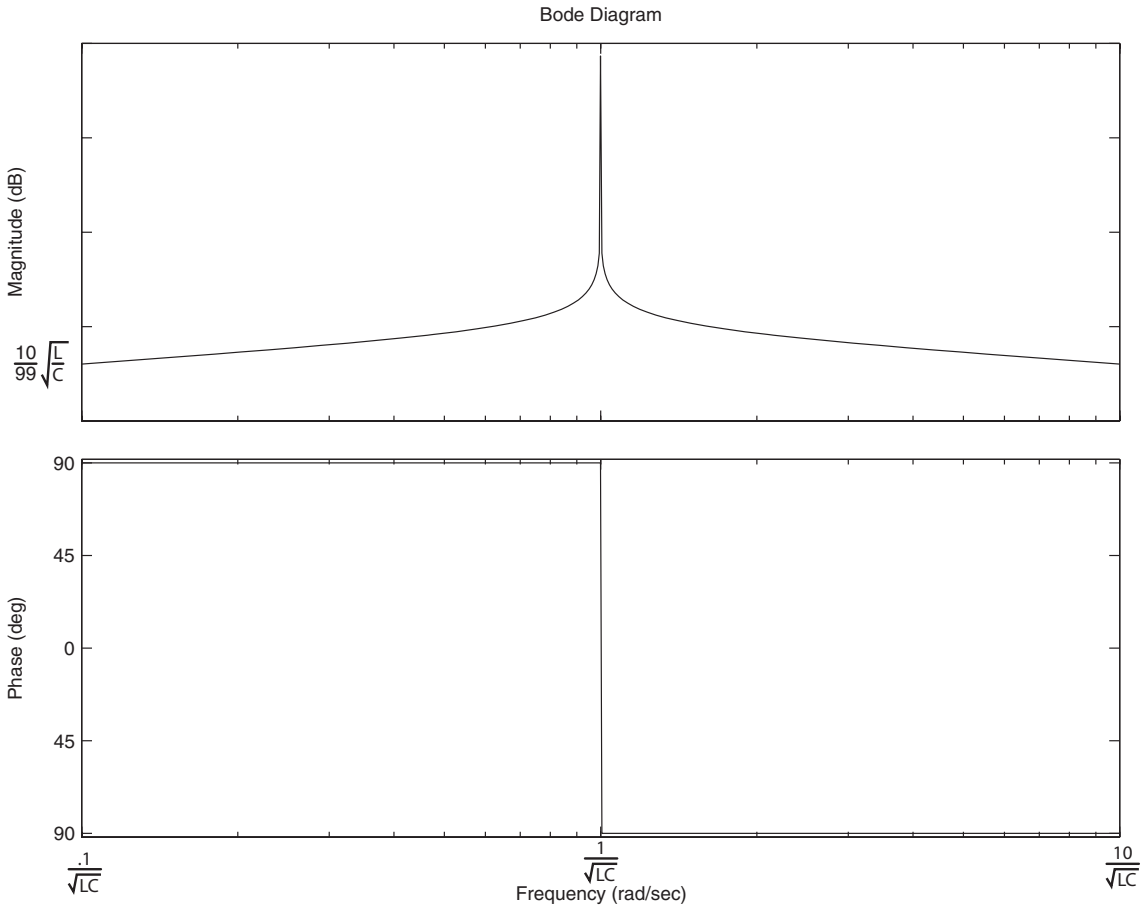


Figure 11.3.3: Bode plot

11.4 LRC Circuit Frequency Response

(a) It is easiest to use impedances for this problem. Combine the resistor, capacitor and inductor into an equivalent impedance Z.

$$\begin{aligned} Z &= R//Ls//\frac{1}{Cs} \\ \frac{1}{Z} &= \frac{1}{R} + \frac{1}{Ls} + Cs \\ Z &= \frac{1}{\frac{1}{R} + \frac{1}{Ls} + Cs} \\ &= \frac{Ls}{LCs^2 + \frac{L}{R}s + 1} \end{aligned}$$

The constitutive relationship for this equivalent impedance is $V_z(s) = ZI_z(s)$. By definition, $V_z(s) = V_c(s)$ and $I_z(s) = I_s(s)$. Combine these equations and take the inverse Laplace transform to derive the differential equation.

$$\begin{aligned} V_c(s) &= \frac{Ls}{LCs^2 + \frac{L}{R}s + 1} I_s(s) \\ (LCs^2 + \frac{L}{R}s + 1)V_c(s) &= LsI_s(s) \\ LC\ddot{v}_c + \frac{L}{R}\dot{v}_c + v_c &= L\dot{I}_s \end{aligned}$$

The characteristic equation for this differential equation is $LCs^2 + \frac{L}{R}s + 1$. This is of the standard form $\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n} + 1$. This yields

$$\begin{aligned} \omega_n &= \sqrt{\frac{1}{LC}} \text{ [rad/sec]} \\ \zeta &= \frac{L\omega_n}{2R} = \frac{1}{2R}\sqrt{\frac{L}{C}} \\ \omega_d &= \sqrt{\frac{1}{LC}}\sqrt{1 - \left(\frac{1}{2R}\sqrt{\frac{L}{C}}\right)^2} = \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}} \text{ [rad/sec]} \\ \sigma &= \zeta\omega_n = \frac{1}{2RC} \text{ [1/sec]} \end{aligned}$$

The pole zero plot is shown in Figure 11.4.1

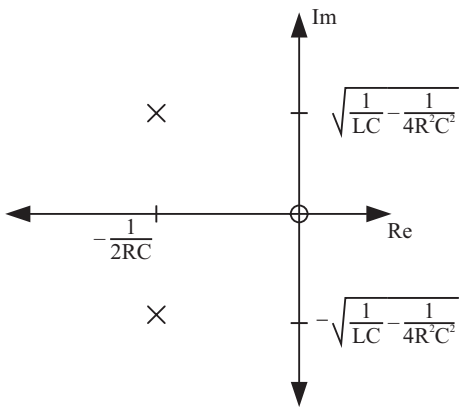


Figure 11.4.1: Pole Zero Plot

(b) With zero initial conditions, the transfer function of this system is simply the system's equivalent impedance. Substitute $I_s(s) = 1/s$ and take the inverse Laplace transform to find an expression for $v_c(t)$.

$$\begin{aligned}
\frac{V_c(s)}{I_s(s)} &= \frac{Ls}{LCs^2 + \frac{L}{R}s + 1} \\
V_c(s) &= \frac{L}{LCs^2 + \frac{L}{R}s + 1} \\
&= \frac{\frac{1}{C}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \\
&= \frac{1}{C\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}} \frac{\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}}{\left(s + \frac{1}{2RC}\right)^2 + \left(\frac{1}{LC} - \frac{1}{4R^2C^2}\right)} \\
v_c(t) &= \frac{1}{C\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}} e^{-\frac{t}{2RC}} \sin\left(\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}\, t\right)
\end{aligned}$$

This function is plotted in Figure 11.4.2

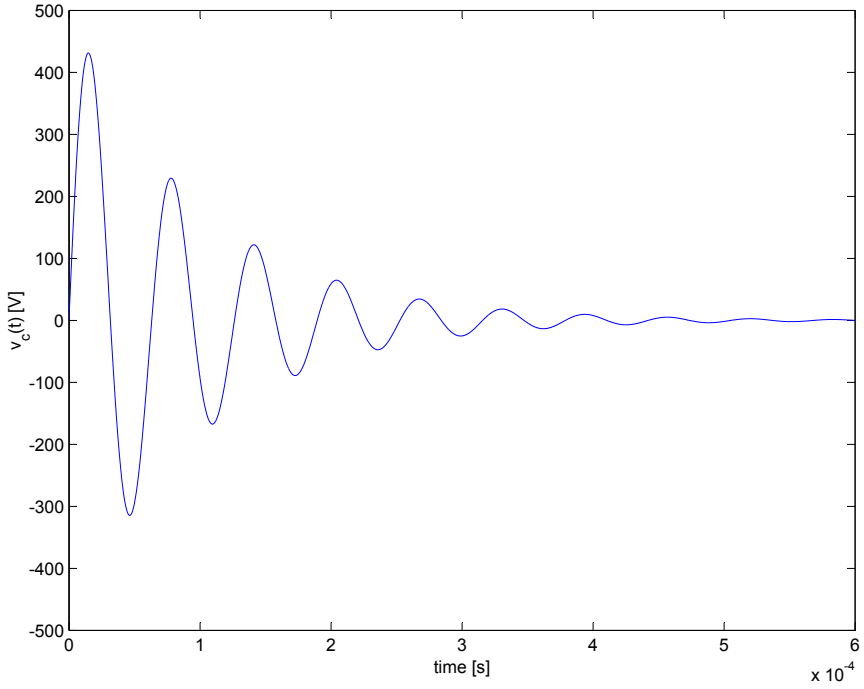


Figure 11.4.2: Step Response with $v_c(0) = 0$ [V] and $i_L(0) = -1$ [A]

(c) M is simply the magnitude of the transfer function evaluated at $s = j\omega$.

$$M = \left|\frac{V_c(s)}{I_s(s)}\right| = \frac{L\omega}{\sqrt{(1 - LC\omega^2)^2 + \left(\frac{L}{R}\omega\right)^2}}$$

The phase is

$$\begin{aligned}\phi &= \tan^{-1}\frac{L\omega}{0} - \tan^{-1}\frac{\frac{L}{R}\omega}{1 - LC\omega^2} \\ &= 90 - \tan^{-1}\frac{\frac{L}{R}\omega}{1 - LC\omega^2}\end{aligned}$$

(d) The plot is shown in Figure 11.4.3. The peak frequency occurs when the system is excited at its natural frequency $w_n = \sqrt{\frac{1}{LC}} = 10^5$ [rad/sec]. The peak magnitude is found by plugging in $\omega = 10^5$ into the magnitude expression. This evaluates to R.

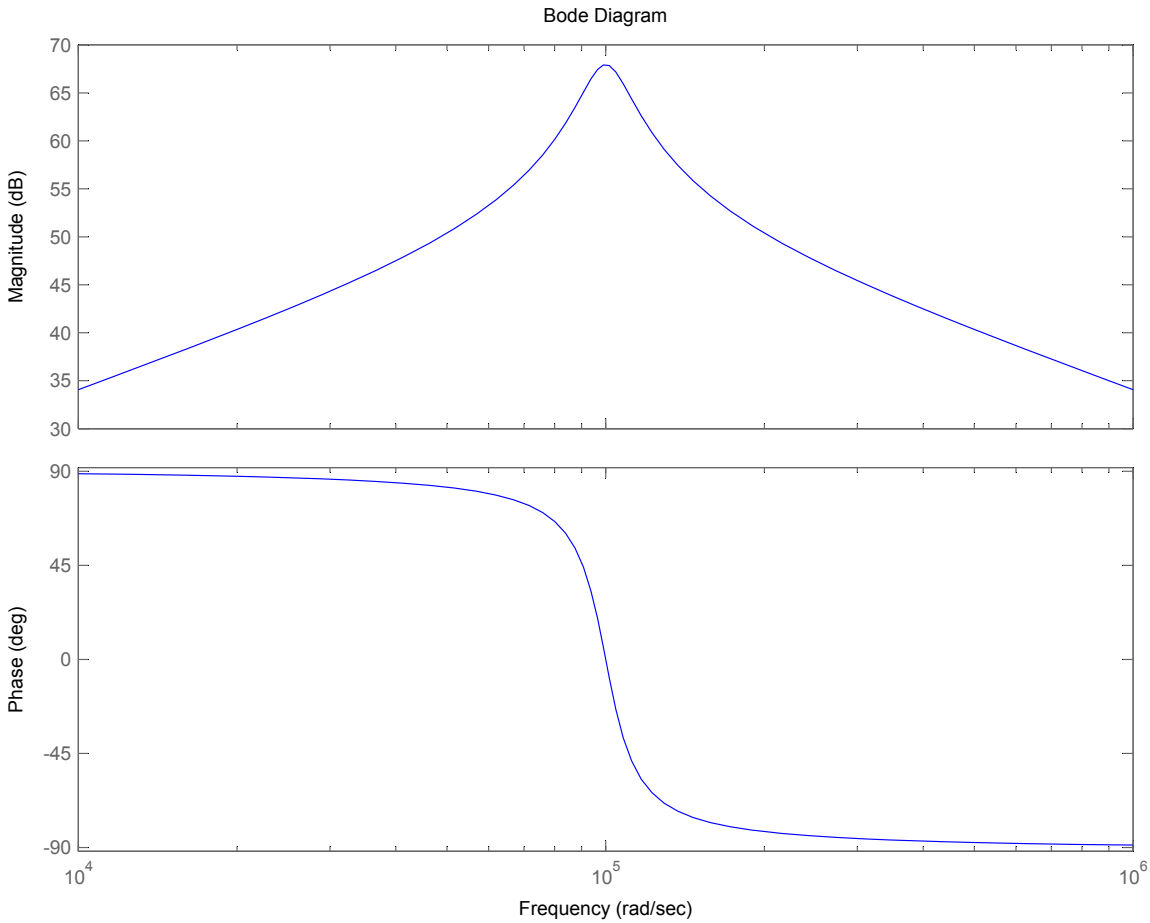


Figure 11.4.3: Bode plot

11.5 Pole-Zero Plots

The solution for this problem is not available.

12 Signals, Transforms, and Transfer Functions

12.1 First Order Zeros

In this problem, we are asked to consider a system with a differential equation

$$\tau\dot{y} + y = a\dot{u} + u \tag{1}$$

The system has pole at $s = -1/\tau$ and a zero at $s = -1/a$.

(a) Integrate the equation from 0^- to 0^+, we get

$$\int_{t=0^-}^{t=0^+} (\tau\dot{y} + y)dt = \int_{t=0^-}^{t=0^+} (a\dot{u} + u)dt \tag{2}$$

$$\tau(y(0^+) - y(0^-)) = a(u(0^+) - u(0^-)) \tag{3}$$

$$y(0^+) = \frac{a}{\tau}u(0^+) = \frac{a}{\tau} \tag{4}$$

The particular solution and the homogenous solution are:

$$y_p(t) = 1 \tag{5}$$

$$y_h(t) = Ae^{-\frac{t}{\tau}} \tag{6}$$

The complete solution is

$$y(t) = y_p(t) + y_h(t) = 1 + Ae^{-\frac{t}{\tau}} \tag{7}$$

Combining the initial condition

$$y(0^+) = \frac{a}{\tau} = 1 + A \tag{8}$$

$$A = \frac{a}{\tau} - 1 \tag{9}$$

$$y(t) = 1 + (\frac{a}{\tau} - 1)e^{-\frac{t}{\tau}} \tag{10}$$

(b) Figures 12.1.1 and 12.1.2 give the response and pole-zero plots.

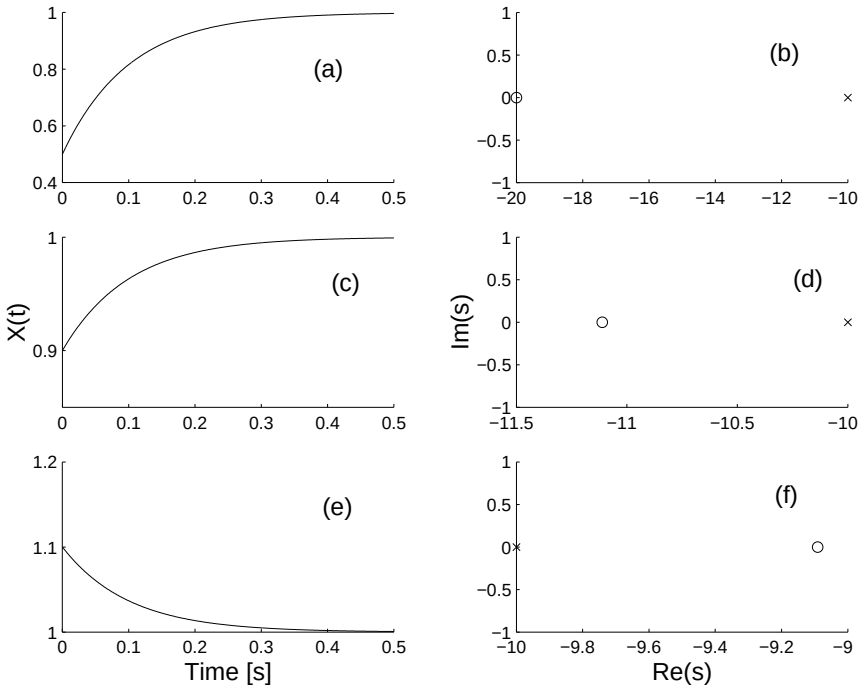


Figure 12.1.1: Step response and pole-zero plots for a =0.05, 0.09, 0.11. (a) Step response for $a = 0.05$. (b) Pole-zero plot for $a = 0.05$. (c) Step response for $a = 0.09$. (d) Pole-zero plot for $a = 0.09$. (e) Step response for $a = 0.11$. (f) Pole-zero plot for $a = 0.11$.

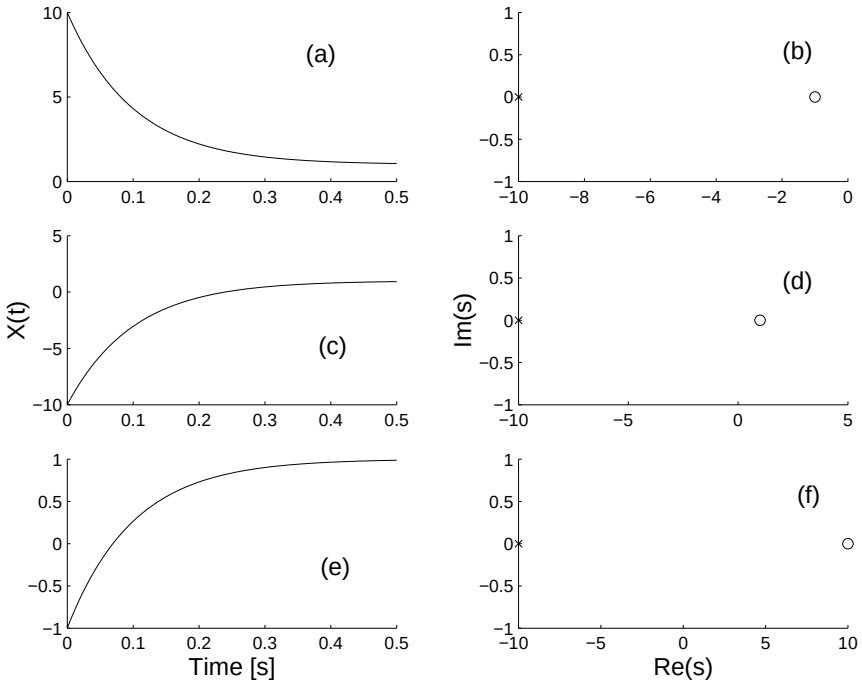


Figure 12.1.2: Step response and pole-zero plots for a =1, -1, -0.1. (a) Step response for $a = 1$. (b) Pole-zero plot for $a = 1$. (c) Step response for $a = -1$. (d) Pole-zero plot for $a = -1$. (e) Step response for $a = -0.1$. (f) Pole-zero plot for $a = -0.1$.

(c) Figure 12.1.3 shows that as $a \to \tau$, $H(s) \to 1$.

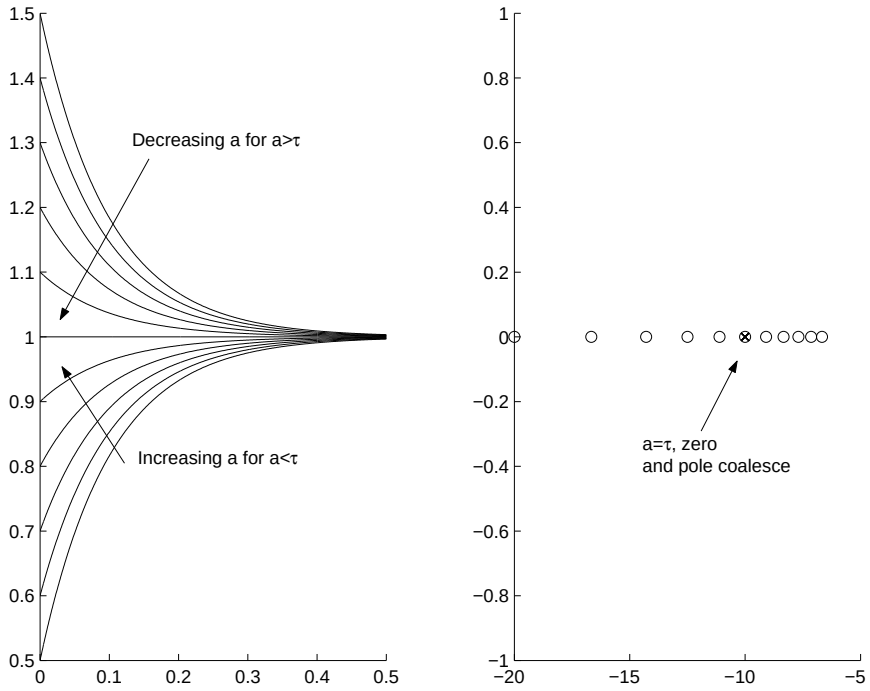


Figure 12.1.3: Step response and pole-zero plots for $a \to \tau = 0.1$.

(d) We want to show that a zero causes certain exponential inputs to have zero response. We consider an exponential input of the form $u(t) = e^{s_1 t}$. We begin the equation in the time domain

$$\tau \dot{y} + y = a\dot{u} + u \tag{11}$$

If we desire $y = 0$, then $a\dot{u} + u$ must equal zero for all time. Therefore, the input must be

$$u(t) = Ae^{-t/a} \tag{12}$$

Comparing this input with $u(t) = e^{s_1 t}$, we find that $\boxed{s_1 = -1/a}$. In the complex plane, the zero and s_1 are in the same location.

(e) The response is the same as part (a).

12.2 Laplace Practice

The Laplace Transform of $x(t)$is defined as

$$\mathcal{L}[x(t)] = X(s) = \int_0^\infty x(t)e^{-st}dt$$

therefore

$$\begin{aligned} W(s) &= \int_0^\infty e^{-st}dt - \left(\int_0^T 0dt + \int_T^\infty e^{-st}dt\right) \\ &= -\frac{1}{s}e^{-st}\Big|_0^\infty - \left(0 + -\frac{1}{s}e^{-st}\Big|_T^\infty\right) \\ &= \frac{1}{s} - \frac{1}{s}e^{-sT} \end{aligned}$$

12.3 Step-Zero

In this problem, we consider the second-order system with a real zero

$$H(s) = \frac{s/\alpha\zeta + 1}{s^2 + 2\zeta s + 1} \tag{1}$$

A useful way method of analyzing the behavior of this transfer function is to write it as a sum of two transfer functions

$$H(s) = \frac{s/\alpha\zeta}{s^2 + 2\zeta s + 1} + \frac{1}{s^2 + 2\zeta s + 1} \tag{2}$$

If we define $H_1(s)$ as

$$H_1(s) = \frac{1}{s^2 + 2\zeta s + 1} \tag{3}$$

and $H_2(s)$ as

$$H_2(s) = \frac{s}{s^2 + 2\zeta s + 1} = sH_1(s) \tag{4}$$

then we notice that $H_2(s)$ is the derivative of $H_1(s)$ in the Laplace domain. Thus, we may write the transfer function as

$$H(s) = \left(\frac{s}{\alpha\zeta} + 1\right) H_1(s) \tag{5}$$

For our purposes, we consider only negative real zeros. If the zero is close to the imaginary axis, then the transfer function closely resembles the derivative term only. If the zero is far away from the origin, then the $H(s)$ behaves primarily like $H_1(s)$. Using partial fractions, we find that step response is given by

$$y(t) = 1 + e^{-\zeta t}\left(\left(\frac{1}{a\zeta\sqrt{1-\zeta^2}} - \frac{\zeta}{\sqrt{1-\zeta^2}}\right)\sin\sqrt{1-\zeta^2}t - \cos\sqrt{1-\zeta^2}t\right) \tag{6}$$

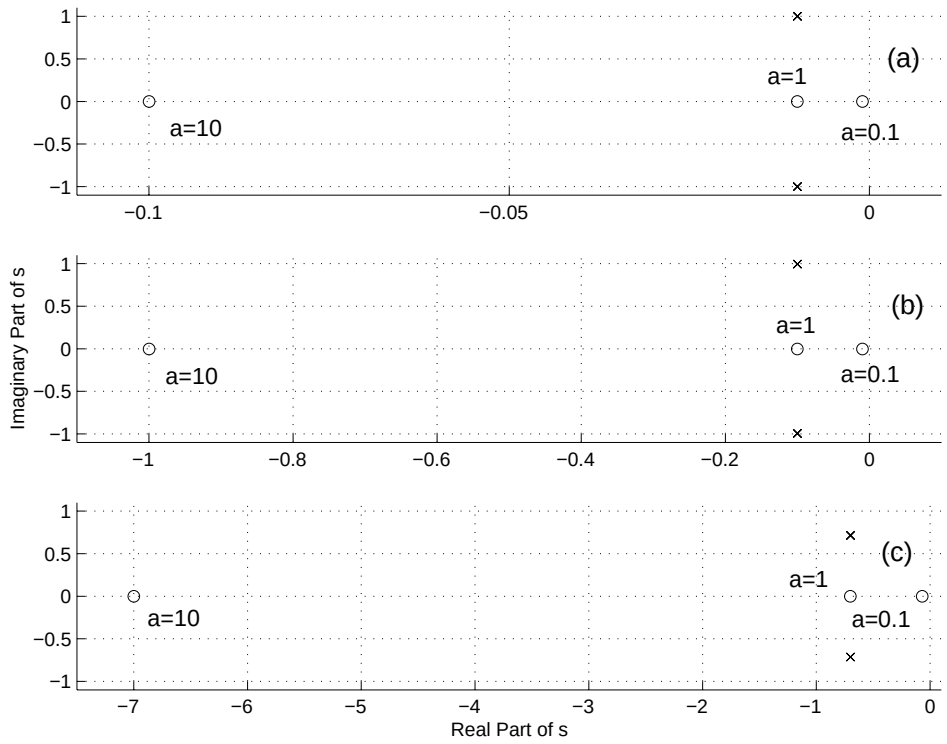


Figure 12.3.1: Pole Zero Plots. (a) $/zeta = 0.01$. (b) $/zeta = 0.1$. (c) $/zeta = 0.7$.

12.4 Laplace to Time

(a) This transform is simple enough to be in most tables. However, we can simplify it with partial fractions:

$$Y(s) = \frac{1}{(s+a)(s+b)} = \frac{C}{s+a} + \frac{D}{s+b}$$

find the constants C and D by setting $s = -a$ and $s = -b$

$$\begin{aligned} \frac{1}{(s+a)(s+b)} &= \frac{C}{s+a} + \frac{D}{s+b} \\ 1 &= C(s+b) + D(s+a) \\ C &= \frac{1}{b-a} \\ D &= \frac{1}{a-b} \end{aligned}$$

therefore

$$Y(s) = \frac{1}{b-a}\frac{1}{s+a} - \frac{1}{b-a}\frac{1}{s+b}$$

By looking up the inverse Laplace Transform of $\frac{1}{s+b}$, we find the total solution $y(t)$

$$y(t) = \frac{1}{b-a} \quad e^{-at} - e^{-bt}$$

(b) First, note that the transform is

$$\begin{aligned} Y(s) &= \frac{s}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1} \\ &= s \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

This Laplace Transform is in a standard form, but not on all tables. We will solve this problem using the property

$$\frac{df}{dt} = sF(s) - f(0)$$

therefore

$$\begin{aligned} y(t) &= \frac{d}{dt}\left(\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right) \\ &= \omega_n^2 e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) - \frac{\zeta\omega_n^2}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) \end{aligned}$$

remember that for this form to be correct, ζ must be less than 1.

12.5 Time to Laplace

(a) This function is one of the most widely used in dynamic systems, so memorize it's transform!

$$Y(s) = \frac{1}{s+a}$$

(b)

$$Y(s) = \frac{\omega_d}{(s+\sigma)^2 + \omega_d^2}$$

(c)

$$Y(s) = \frac{s+\sigma}{(s+\sigma)^2 + \omega_d^2}$$

12.6 Laplace Transform

Each term of $X(s)$ can be evaluated directly using the Table of Laplace Transforms.

$$\mathcal{L}^{-1}\{b\} = b\delta(t)$$

and

$$\mathcal{L}^{-1}\left\{\frac{a}{s(s+a)}\right\} = 1 - e^{-at}$$

The final result is then

$$\mathcal{L}^{-1}\left\{X(s)\right\} = b\delta(t) + 1 - e^{-at}$$

12.7 Signal in Time and Frequency Domain

(a) Sketch of x(t).

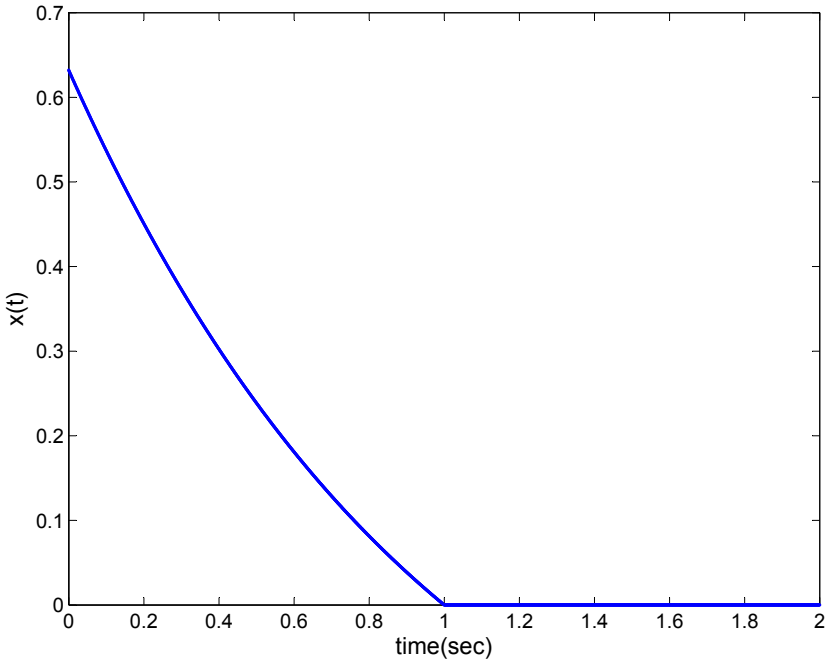


Figure 12.7.1: Signal x(t) plotted as a function of time

(b) Simplify the expression in to a sum of terms,

$$x(t) = e^{-t}u_s(t) - e^{-1}u_s(t) - e^{-t}u_s(t-1) + e^{-1}u_s(t-1)$$

Now take the Laplace transform of the first, second and fourth terms,

$$X(s) = \frac{1}{s+1} - \frac{e^{-1}}{s} - \mathcal{L}e^{-t}u_s(t-1) + \frac{e^{-1}e^{-s}}{s} \tag{1}$$

The third term requires some massaging to get it in a form available on the table. The term can be modified into the form of a time delay, by factoring out e^{-1}.

$$\mathcal{L}\left\{e^{-t}u_s(t-1)\right\} = e^{-1}\mathcal{L}\left\{e^{-(t-1)}u_s(t-1)\right\}$$

Now applying the Laplace Transform for a time delay from the table

$$e^{-1}\mathcal{L}\left\{e^{-(t-1)}u_s(t-1)\right\} = \frac{e^{-1}e^{-s}}{s+1}$$

Substituting this piece back into Equation (1) gives the solution

$$X(s) = \frac{1}{s+1} - \frac{e^{-1}}{s} - \frac{e^{-1}e^{-s}}{s+1} + \frac{e^{-1}e^{-s}}{s}$$

12.8 Transfer Function and Time Constants

(a) The transfer function for the circuit is easily found using the impedance method.

$$Z_{eq1} = R_1||C_1 = \frac{R_1}{R_1C_1s+1}$$

and Z_{eq2} is similar. The circuit is now a voltage divider with complex impedances.

$$v_o(s) = \frac{Z_{eq1}}{Z_{eq1}+Z_{eq2}}v_i(s)$$

Simplifying results in the transfer function

$$\frac{v_o(s)}{v_i(s)} = \frac{R_1(R_2C_2s+1)}{R_1R_2(C_1+C_2)s+R_1+R_2}$$

To get the transfer function in the desired form we factor $R1+R2$ out of the denominator, while the numerator is already in the desired form. We now have

$$\frac{v_o(s)}{v_i(s)} = \frac{R_1}{R_1+R_2}\frac{R_2C_2s+1}{\frac{R_1R_2(C_1+C_2)}{R_1+R_2}+1}$$

so

$$K = \frac{R_1}{R_1+R_2}$$
$$\tau_1 = R_2C_2$$
$$\tau_2 = \frac{R_1R_2(C_1+C_2)}{R_1+R_2}$$

Using the Laplace initial value theorem with $v_i = 1/s$

$$v_o(0^+) = \lim_{s\to\infty} sv_o(s) = \lim_{s\to\infty} s\frac{1}{s}\frac{R_1}{R_1+R_2}\frac{R_2C_2s+1}{\frac{R_1R_2(C_1+C_2)}{R_1+R_2}s+1}$$

After simplifying, we can now factor s out of the numerator and denominator, which cancel, resulting in

$$v_o(0^+) = \lim_{s\to\infty} \frac{R_1}{R_1+R_2}\frac{R_2C_2+1/s}{\frac{R_1R_2(C_1+C_2)}{R_1+R_2}+1/s}$$

Now when taking the limit, the second terms in the numerator and denominator go to zero, and after simplifying

$$v_o(0^+) = \frac{C_2}{C_1+C_2}$$

Using the Laplace final value theorem with $v_i = 1/s$

$$v(\infty) = \lim_{s\to 0} sv_o(s) = \lim_{s\to 0} s\frac{1}{s}\frac{R_1}{R_1+R_2}\frac{R_2C_2s+1}{\frac{R_1R_2(C_1+C_2)}{R_1+R_2}s+1}$$

and simplifying

$$v(\infty) = \frac{R_1}{R_1+R_2}$$

The x in Figure 12.8.1 shows the pole location and the o shows the zero location. Plotting is made possible by assuming $\tau_1 > \tau_2$, but assuming $\tau_1 < \tau_2$ would have been just as valid for plotting purposes.

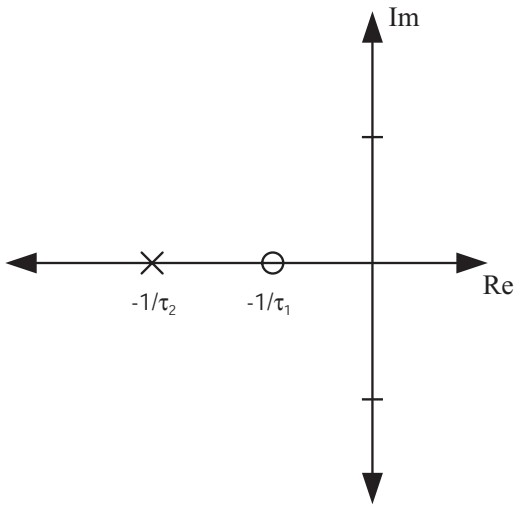


Figure 12.8.1: Plot of pole and zero of system in terms of τ_1 and τ_2

When $R_1C_1 = R_2C_2$ the circuit becomes a voltage divider independent of frequency.

(b) Letting $v_i(s) = 1/s$, a unit step, and assuming initial rest conditions, and continuing with the K, τ_1, τ_2 form,

$$v_o(s) = \frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)}$$

In order to calculate $v_o(t)$, we first perform partial fraction decomposition, which results in the following:

$$v_o(s) = \frac{K}{s} + \frac{K(\tau_1 - \tau_2)}{\tau_2(s + 1/\tau_2)}$$

We can now calculate the inverse Laplace transform for each term directly from the Table of Laplace Transforms.

$$\mathcal{L}^{-1}\{v_o(t)\} = Ku_s(t) + \frac{K(\tau_1 - \tau_2)}{\tau_2} e^{-t/\tau_2}$$

After substituting K, τ_1, and τ_2 in we have $v_o(t)$ as a function of R_1, R_2, C_1, and C_2 for a step input.

$$v_o(t) = \frac{R_1}{R_1 + R_2} u_s(t) + \left[\frac{C_2}{C_1 + C_2} - \frac{R_1}{R_1 + R_2}\right] e^{\frac{-t(R_1+R_2)}{R_1R_2(C_1+C_2)}}$$

The step response and pole locations are now plotted for the following sets of parameter values:

(i) $R_1 = 1k\Omega$, $R_2 = 9k\Omega$, $C_1 = 1\mu F$, $C_2 = 9\mu F$

(ii) $R_1 = 1k\Omega$, $R_2 = 9k\Omega$, $C_1 = 9\mu F$, $C_2 = 1\mu F$

(iii) $R_1 = 9k\Omega$, $R_2 = 1k\Omega$, $C_1 = 9\mu F$, $C_2 = 1\mu F$

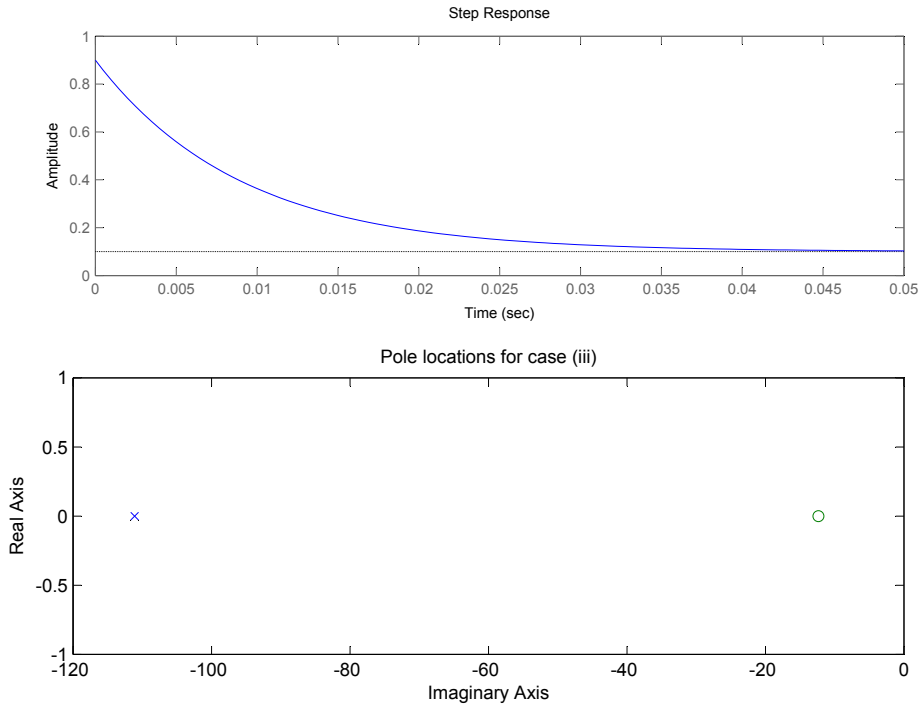


Figure 12.8.2: Step response and pole locations for case (i)

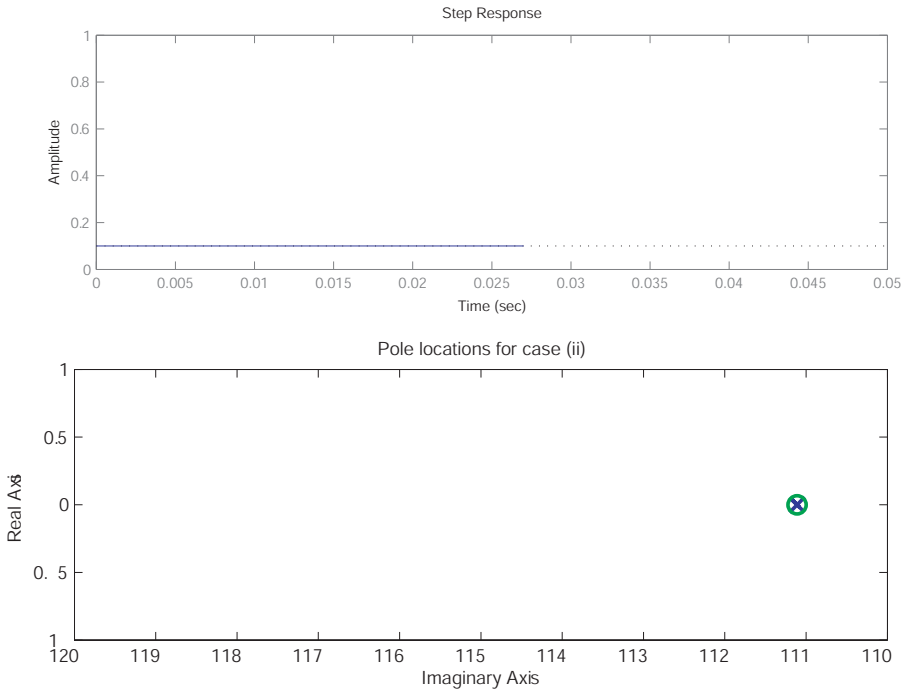


Figure 12.8.3: Step response and pole locations for case (ii). Pole and zero are collocated.

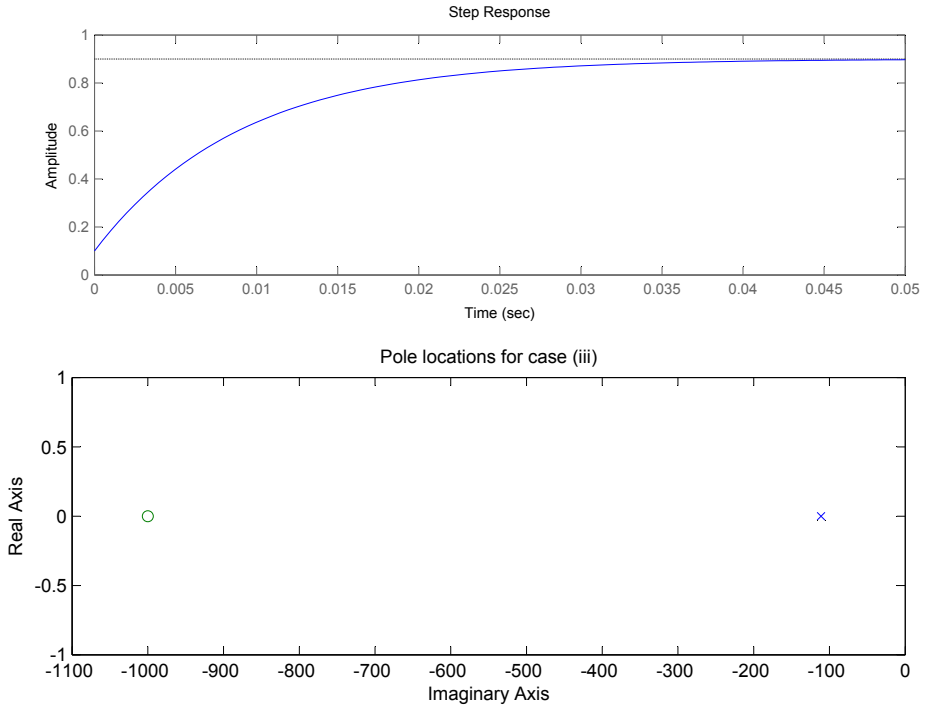


Figure 12.8.4: Step response and pole locations for case (iii)

12.9 Fourier Series Input

PROBLEM 3

a) $u(t) = \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{(-1)^{(n-1)/2}}{n^2} \sin\left(\frac{n\pi t}{T/2}\right)$ (TRIANGLE WAVE FOURIER SERIES)

WHERE $T = 0.628$ sec.

ONLY THE FIRST FIVE TERMS:

$$u(t) = \frac{8}{\pi^2}\left[\sin\left(\frac{\pi t}{T/2}\right) - \frac{1}{9}\sin\left(\frac{3\pi t}{T/2}\right) + \frac{1}{25}\sin\left(\frac{5\pi t}{T/2}\right) \cdots \right.$$

$$\left. \cdots - \frac{1}{49}\sin\left(\frac{7\pi t}{T/2}\right) + \frac{1}{81}\sin\left(\frac{9\pi t}{T/2}\right)\right]$$

b) SEE ATTACHED BODE PLOTS

c) $G_1(s)$: FROM PLOT:

$$y(t) = \frac{8}{\pi^2}\left[0.05\sin(10t - \pi/2) - 0.0019\sin(30t - \pi/2) \cdots \right.$$

$$+ 0.0004\sin(50t - \pi/2) - 0.000146\sin(70t - \pi/2) \cdots$$

$$\left. + 0.000069\sin(90t - \pi/2)\right]$$

$G_2(s)$: FROM PLOT:

$$y(t) = \frac{8}{\pi^2}\left[40\times10^{-6}\sin(10t - \pi) - 492\times10^{-9}\sin(30t - \pi) \cdots \right.$$

$$+ 64\times10^{-9}\sin(50t - \pi) - 17\times10^{-9}\sin(70t - \pi) \cdots$$

$$\left. + 6.17\times10^{-9}\sin(90t - \pi)\right.$$

Figure 12.9.1:

$G_3(s)$: FROM PLOT:

$$y(t) = \frac{8}{\pi^2}\left[10.2\sin(10t+1.37) - 3.8\sin(30t+0.82) - \cdots + 2\sin(50t) - 0.84\sin(70t-0.60) - \cdots + 0.39\sin(90t-0.89)\right]$$

$G_4(s)$: FROM PLOT:

$$y(t) = \frac{8}{\pi^2}\left[9.86\times10^{-4}\sin(10t+1.28) - 0.0078\sin(30t+2.23) - \cdots + 0.0069\sin(50t+1.73) - 0.0054\sin(70t+1.29) - \cdots + 0.0041\sin(90t+0.93)\right]$$

d) SEE ATTACHE M CODE AND PLOTS

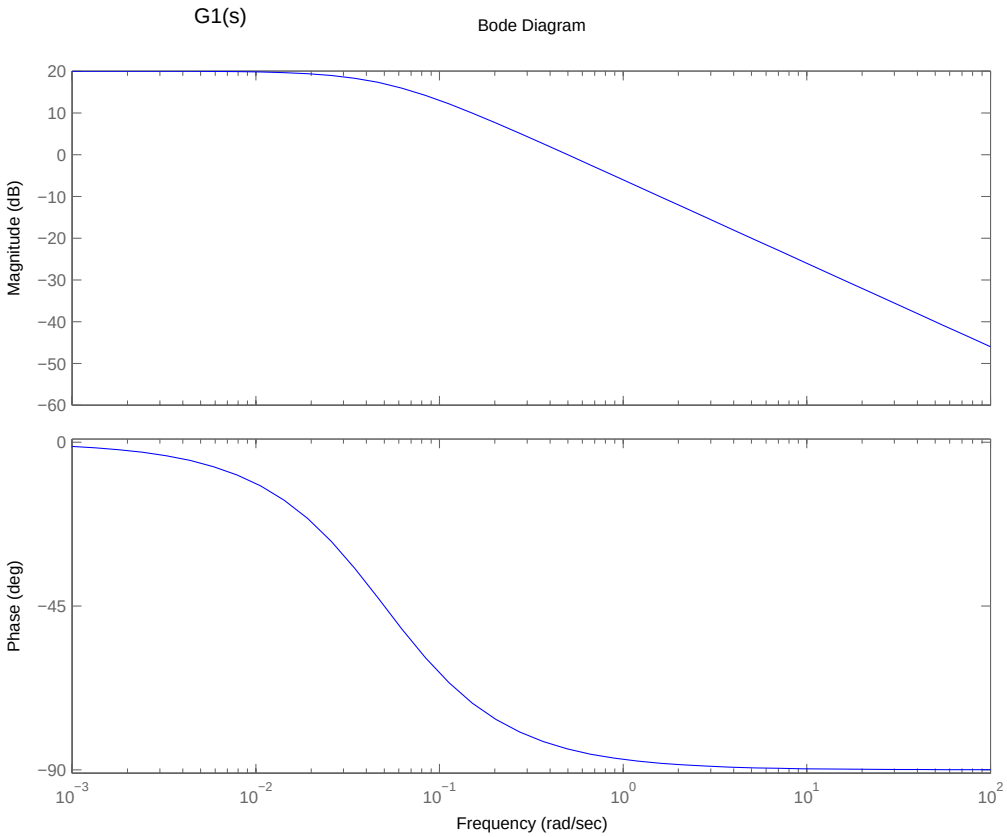
e) $G(s) = \dfrac{K.s.\omega_n^2}{s^2+2\zeta\omega_n.s+\omega_n^2}$; $\zeta = 0.01$, $\omega_n = 30$.
$K = (1/1500)\times(\pi^2/8)$

$$y(t) = \frac{8}{\pi^2}\left[0.083\sin(10t-0) - \frac{11.109}{9}\sin(30t) - \cdots + \frac{0.208}{25}\sin(50t-\pi/2) - \frac{0.1166}{49}\sin(70t-\pi/2) - \cdots + \frac{0.083}{81}\sin(90t-\pi/2)\right]$$

Figure 12.9.2:

```
%2.003 Problem Set #9
%Problem 3

T=0.628;
w=2*pi/T*[1 3 5 7 9];


%Defining laplace variable
s=tf('s');

%G1(s)
K=10;
tau=20;
G1s=K/(tau*s+1);
figure(1);
bode(G1s,{10^-3,100});
title('Bode plot of G1(s)');

%G2(s)
K=1;
tau1=25;
tau2=10;
G2s=K/(tau1*s+1)/(tau2*s+1);
figure(2);
bode(G2s,{10^-3,100});
title('Bode plot of G2(s)');


%G3s
zeta=0.5;
wn=50;
G3s=s*wn^2/(s^2+2*zeta*wn*s+wn^2);
figure(3);
bode(G3s);
title('Bode plot of G3(s)');


%G4s
K=1;
tau=0.01;
zeta=1;
wn=100;
zetaz=0.05;
wzn=10;
G4s=K/(tau*s+1)*(s^2+2*zetaz*wzn*s+wzn^2)/(s^2+2*zeta*wn*s+wn^2);
figure(4);
bode(G4s);
title('Bode plot of G4(s)');
```

Figure 12.9.3:


G1(s)
Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

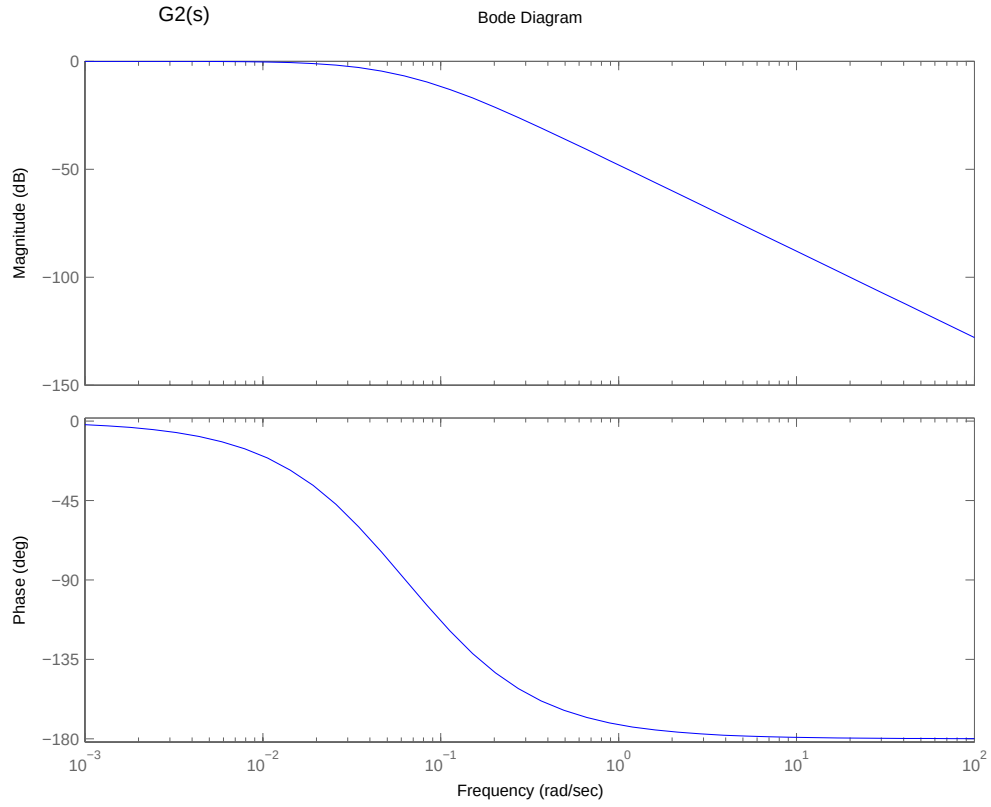

G2(s)
Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

Figure 12.9.4:

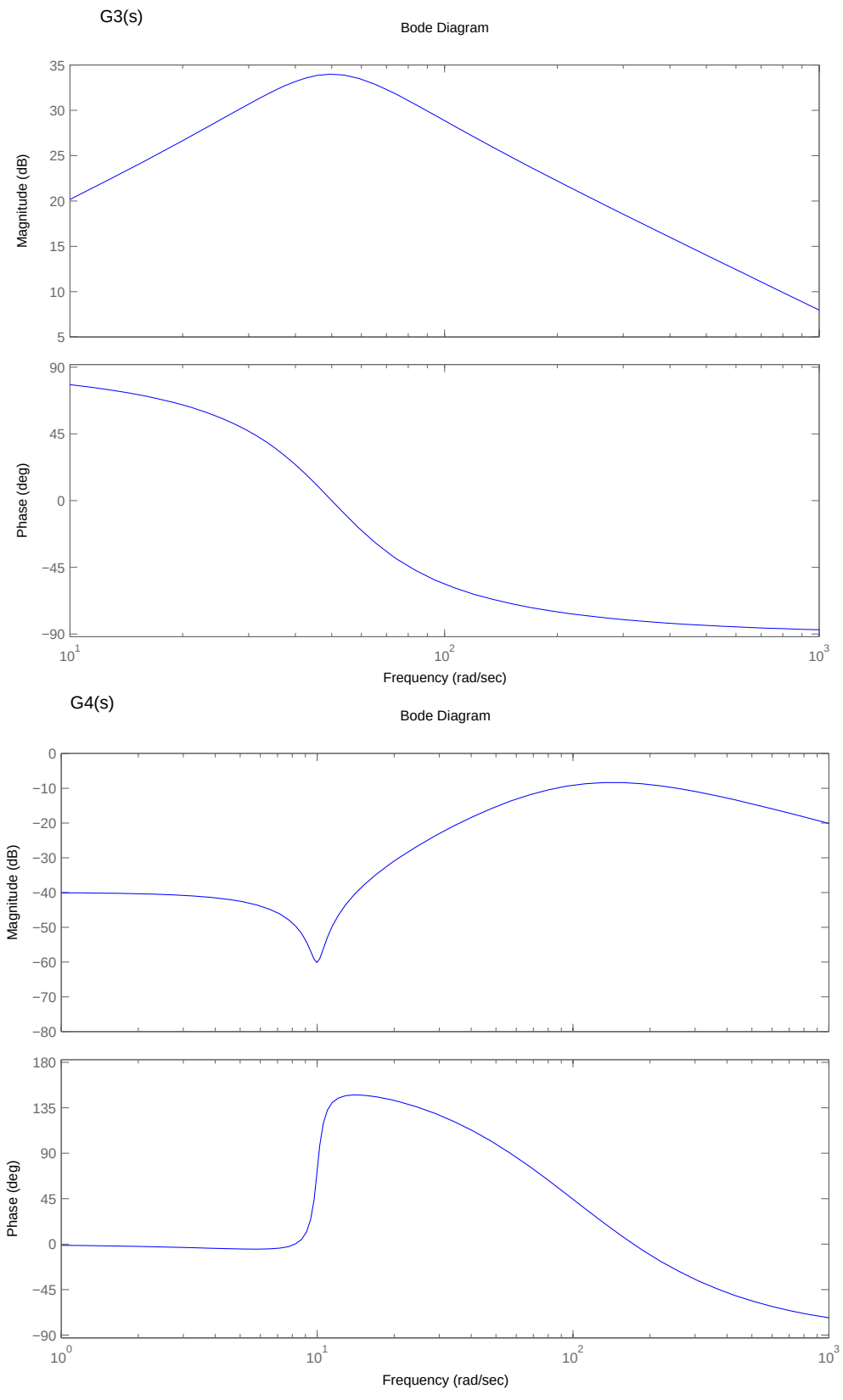

G3(s)
Bode Diagram
Magnitude (dB)
35
30
25
20
15
10
5
Phase (deg)
90
45
0
−45
−90
10^1
10^2
10^3
Frequency (rad/sec)
G4(s)
Bode Diagram
Magnitude (dB)
0
−10
−20
−30
−40
−50
−60
−70
−80
Phase (deg)
180
135
90
45
0
−45
−90
10^0
10^1
10^2
10^3
Frequency (rad/sec)

Figure 12.9.5:

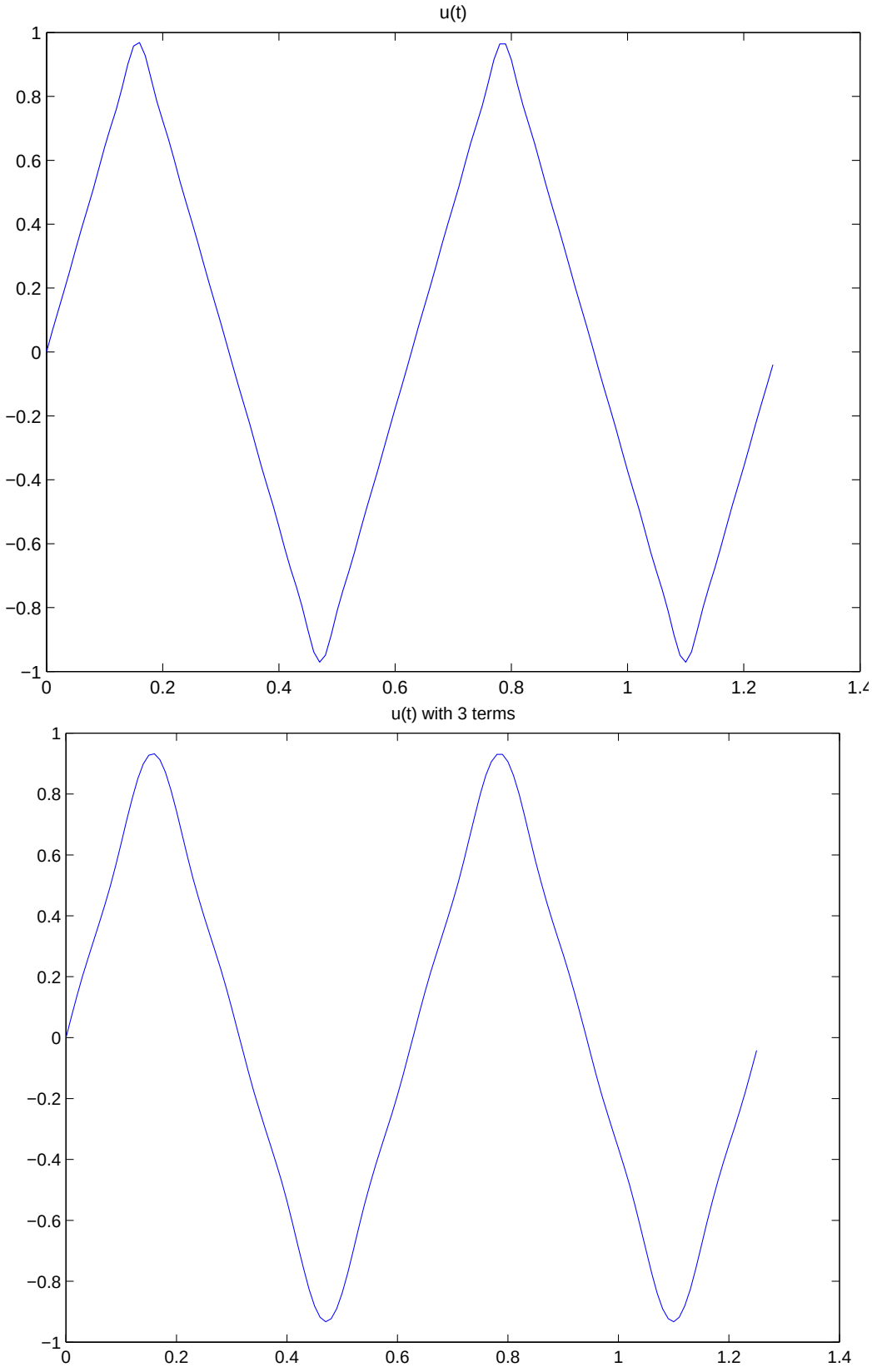
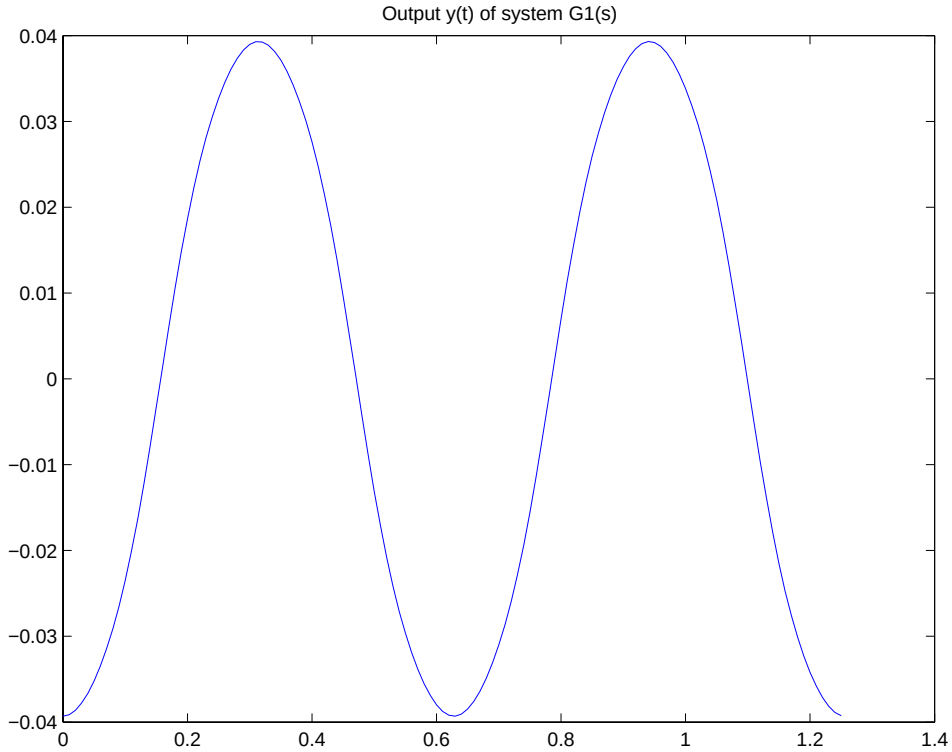

u(t)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.2
0.4
0.6
0.8
1
1.2
u(t) with 3 terms
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.2
0.4
0.6
0.8
1
1.2
1.4

Figure 12.9.6:


Output y(t) of system G1(s)
0.04
0.03
0.02
0.01
0
−0.01
−0.02
−0.03
−0.04
0
0.2
0.4
0.6
0.8
1
1.2
1.4

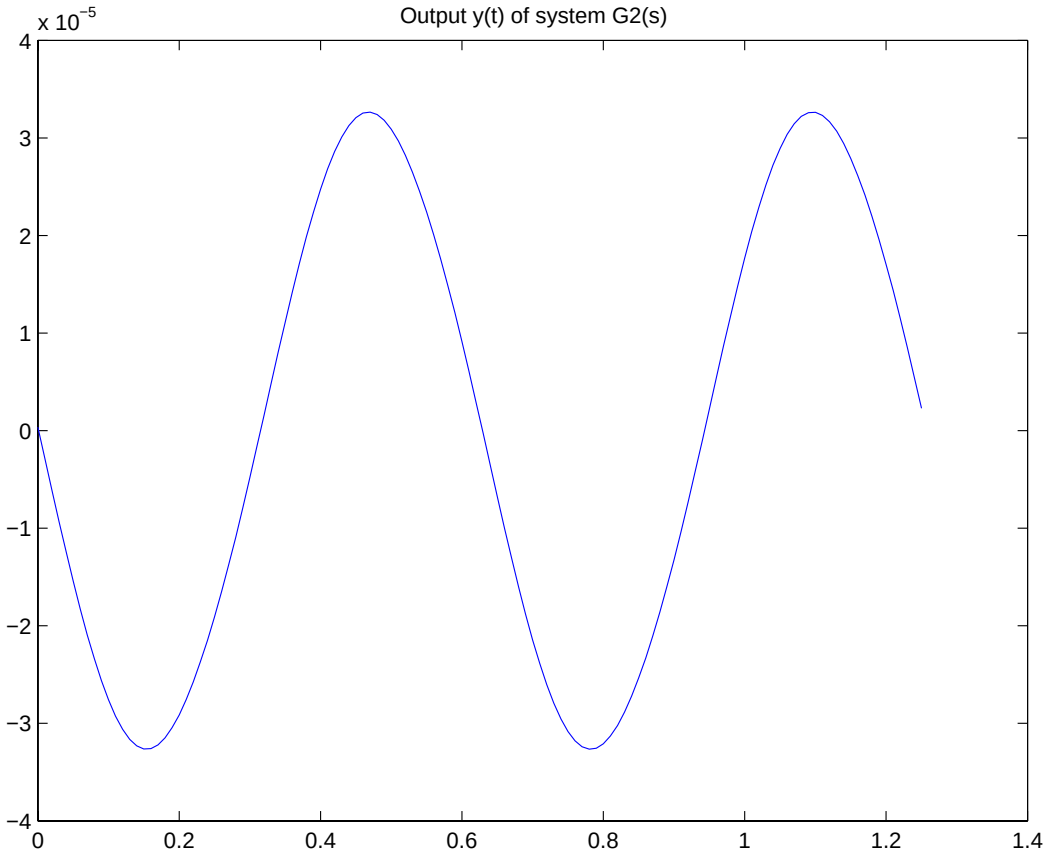
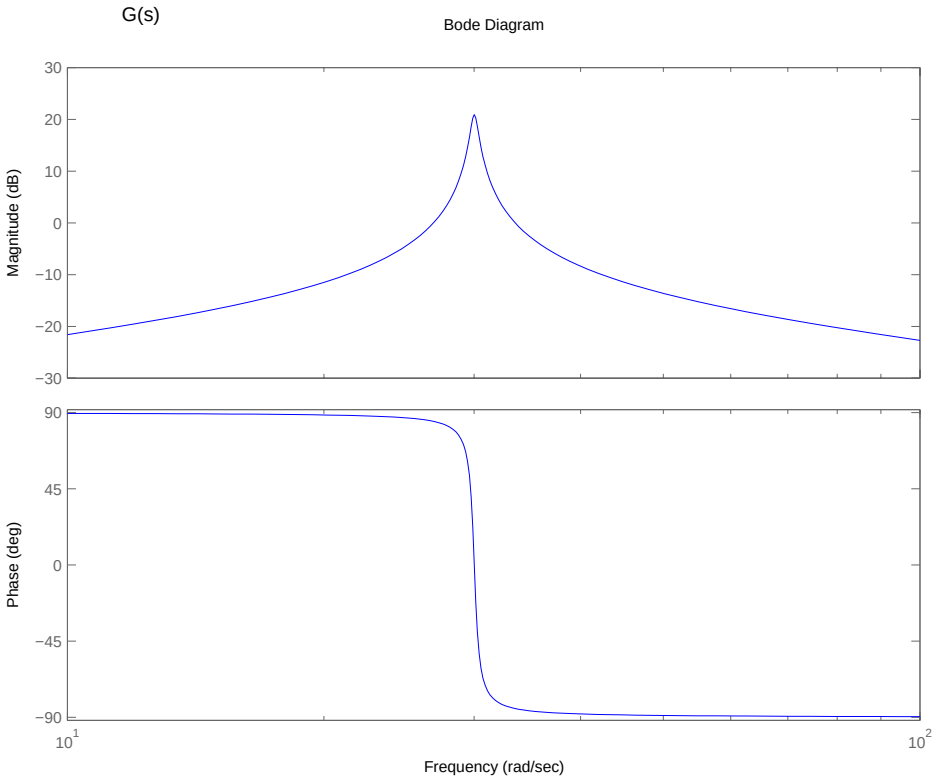

Output y(t) of system G2(s)
x 10^−5
4
3
2
1
0
−1
−2
−3
−4
0
0.2
0.4
0.6
0.8
1
1.2
1.4

Figure 12.9.7:


G(s)
Bode Diagram
Magnitude (dB)
30
20
10
0
−10
−20
−30
Phase (deg)
90
45
0
−45
−90
10^1
10^2
Frequency (rad/sec)

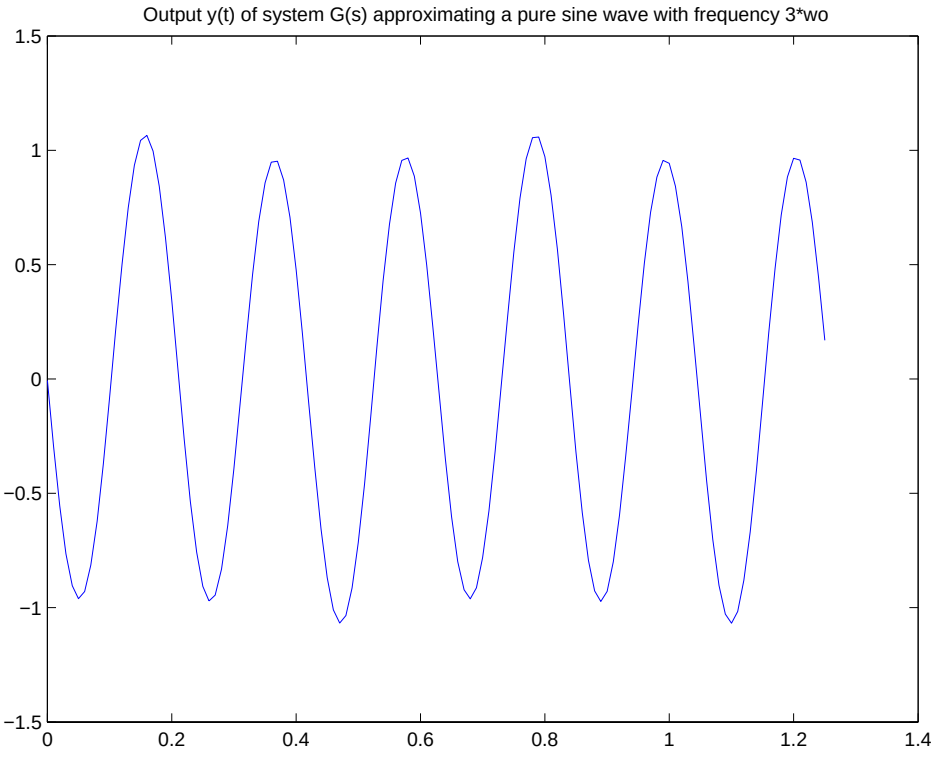

Output y(t) of system G(s) approximating a pure sine wave with frequency 3*wo
1.5
1
0.5
0
−0.5
−1
−1.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4

Figure 12.9.8:

12.10 Spring Mass System

(a)

$$\begin{aligned} f(t) &= u(t-2) - u(t-10) \\ F(s) &= \frac{1}{s}(e^{-2s} - e^{-10s}) \end{aligned}$$

(b)

$$m\ddot{x} + b\dot{x} + kx = f(t) = u(t-2) - u(t-10)$$

The Laplace transform of this system is

$$\begin{aligned} m[s^2X(s) - sx(0) - \dot{x}(0)] + b[sX(s) - x(0)] + kX(s) &= \frac{1}{s}(e^{-2s} - e^{-10s}) \\ \text{Assuming } x(0) = \dot{x}(0) &= 0 \\ (ms^2 + bs + k)X(s) &= \frac{1}{s}(e^{-2s} - e^{-10s}) \\ X(s) &= \frac{1}{s(s^2+s+1)}(e^{-2s} - e^{-10s}) \end{aligned}$$

The inverse laplace transform of $1/(s(s^2+s+1))$ can easily be seen to be a decaying sinusoid. e^{-2s} and $-e^{-10s}$ are just time delays in the real time. Thus

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s(s^2+s+1)}\right] &= 1 - \frac{1}{\sqrt{1-0.5^2}}e^{-0.5t}\sin(\sqrt{1-0.5^2}t + \phi) \\ x(t) &= u(t-2)[1 - \frac{1}{\sqrt{1-0.5^2}}e^{-0.5(t-2)}\sin(\sqrt{1-0.5^2}(t-2) + \phi)] \\ &\quad -u(t-10)[1 - \frac{1}{\sqrt{1-0.5^2}}e^{-0.5(t-10)}\sin(\sqrt{1-0.5^2}(t-10) + \phi)] \\ \phi &= \tan^{-1}(\sqrt{1-0.5^2}/0.5) \end{aligned}$$

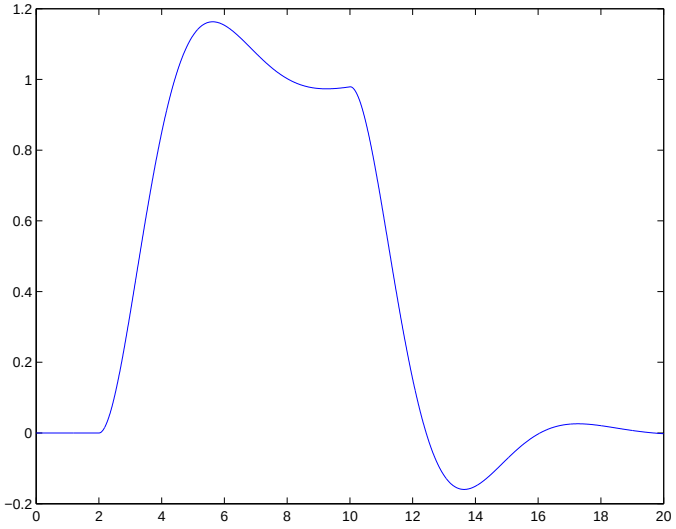


Figure 12.10.1: x(t) for Part c

(c) Well there are a couple of ways to get this plot using Matlab. One is to use the lsim command in which you specify the system transfer function (we have not talked about this in class yet), the input vector (0 t<2, 1 2<t<10, 0 t>10), and the calculation time vector (t=[0:0.01:20]). Note this method determines the solution by integrating from one time step to the next, thus for "Good" results you need to have a relatively fine time vector. The advantage of this method is a relatively compact .M file.

I calculated the response by nesting an if loop inside a for loop and calculated the exact response for each time index. I used the script 'springmasssol.m' at the end of this solution. Figure 12.10.1 shows the resulting plot.

(d) The plot is basically the sum of two separate decaying sinusoids. By this I mean, the dynamic response to the first step input has largely died away before the response to the negative step begins.

(e) If the pulse were 2 sec. instead of 8 sec., the two sinusoids would act destructively with each other. The exact response would depend on the phase of the two sinusoid relative to one another but the system would not reach 1 but still oscillate about zero. Figure 12.10.2 shows this response.

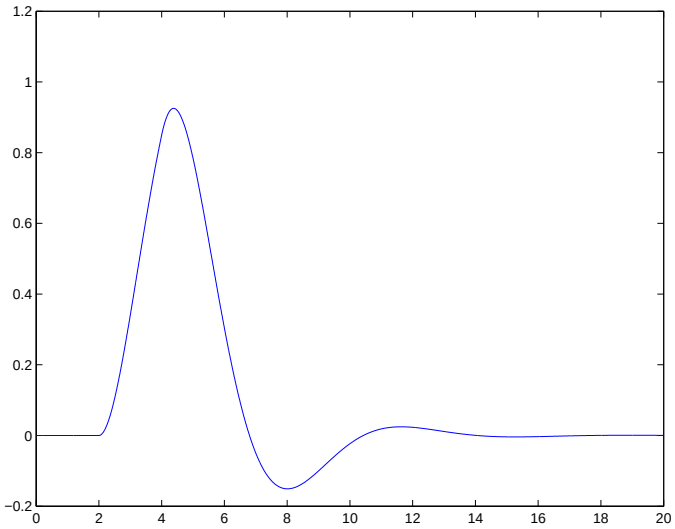


Figure 12.10.2: x(t) for Part e

12.11 Transfer Function

(a) The zeros are located at

$$\begin{aligned} s_1 &= -1 \\ s_2 &= -10 \\ s_3 &= -99 \end{aligned}$$

The poles are located at

$$\begin{aligned} s_{1,2} &= -0.5 \pm 0.866j \\ s_{3,4} &= -0.5 \pm 7.053j \end{aligned}$$

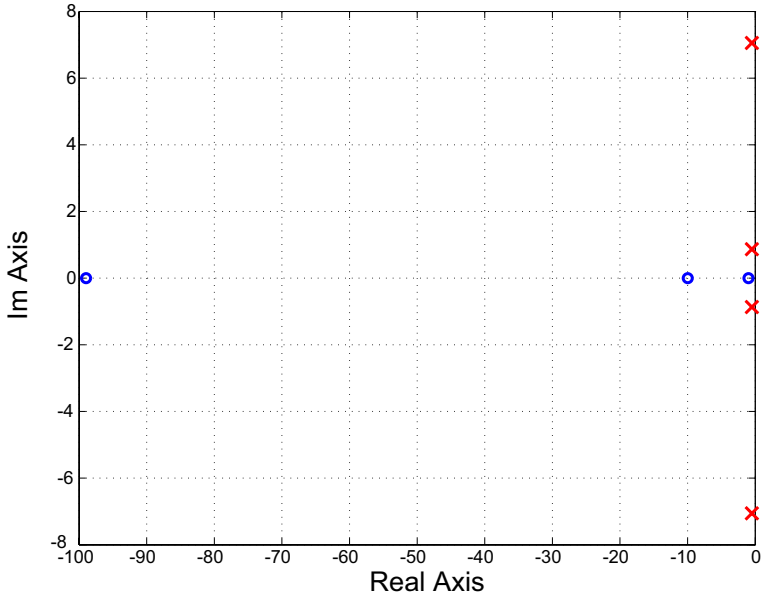


Figure 12.11.1: Pole zero plot

(b) Figure 12.11.1 shows the pole/zero plot for this system.

(c) From the pole locations, we know that the response will be a sum of sinusoids with a damped natural frequency of 0.87 r/s and 7 r/s. The higher frequency sinusoid is more lightly damped but since σ is the same for both sinusoids, they will decay out in the same time period. In this case, we would might assume that the energy contained in each of the sinusoids is approximately equal. Since kinetic energy is $\frac{1}{2}mv^2$ and $v = \frac{dx}{dt} = \omega_d \cos \omega_d t$, we would expect the magnitude of the faster sinusoid to be smaller than the magnitude of the slower sinusoid. Thus the response will look like a large slow sinusoid with a smaller faster sinusoid added on top. The impact of the zeros is a little harder to predict. The zero at -100 is so fast that any effect will be over so quickly that it will not effect the shape of the response. The zeros as -1 and -10 will impact the system response. Generally, zeros near the frequency of a complex pole pair cause the system to overshoot more than predicted by the pole pair ζ. Thus the

zero at -10 will cause the 7 r/s sinusoid to overshoot more than predicted while the zero at -1 will cause the 0.86 r/s sinusoid to have more overshoot. Note: if the zeros were in the right half plane they would cause the system to initially move in the negative direction.

(d) Figure 12.11.2 shows the step response for this system. My Matlab code for this problem is shown at the end of the solution in an m-file called "tfsol.m".

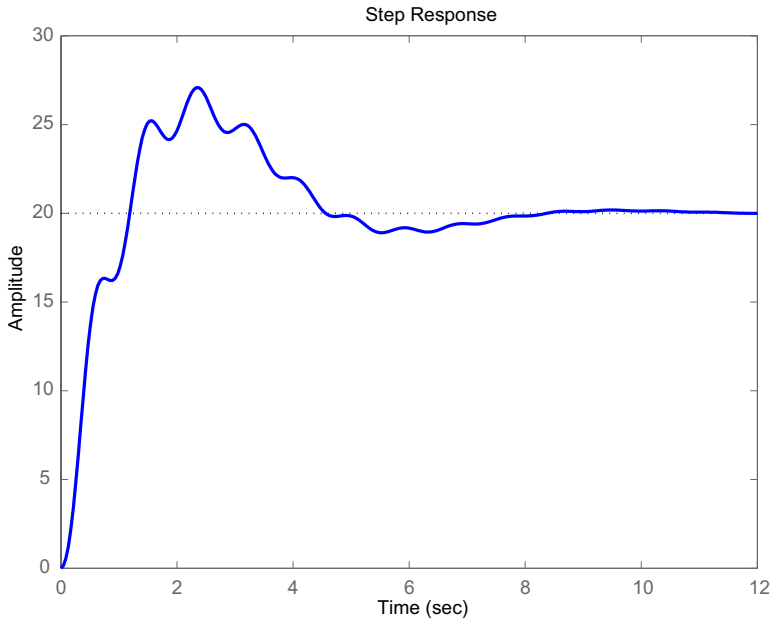


Figure 12.11.2: Step Response

13 Controls

13.1 Rooftop Antenna

(a) To start this problem we need to break the analysis into steps. In step 1, we analysis antenna system. Figure 13.1.1-A shows shows the free body diagram for the antenna. The control torque T_m and a disturbance T_d act on the antenna inertia:

$$\begin{aligned} J\dot{\omega}_o &= T_m + T_d \rightarrow \omega_o = \frac{T_m + T_d}{Js} \\ \theta_o &= \frac{\omega_o}{s} \end{aligned}$$

The equivalent block diagram is shown beside the FBD. Next, we need to evaluate the motor. Figure 13.1.1-B shows the circuit model of the motor:

$$\begin{aligned} V_m &= R_a I_a + 10\omega_o \\ I_a &= \frac{V_m - 10\omega_o}{R_a} \\ T_m &= K_m I_a \end{aligned}$$

The equivalent block diagram for the motor is shown next to the circuit. We can now combine the block diagrams in Figure 13.1.1 -A&B to form the block diagram in Figure 13.1.1-C. The amplifier has the following characteristics:

$$V_m = G(\theta_i - \theta_o)$$

with the equivalent block diagram in Figure 13.1.1-D. Combining Figure 13.1.1 -C&D yields the block diagram for the complete system Figure 13.1.1-E.

In order to find the transfer function $V_o/V_i(s)$, we set the disturbance torque $T_d = 0$. We start by reducing the inner loop of the block diagram to:

$$\frac{\omega_o}{V_m}(s) = \frac{\frac{K_m}{R_a Js}}{1 + \frac{10K_m}{R_a Js}} = \frac{K_m}{R_a Js + 10K_m}$$

We can now address the outer loop

$$\frac{\theta_o}{\theta_i}(s) = \frac{\frac{GK_m}{s(R_a Js + 10K_m)}}{1 + \frac{GK_m}{s(R_a Js + 10K_m)}} = \frac{GK_m}{R_a Js^2 + 10K_m s + GK_m}$$

To determine the transfer function $\theta_o/T_d(s)$, we set $\theta_i = 0$. It is helpful to reorganize the block diagram into the form shown in Figure 13.1.1-F.

$$\frac{\theta_o}{T_d}(s) = \frac{\frac{1}{Js^2}}{1 + \frac{K_m}{R_a Js^2}(G + 10s)} = \frac{R_a}{R_a Js^2 + 10K_m s + GK_m}$$

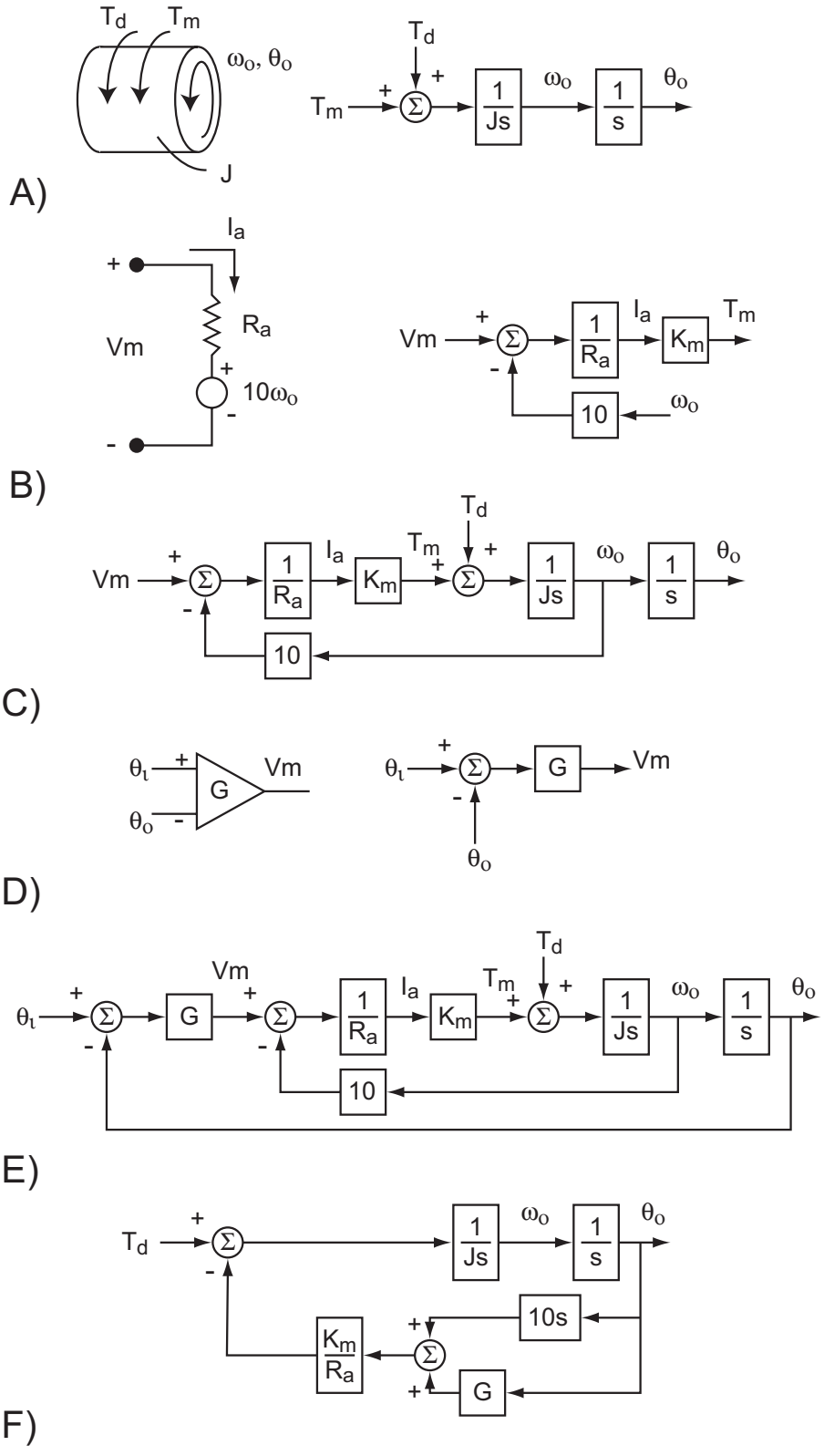


Figure 13.1.1: Block diagrams for antenna analysis

(b)

$$\begin{aligned}
\frac{\theta_o}{\theta_i}(s) &= \frac{100}{10s^2 + 100s + 100} \\
s_1 &= -8.873 \\
s_2 &= -1.127 \\
\theta_o(t) &= 1 + 0.145e^{-8.873t} - 1.145e^{-1.127t}
\end{aligned}$$

Figure 13.1.2 shows the pole map and step response for this system

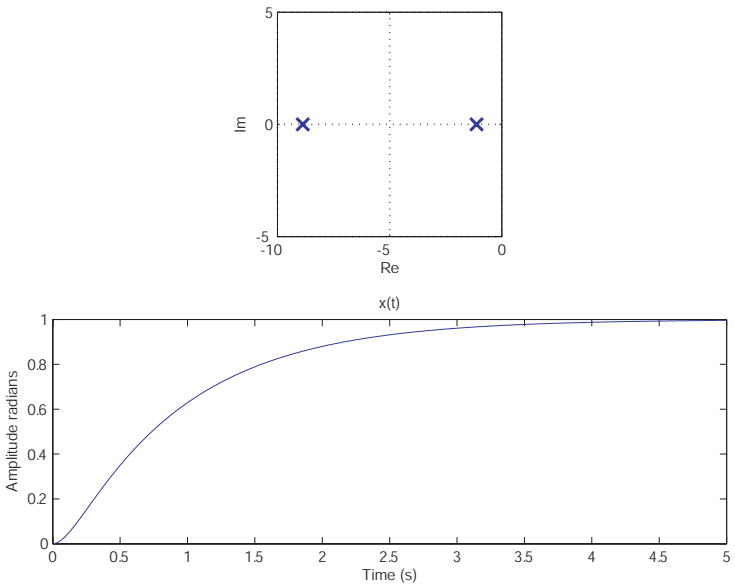


Figure 13.1.2: Pole map $\omega_o/\omega_i(s)$ and step response $\omega_o(t)$ for $u(t)\omega_i$

(c) The solid line in Figure 13.1.3 shows the frequency response.

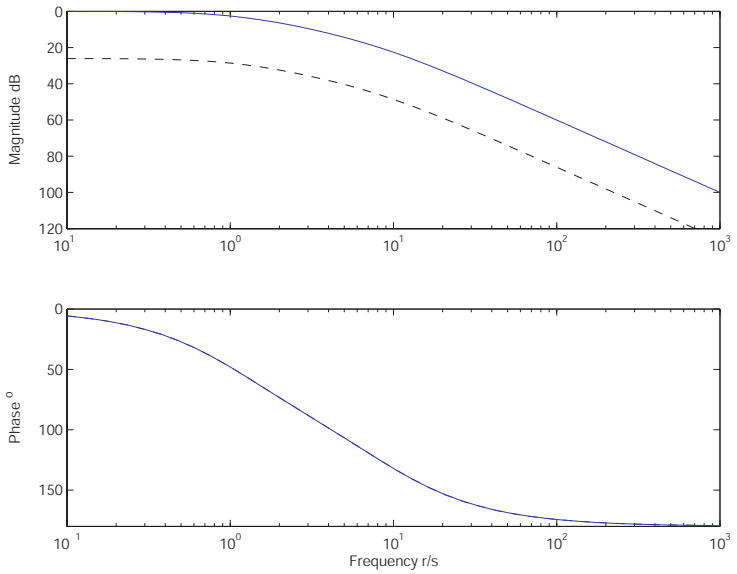


Figure 13.1.3: Frequency response $\omega_o/\omega_i(s)$ (solid line) and $\omega_o/T_d(s)$ (dashed line)

(d)

$$\begin{aligned}\frac{\theta_o}{T_d}(s) &= \frac{5}{10s^2+100s+100}\\ \theta_o(t) &= \frac{1}{20}(1+0.145e^{-8.873t}-1.145e^{-1.127t})\end{aligned}$$

The step response is identical to that shown in Figure 13.1.2 except the magnitude is scale down to 0.05. The frequency response is the dashed line in Figure 13.1.3.

(e) The negative of the loop transmission for this system is

$$\begin{aligned}L(s) &= \frac{GK_m}{s(R_aJs+10K_m)} = \frac{100}{10s^2+100s}\\ \omega_c &= 1\,r/s\\ \phi_c &= -95.8^o \;\rightarrow\; \phi_m = 84.2^o\end{aligned}$$

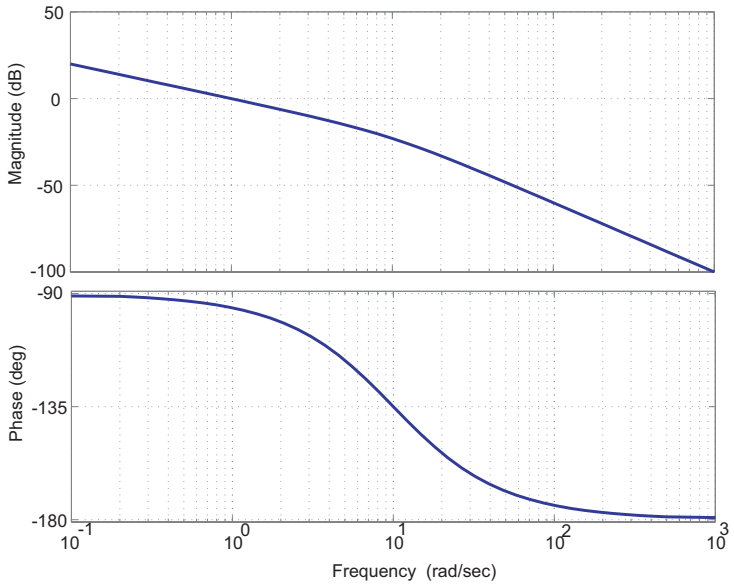


Figure 13.1.4: Bode plot for the negative of the loop transmission $L(s)$ for G=10.

Figure 13.1.4 shows the bode plot of $L(s)$. The crossover frequency ω_c is the frequency where the magnitude of $L(s) = 1$ (0 dB). The phase margin $\phi_m = 180^o - \phi_c$ where ϕ_c is the phase of $L(s)$ at ω_c.

(f)

$$
\begin{aligned}
\frac{\theta_o}{\theta_i}(s) &= \frac{1000}{10s^2 + 100s + 1000} \\
s_{1,2} &= -5 \pm 8.66j \\
\theta_o(t) &= 1 + 1.15e^{-5t}\sin(8.66t - 2.094) \text{ for u(t)}\omega_i \\
\frac{\theta_o}{T_d}(s) &= \frac{5}{10s^2 + 100s + 1000} \\
\theta_o(t) &= \frac{1}{200}(1 + 1.15e^{-5t}\sin(8.66t - 2.094)) \text{ for u(t)}T_d \\
L(s) &= \frac{1000}{10s^2 + 100s} \\
\omega_c &= 7.8\, r/s \\
\phi_c &= -128^o \;\rightarrow\; \phi_m = 52^o
\end{aligned}
$$

Figure 13.1.5 shows the pole map and step response for ω_o/ω_i with a G=100. The $\omega_o/T_d(s)$ step response is 1/200 the ω_o/ω_i step response. Figure 13.1.6 shows the frequency response for ω_o/ω_i (solid line) and $\omega_o/T_d(s)$ (dashed line). Figure 13.1.7 shows the bode plot of the negative of the loop transmission $L(s)$. As we can see from the step response, increasing G speeds up the closed-loop dynamics and reduces the damping ratio of the response. Increasing G impacts the frequency response by increasing the range of frequencies for which the magnitude is at or near one. This means that the range of frequencies where we get good command following is increased. We also notice in the frequency response that the phase transition is sharper

for the higher gain system reflecting the reduced damping ration. Lastly we note, that the higher gain system rejects disturbance much better (10X better to be exact). Increasing the gain has transformed the two real poles into a complex pole pair. Further increasing the G will move the poles further up along the imaginary axis (i.e. increase ω_d) without changing the real component of the pole (i.e. further reduce the damping ratio).

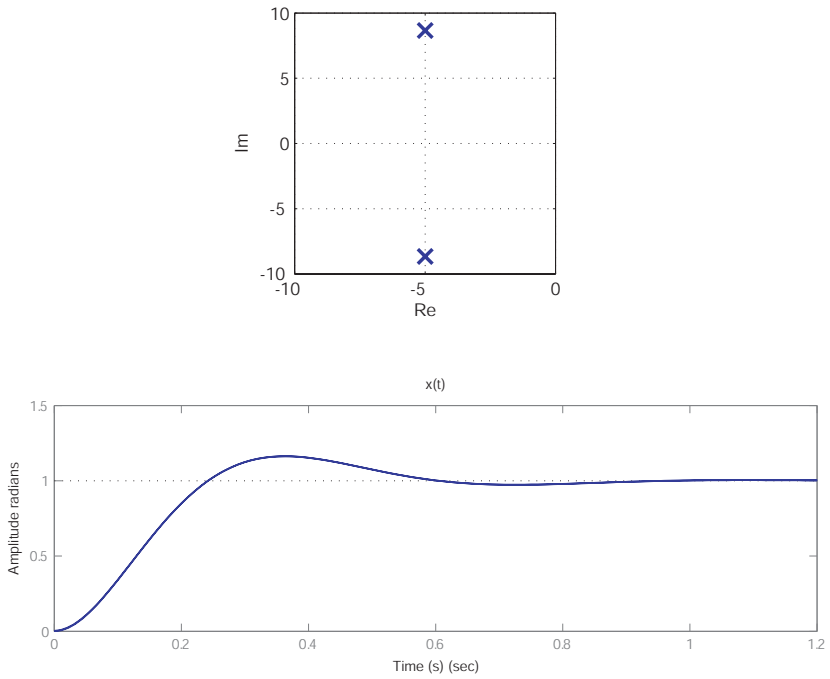


Figure 13.1.5: Pole map $\omega_o/\omega_i(s)$ and step response $\omega_o(t)$ for $u(t)\omega_i$ for $G = 100$.

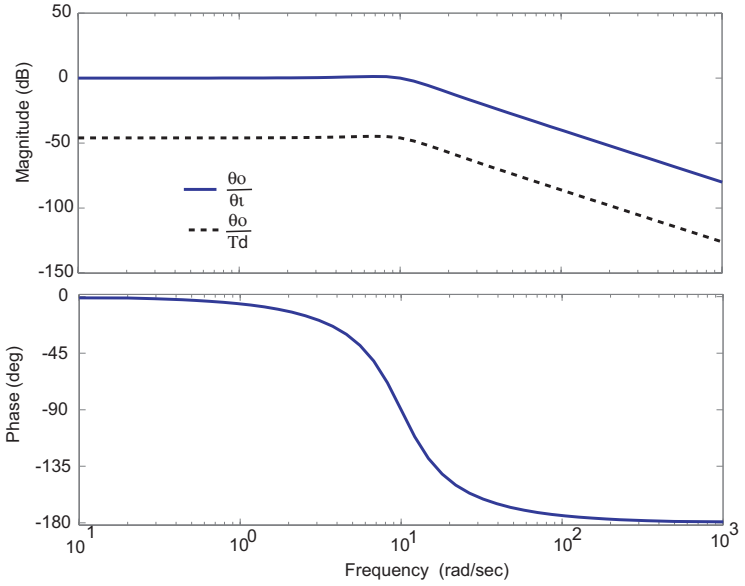


Figure 13.1.6: Frequency response $\omega_o/\omega_i(s)$ (solid line)and $\omega_o/T_d(s)$ (dashed line) for G=100

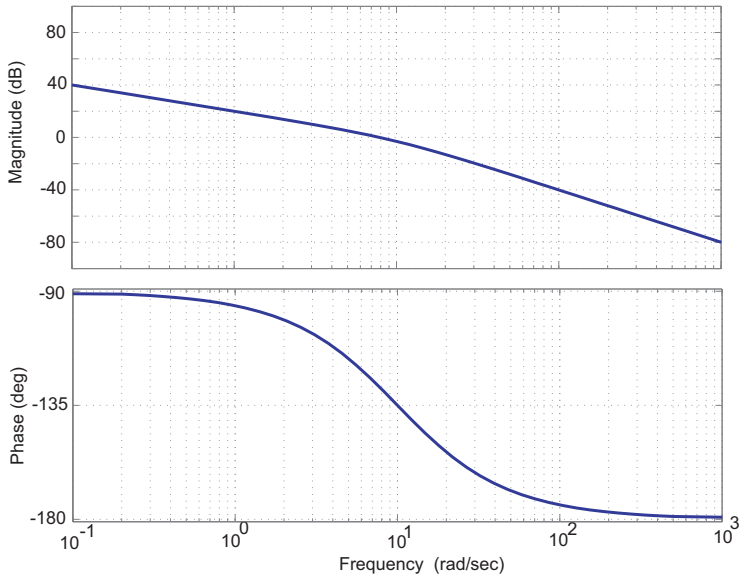


Figure 13.1.7: Bode plot for the negative of the loop transmission $L(s)$ for G=100.

(g)

$$
\begin{aligned}
\frac{\theta_o}{\theta_i}(s) &= \frac{10000}{10s^2+100s+10000} \\
s_{1,2} &= -5 \pm 31.2j \\
\theta_o(t) &= 1 + 1.01e^{-5t}\sin(31.2t - 1.73) \text{ for u(t)}\omega_i \\
\frac{\theta_o}{T_d}(s) &= \frac{5}{10s^2+100s+10000} \\
\theta_o(t) &= \frac{1}{2000}(1 + 1.01e^{-5t}\sin(31.2t - 1.73)) \text{ for u(t)}T_d \\
L(s) &= \frac{10000}{10s^2+100s} \\
\omega_c &= 30.7\, r/s \\
\phi_c &= -162^o \;\rightarrow\; \phi_m = 18^o
\end{aligned}
$$

Figure 13.1.8 shows the pole map and step response for ω_o/ω_i with a G=1000. The $\omega_o/T_d(s)$ step response is 1/200 the ω_o/ω_i step response. Figure 13.1.9 shows the frequency response for ω_o/ω_i (solid line) and $\omega_o/T_d(s)$ (dashed line). Figure 13.1.10 shows the bode plot of the negative of the loop transmission $L(s)$.

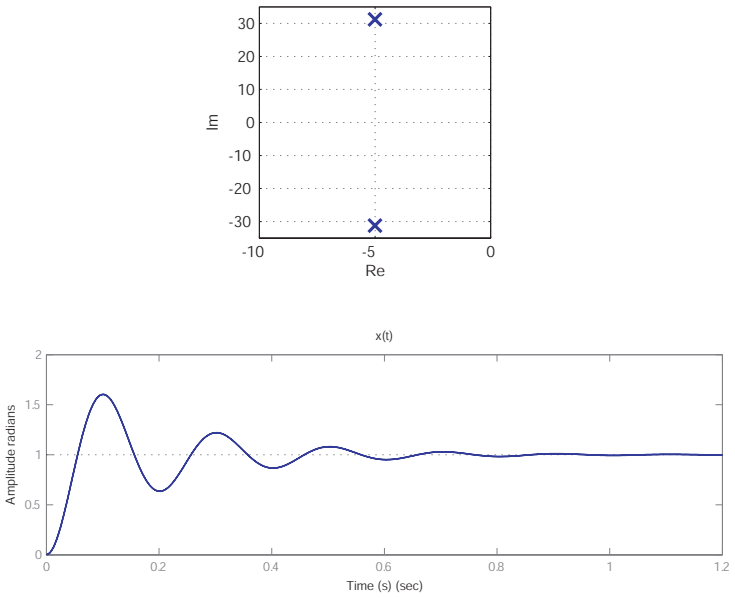


Figure 13.1.8: Pole map $\omega_o/\omega_i(s)$ and step response $\omega_o(t)$ for $u(t)\omega_i$ for G=1000.

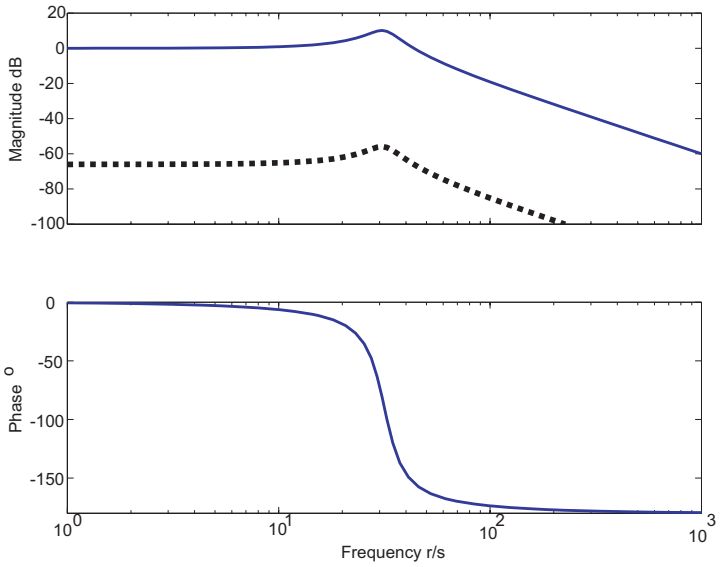


Figure 13.1.9: Frequency response $\omega_o/\omega_i(s)$ (solid line) and $\omega_o/T_d(s)$ (dashed line) for G=1000.

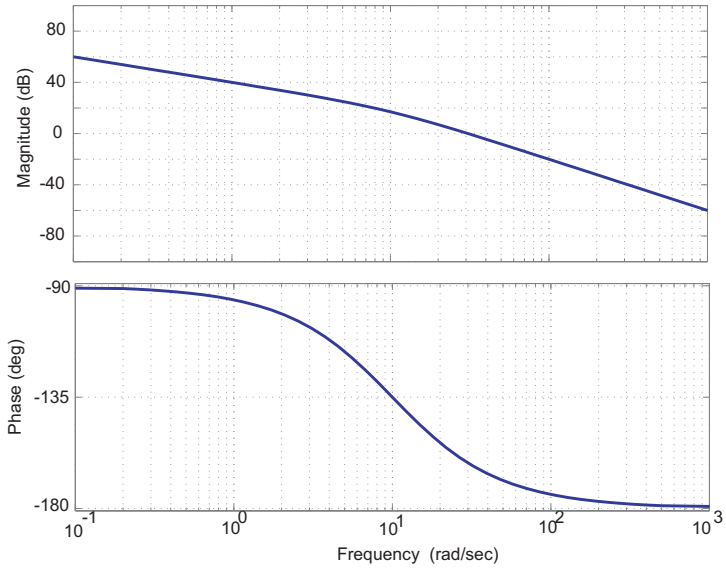


Figure 13.1.10: Bode plot for the negative of the loop transmission $L(s)$ for G=1000.

13.2 Lead Controller

The solution for this problem is not available.

13.3 Third Order

Symmetrical mechanical drive. See Fig. 13.3.1. Two equal inertias I with equal linear friction coefficients B are coupled by a shaft with torsional stiffness K_c and torsional damping coefficient B_c.

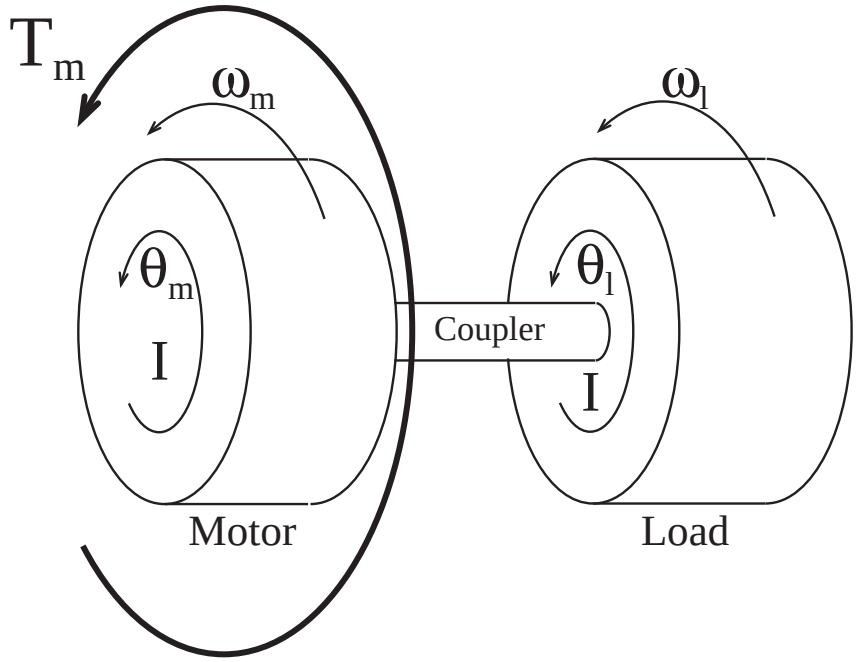


Figure 13.3.1: Symmetrical Mechanical Drive

(a) The fundamental requirements are:
Geometrical Compatibility: $\Delta\theta = \theta_m - \theta_l, \qquad d\Delta\theta/dt = \omega_m - \omega_l$

Constitutive Equations: $T_k = K_c\Delta\theta, \quad T_d = B_c d\Delta\theta/dt, \quad T_{f,m} = B\omega_m,$
$T_{f,l} = B\omega_l, \quad T_{I,m} = Id\omega_m/dt, \quad T_{I,l} = Id\omega_l/dt.$

Torque Balance: Motor: $T_{I,m} = T_m - K_c\Delta\theta - B\omega_m - B_c d\Delta\theta/dt,$
Load: $T_{I,l} = K_c\Delta\theta - B\omega_l + B_c d\Delta\theta/dt$

In the following state equations , the first equation expresses the geometric compatibility requirement, the second equation is the result of inserting the constitutive equations into the torque balance for the motor, and the third is the result of substituting the constitutive equations into the torque balance for the load.

$$\frac{d}{dt}\begin{Bmatrix} \Delta\theta \\ \omega_m \\ \omega_l \end{Bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -\frac{K_c}{I} & -\frac{B+B_c}{I} & \frac{B_c}{I} \\ \frac{K_c}{I} & \frac{B_c}{I} & -\frac{B+B_c}{i} \end{bmatrix} \begin{Bmatrix} \Delta\theta \\ \omega_m \\ \omega_l \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{I} \\ 0 \end{Bmatrix} T_m$$

(b) The new variables $\Delta\omega$ and ω_{avg} satisfy the following equations:

$$\begin{aligned} 2\omega_{avg} &= \omega_m + \omega_l \\ \Delta\omega &= \omega_m - \omega_l \end{aligned}$$

By adding and subtracting these equations we find

$$\begin{aligned} \omega_m &= \omega_{avg} + \Delta\omega/2 \\ \omega_l &= \omega_{avg} - \Delta\omega/2 \end{aligned}$$

(c) We can construct the state equations for the new variables $\Delta\theta$, $\Delta\omega$ and ω_{avg} one at a time. For $\Delta\theta$, we have

$$\frac{d\Delta\theta}{dt} = \omega_m - \omega_l = \Delta\omega$$

For $\Delta\omega$, we write

$$\begin{aligned} \frac{d\Delta\omega}{dt} = \frac{d\omega_m}{dt} - \frac{d\omega_l}{dt} &= \left[-\frac{K_c}{I}\Delta\theta - \frac{B+B_c}{I}(\omega_{avg} + \Delta\omega/2) + \frac{B_c}{I}(\omega_{avg} - \Delta\omega/2)\right] + \frac{T_m}{I} \\ &\quad - \left[\frac{K_c}{I}\Delta\theta - \frac{B+B_c}{I}(\omega_{avg} - \Delta\omega/2) + \frac{B_c}{I}(\omega_a vg + \Delta\omega/2)\right] \\ &= -\frac{2K_c}{I}\Delta\theta - \frac{B+2B_c}{I}\Delta\omega + \frac{T_m}{I} \end{aligned}$$

Similarly, for ω_{avg}, we write

$$\begin{aligned} \frac{d\omega_{avg}}{dt} = \frac{1}{2}\left(\frac{d\omega_m}{dt} + \frac{d\omega_l}{dt}\right) &= \frac{1}{2}\left[-\frac{K_c}{I}\Delta\theta - \frac{B+B_c}{I}(\omega_{avg} + \Delta\omega/2) + \frac{B_c}{I}(\omega_{avg} - \Delta\omega/2)\right] + \frac{T_m}{I} \\ &\quad + \frac{1}{2}\left[\frac{K_c}{I}\Delta\theta - \frac{B+B_c}{I}(\omega_{avg} - \Delta\omega/2) + \frac{B_c}{I}(\omega_a vg + \Delta\omega/2)\right] \\ &= -\frac{B}{I}\omega_{avg} + \frac{T_m}{I} \end{aligned}$$

These state equations for the new variables can be arranged in matrices of a third-order sytem:

$$\frac{d}{dt}\begin{Bmatrix} \Delta\theta \\ \Delta\omega \\ \omega_{avg} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2\frac{K_c}{I} & -\frac{B+2B_c}{I} & 0 \\ 0 & 0 & -\frac{B}{I} \end{bmatrix} \begin{Bmatrix} \Delta\theta \\ \Delta\omega \\ \omega_{avg} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{I} \\ \frac{1}{I} \end{Bmatrix} T_m$$

or, as a combination of a second-order sub-system

$$\frac{d}{dt}\begin{Bmatrix} \Delta\theta \\ \Delta\omega \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -2\frac{K_c}{I} & -\frac{B+2B_c}{I} \end{bmatrix} \begin{Bmatrix} \Delta\theta \\ \Delta\omega \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{I} \end{Bmatrix} T_m$$

describing the relative motion of the motor and the load, and a first-order sub-system

$$\frac{d\omega_{avg}}{dt} = -\frac{B}{I}\omega_{avg} + \frac{1}{I}T_m$$

describing the average speed of motor and load. Both sub-systems have the motor torque as input, but the second-order sub-system is independent of the average speed ω_{avg}, and the first-order sub-system is independent of the *relative* motion between the motor and the load.

(d) The first-order sub-system equation can be rewritten as

$$\frac{I}{B}\frac{d\omega_{avg}}{dt} + \omega_{avg} = \frac{T_m}{B}$$

from which we identify the *decay time-constant*

$$\tau = \frac{I}{B} = \frac{\text{5e-5}}{\text{1.0e-4}} = 0.5 \text{ sec}$$

and the *steady-state speed*

$$\omega_{ss} = \frac{T_m}{B} = \frac{\text{6e-3}}{\text{1.0e-4}} = 60 \text{ rad/sec}$$

(e) From the second-order sub-system equations we identify

$$\omega_o^2 = \frac{2K_c}{I} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{B + 2B_c}{I}$$

from which we obtain the behavioral parameters of the second-order sub-system

$$\omega_o = \sqrt{\frac{2(\text{1.24e-2})}{\text{5e-5}}} = 22.3 \text{ rad/sec} \qquad \text{and} \qquad \zeta = \frac{\text{1.0e-4} + 2(\text{2e-5})}{2(22.3)(\text{5e-5})} = 0.0629$$

13.4 Error Constants

A)

$$\frac{E}{Y}(s) = -1$$

B)

$$\begin{aligned} \frac{E}{R}(s) &= \frac{1}{1+KG(s)} \\ \text{Step } E(s) &= \frac{1}{s(1+KG(s))} \\ \text{Ramp } E(s) &= \frac{1}{s^2(1+KG(s))} \\ \text{Parabola } E(s) &= \frac{1}{s^3(1+KG(s))} \end{aligned}$$

C) For this section $G(0) = K_g$.

	Type 0	Type 1	Type 2
Step Input	$\frac{1}{1+KK_g}$	0	0
Ramp	∞	$\frac{1}{KK_g}$	0
Parabola	∞	∞	$\frac{1}{KK_g}$

Figure 13.4.1:

13.5 System Type

A) Figure 1 shows the negative loop transmission for this system.
B) Note: To get a phase margin of 50^o, we would need the crossover frequency to be 14.9 r/s. It is not possible to get a crossover at 14.9 r/s because of the peak from the complex pair of poles. If we set K=8.6 (the gain needed to raise the magnitude of the loop transmission to 1 at 14.9 r/s), we find that the real crossover occurs at 16.9 r/s with a resulting phase margin of 44^o.

$$\begin{aligned} K_g &= 1 \\ \text{Error} &= \frac{1}{K * K_g} = \frac{1}{8.6} = 0.11 \end{aligned}$$

C) Figure 2 shows the loop transmission for the type 2 system. As we can see, this is not a good thing to do since we now have system where the phase is always below -180^o. Now it is possible to get stable closed loop systems where the loop transmission phase is below -180^o but these systems must have magnitudes less than 1 when the phase passes through -180^o. Since this system has 2 poles at $\omega = 0$ we essentially have an open loop gain of ∞ when we pass through -180^o resulting in an unstable closed loop system.

Figure 13.5.1:

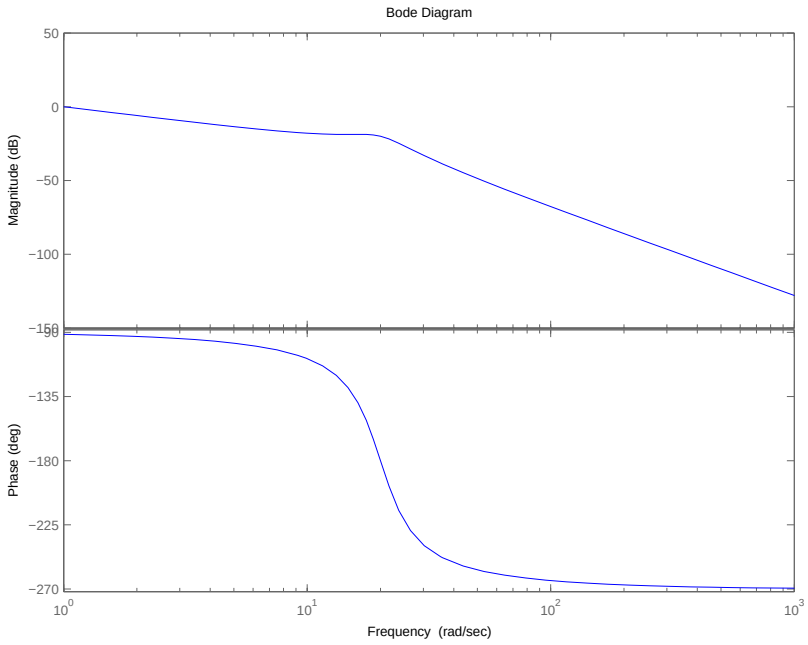


Figure 1: Bode diagram for Problem 2-A

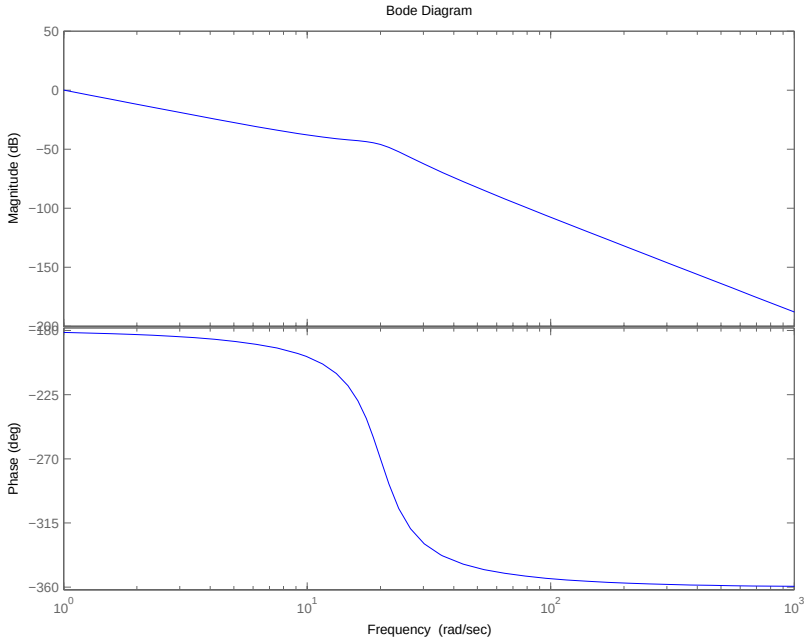


Figure 2: Bode diagram for Problem 2-C

Figure 13.5.2:

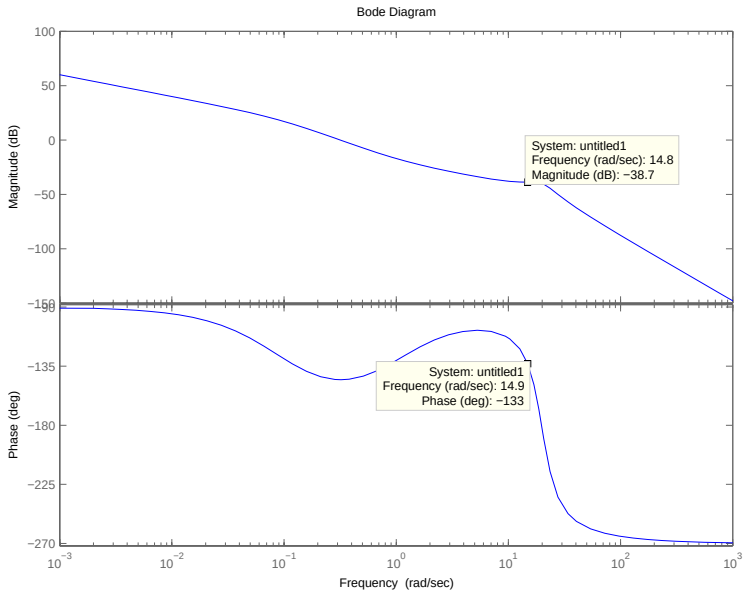


Figure 3: Intermidiate Bode plot for problem 2-D

D To get an error=2%, $0.02 \leq \frac{1}{K} \rightarrow K \geq 50$. From part B, we know that without the lag compensator is 8.6, so if we reduce the magnitude of the transfer function by a factor of ten we will raise the required gain to 86 (this is greater than 50). The transfer function for a lag compensator is

$$G_{lag}(s) = K\frac{\tau s + 1}{\alpha\tau s + 1}$$

For a lag compensator the pole occurs at a lower frequency than the zero, thus for the region between the pole and the zero the lag compensator has a -1 slope. We want to drop the magnitude by a factor of 10 (1 dec) so that tells us that $\alpha = 10$ so that the lag compensator acts over 1 decade. Now we need to figure out τ. We do not want the phase of the lag compensator to effect the phase at our desired crossover, so τ can be any value which places the zero of the lag a decade or more below the crossover frequency.

$$\tau \geq \frac{1}{1.49} = 0.67$$

To make life easy, I have elected to make $\tau = 1$. Figure 3 shows the bode plot for $tau = 1$. As we can see, the magnitude is the where we wanted it (required gain 86) but we have actually lost some phase. This is because the lag compensator is close enough to have some impact on the phase. If I were designing this for myself, I would just leave it but to fix this for the problem set I have set $tau = 2$ and recovered all but 1^o of phase. Figure 4 shows the negative loop transmission of this system with $tau = 2$, $K = 86$, and $\alpha = 10$.

Figure 13.5.3:

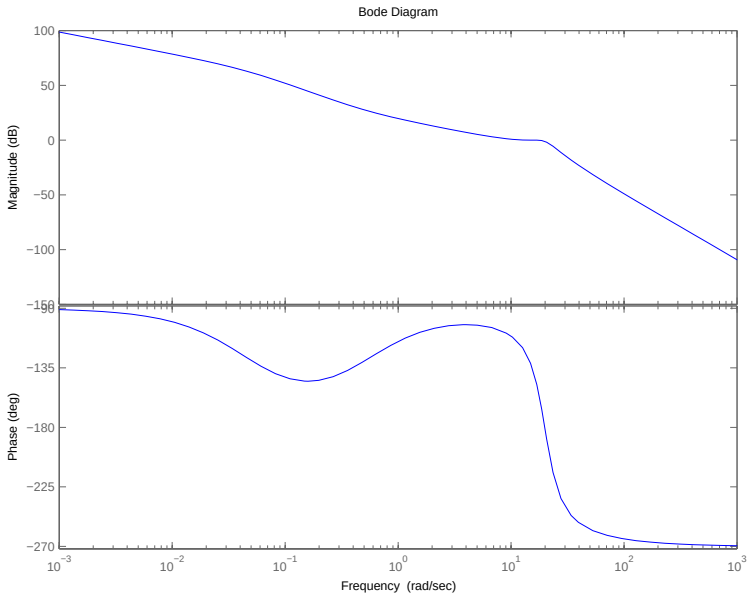


Figure 4: Bode plot for problem 2-D

On a design note here, I could get 0 steady state error by using the following "lag" compensator

$$G_c(s) = K\frac{\tau s + 1}{s}.$$

You may ask yourself, how is this different than changing the system to a type 2 system. The answer is that because we have added a zero in addition to the pole the phase of the system never actually equals -180^o at low frequencies, just gets really close. The major issue with this approach is that if there are any un-modelled dynamics such as time delay which cause the phase to decrease, the system will be unstable.
E Figure 5 shows the step response of the error function for the controller designed in part d. Well the damping is much lower than expected and the natural frequency is higher as well. The crossover frequency is higher than expected due to the resonant peak as discussed in part b. In the case of the lag controller, we find that not only is the actual crossover frequency about 16.5 r/s but that lag controller has further reduced our phase margin to 36^o.
F Figure 6 shows the error of my controller to a ramp input. Since, I designed my controller to have an error less than 2% and my actual error is about 1.6% I am happy. If customer (Prof. Hardt) really wanted 1% he should have told me that before I built the thing.

Figure 13.5.4:

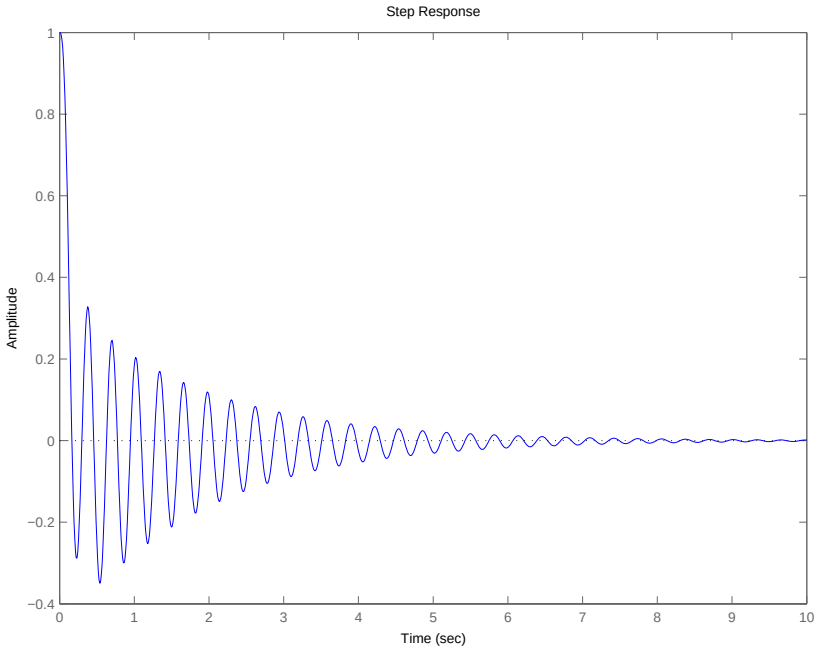


Figure 5: Step response of the error for lag controller design in D

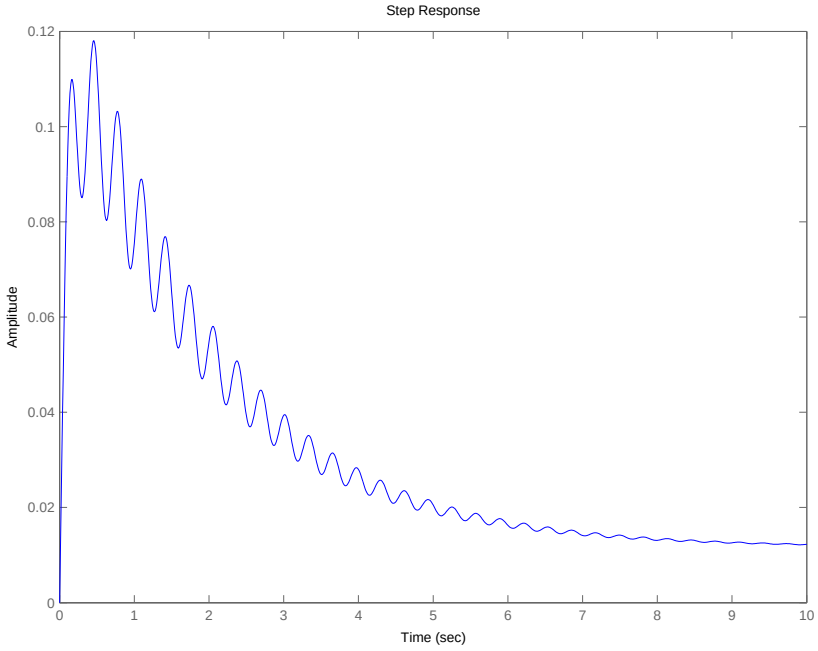
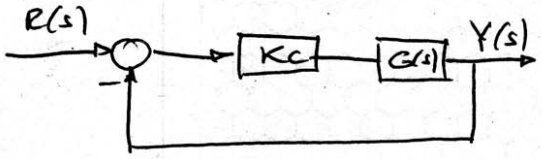


Figure 6: Error to a ramp input

Figure 13.5.5:

13.6 Unity Feedback

PROBLEM 4



$G(s) = 1/s$

$G_c(s) = K_c$

a) $T(s) = \dfrac{Y(s)}{R(s)} = \dfrac{K_c \cdot G(s)}{1 + K_c \cdot G(s)} = \dfrac{K_c \cdot (1/s)}{1 + K_c \cdot (1/s)}$

$T(s) = \dfrac{K_c}{K_c + s} = \dfrac{K_c}{s + K_c}$

b) $4\tau = 1\,\text{msec}$ (SINCE IT IS A FIRST ORDER SYSTEM)

$T(s) = \dfrac{1}{\frac{s}{K_c} + 1} = \dfrac{1}{\tau \cdot s + 1}$ so $4\tau = \dfrac{4}{K_c} = 1\,\text{msec}.$

$K_c = 4000$

c) $T_1(s) = \dfrac{1}{s+1}$; $T_2(s) = \dfrac{1}{\frac{s}{10} + 1}$; $T_3(s) = \dfrac{1}{\frac{s}{100} + 1}$

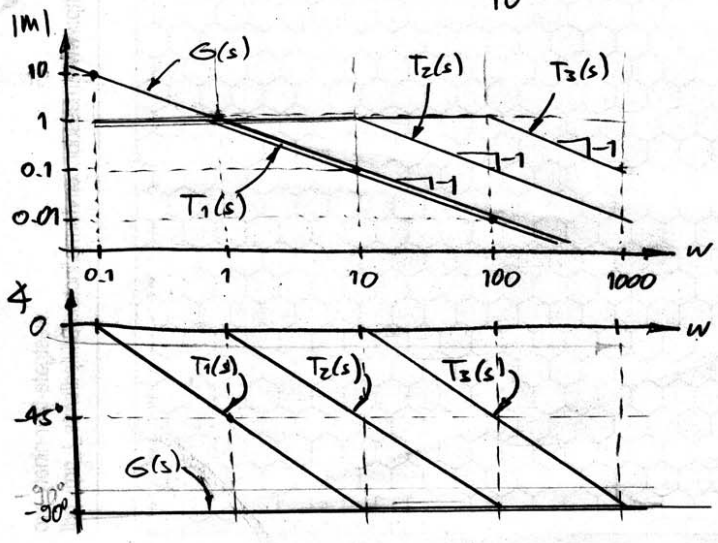


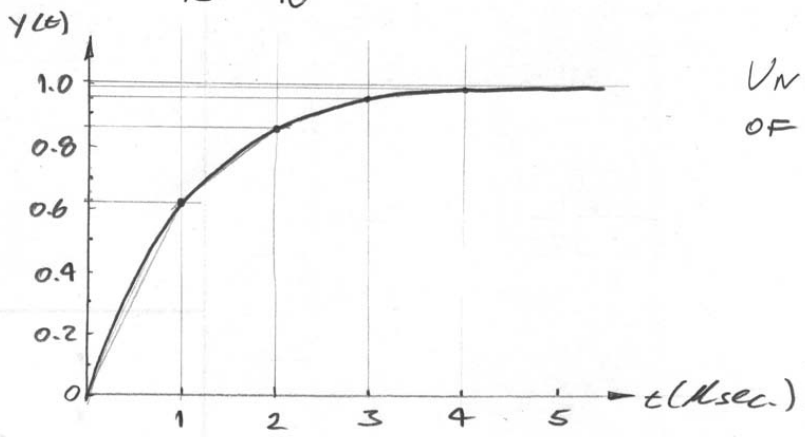
Figure 13.6.1:

d)

	BANDWIDTH (rad/sec.)
$T_1(s)$	$1/(0.5)^1 = 2$
$T_2(s)$	$10/(0.5)^1 = 20$
$T_3(s)$	$100/(0.5)^1 = 200$

e) FOR AN OP-AMP

$$\tau = \frac{1}{K} = \frac{1}{10^6} = 10^{-6}\,\text{sec.} = 1\,\mu\text{sec.}$$



UNIT STEP RESPONSE OF AN OP-AMP

f) THE BANDWIDTH OF THE OP-AMP AS DEFINED BEFORE WILL BE OF $10^6 \times (0.5)^1 = 2 \times 10^6$ rad/sec

Figure 13.6.2:

13.7 P and PI Controllers

The solution for this problem is not available.

14 Motors and Transformers

14.1 Locked Load

With the load clamped so that it cannot rotate, the application of motor torque causes twisting of the coupler and oscillation of the rotor. The system consists of the motor rotor with moment of inertia I_r and the coupler with linear torsional stiffness K_c and linear damping with damping coefficient B_c, acted on by the motor torque T_m and a frictional motor torque with linear damping coefficient B_m, as shown in Fig. 14.1.1.

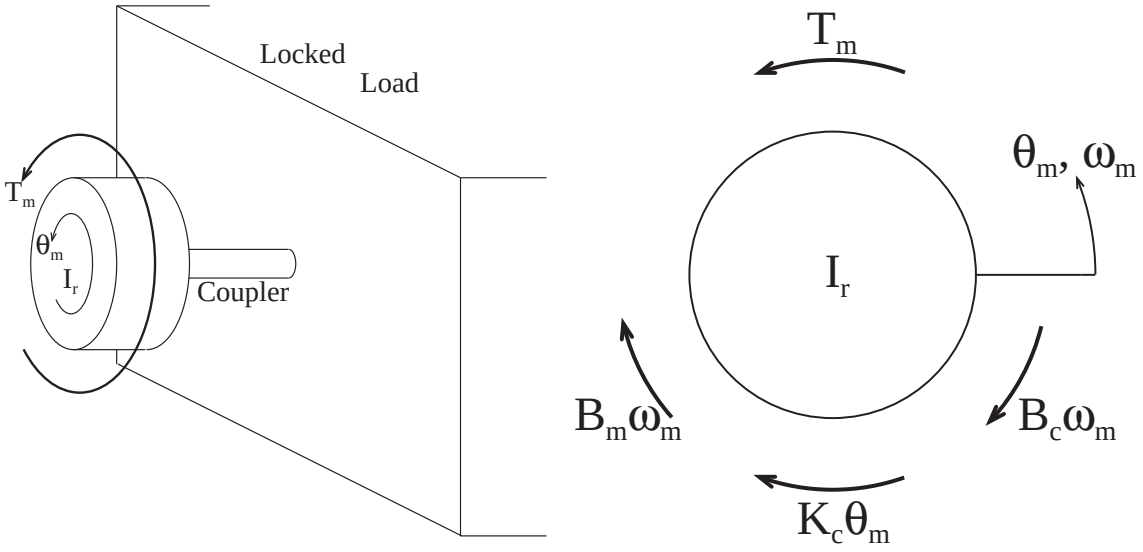


Figure 14.1.1: Motor Rotor connected to Locked Load by Elastic Coupler

(a) The fundamental requirements are:
Geometric compatibility: $\omega_m = d\theta_m/dt$
Constitutive equations: $T_I = I_r d\omega_m/dt, \quad T_f = (B_m + B_c)\omega_m, \quad T_k = K_m\theta_m$
Torque balance: $T_I = T_m - T_f - T_k$
Take θ_m and ω_m as state variables. Then one equation is provided by the geometric compatibility requirement. The second equation is obtained by substituting the constitutive equations into the torque-balance requirement to get

$$I_r \frac{d\omega_m}{dt} = T_m - (B_m + B_c)\omega_m - K_c\theta_m$$

With T_m as the input and θ_m as the desired output, the standard form for a state-determined system

$$\begin{aligned} \tfrac{d}{dt}\mathbf{x} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} + \mathbf{Du} \end{aligned}$$

becomes

$$\frac{d}{dt}\begin{Bmatrix} \theta_m \\ \omega_m \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K_c}{I_r} & -\frac{B_m+B_c}{I_r} \end{bmatrix} \begin{Bmatrix} \theta_m \\ \omega_m \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{I_r} \end{Bmatrix} T_m$$

with output θ_m given by

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} = \{\ 1\ \ 0\ \} \left\{ \begin{array}{c} \theta_m \\ \\ \omega_m \end{array} \right\} + 0 = \theta_m$$

(b) The behavioral parameters ω_o and ζ appear in the matrix $\mathbf{A}$ of second-order systems in the following pattern

$$\mathbf{A} = \left[\begin{array}{cc} 0 & 1 \\ \\ -\omega_o^2 & -2\zeta\omega_o \end{array} \right] \tag{1}$$

By comparing the matrix $\mathbf{A}$ in the equation for the locked-load response with (1) we identify the the following relationships for the behavioral parameters

$$\omega_o^2 = \frac{K_c}{I_r} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{B_m + B_c}{I_r}$$

which lead to

$$\omega_o = \sqrt{\frac{K_c}{I_r}} = \sqrt{\frac{1.24\text{e-}2}{5\text{e-}5}} = 15.75 \text{ rad/sec} \quad \text{and} \quad \zeta = \frac{B_m + B_c}{2\sqrt{K_c I_r}} = \frac{1\text{e-}4 + 2\text{e-}5}{2\sqrt{(1.24\text{e-}2)(5\text{e-}5)}} = 0.0762$$

(i) The damped natural frequency is

$$\omega_d = \omega_o\sqrt{1-\zeta^2} = 15.75\sqrt{1-(0.0762)^2} = 15.70 \text{ rad/sec}$$

(ii) The decrement ratio for a damped sinusoidal oscillation, as defined in the Notes for Lecture 6, is the ratio of the amplitudes of two successive peaks. The time increment between two such peaks is half of the period of the damped oscillation. The envelope of the damped oscillation decays in proportion to $\exp(-\zeta\omega_o t)$, so with $t = \frac{1}{2}T_d = \pi/\omega_d$ the decrement ratio is

$$\text{dec ratio} = \exp(-\zeta\omega_o\pi/\omega_d) = \exp(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}) = \exp(-\frac{0.0762\pi}{\sqrt{1-(0.0762)^2}}) = 0.787$$

(c) The dimensions of G_1 are [torque/angular velocity]. Its units are N-m/r/s or N-m-s/r in the SI system of units. When the control algorithm (with $\omega_{sensed} = \omega_m$),

$$T_M = G_1(r_1 - \omega_m),$$

is substituted in the state equations of the uncontrolled locked-load system, the resulting equations for the velocity-controlled system are

$$\frac{d}{dt}\left\{ \begin{array}{c} \theta_m \\ \\ \omega_m \end{array} \right\} = \left[\begin{array}{cc} 0 & 1 \\ \\ -\frac{K_c}{I_r} & -\frac{B_m+B_c+G_1}{I_r} \end{array} \right] \left\{ \begin{array}{c} \theta_m \\ \\ \omega_m \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ \\ \frac{G_1}{I_r} \end{array} \right\} r_1$$

The input is changed from the motor torque T_m to the reference speed r_1, and the Matrices $\mathbf{A}$ and $\mathbf{B}$ now depend on the gain G_1. The undamped natural frequency ω_o remains unchanged but the damping ratio now depends on the gain G_1. By comparing the matrix $\mathbf{A}$ with (1) we identify

$$2\zeta\omega_o = \frac{B_m + B_c + G_1}{I_r}$$

To obtain the value of G_1 required to yield a critical;ly damped response, set $\zeta = 1$ and solve for G_1

$$G_1 = 2I_r\omega_o - B_m - B_c = 2(5\text{e-}5)(15.75) - 1\text{e-}4 - 2\text{e-}5 = 14.55\text{e-}4 \text{ N-m-s/r}$$

(d) The dimensions of G_2 are [torque/angle]. Its units are N-m in the SI system of units. When the control algorithm (with $\theta_{sensed} = \theta_m$),

$$T_M = G_2(r_2 - \theta_m),$$

is substituted in the state equations of the uncontrolled locked-load system, the resulting equations for the position-controlled system are

$$\frac{d}{dt}\left\{\begin{array}{c}\theta_m\\ \omega_m\end{array}\right\} = \left[\begin{array}{cc}0 & 1\\ -\frac{K_c+G_2}{I_r} & -\frac{B_m+B_c}{I_r}\end{array}\right]\left\{\begin{array}{c}\theta_m\\ \omega_m\end{array}\right\} + \left\{\begin{array}{c}0\\ \frac{G_2}{I_r}\end{array}\right\} r_2$$

By comparing the matrix $\mathbf{A}$ in this equation with (1), we identify the following relationships for the behavioral parameters

$$\omega_o^2 = \frac{K_c + G_2}{I_r} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{B_m + B_c}{I_r}$$

Both the undamped natural frequency ω_o and the damping ratio ζ now depend on the gain G_2.

(i) The undamped natural frequency is

$$\omega_o = \sqrt{\frac{K_c + G_2}{I_r}}$$

(ii) The damping ratio is

$$\zeta = \frac{B_m + B_c}{2I_r\omega_o} = \frac{B_m + B_c}{2\sqrt{(B_m + G_2)I_r}}$$

(e) The control algorithms can be implemented by ordinary mechanical elements connected between a desired-motion driver and the motor rotor.

(i) The velocity feedback control algorithm

$$T_m = G_1(r_1 - \omega_m)$$

states that a torque is applied to the rotor which is proportional to the difference in speeds between the motion driver and the rotor. A linear *friction element* connected between the motion driver and the rotor would apply this same torque if its damping coefficient were G_1.

(ii) The position feedback control algorithm

$$T_m = G_2(r_2 - \theta_m)$$

states that a torque is applied to the rotor which is proportional to the difference in angular position between the motion driver and the rotor. A linear *torsional spring* connected between the motion driver and the rotor would apply this same torque if its stiffness were G_2.

A drawback of these mechanical implementations is the necessity of constructing a *mechanical* motion driver to provide the information about the desired motion.

14.2 Motoring with a Capacitor

Figure 14.2.1 shows the free body diagram for this problem. Note: I have applied all of the torques in the positive direction.

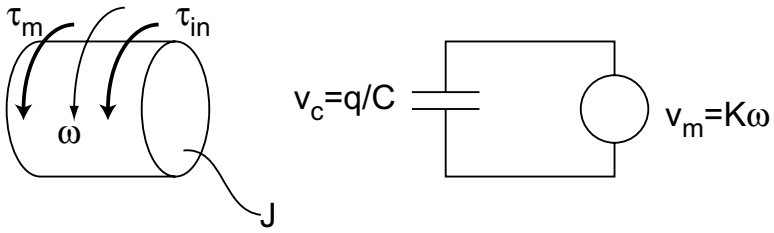


Figure 14.2.1: Free Body Diagram and Circuit Diagram

(a) Based on the free body diagram

$$\begin{aligned} J\dot{\omega} &= \tau_{in} + \tau_m \\ \tau_m &= Ki_m \\ J\dot{\omega} &= \tau_{in} + Ki_m \end{aligned}$$

(b) Evaluating the circuit results in

$$\begin{aligned} 0 = V_m + V_c \rightarrow V_m &= -V_c \\ q &= \int i_m dt \\ V_c = \frac{q}{C} \rightarrow V_c &= \frac{1}{C}\int i_m dt \\ V_m &= K\omega \\ -\frac{1}{C}\int i_m dt = K\omega \rightarrow \frac{i_m}{C} &= -K\dot{\omega} \\ i_m &= -KC\dot{\omega} \end{aligned}$$

(c)

$$\begin{aligned} J\dot{\omega} &= \tau_{in} - K^2 C\dot{\omega} \\ J + K^2 C \;\; \dot{\omega} &= \tau_{in} \end{aligned}$$

(d) The equivalent mechanical model would be $J_{eq}\dot{\omega} = \tau_{in}$ where $J_{eq} = J + R^2C$. This is true because the capacitor acts to increase the apparent inertia of the motor rotor.

(e)

$$\begin{aligned}
\mathcal{L}\left\{\left(J+K^2C\right)\dot{\omega}\right\} &= \Omega(s)\left(J+K^2C\right)+\left(J+K^2C\right)\omega(0)\\
\omega(0) &= 0\\
\mathcal{L}\left\{u(t)\right\} &= \frac{1}{s}\\
\frac{\Omega(s)}{\tau(s)} &= \frac{1}{(J+K^2C)s^2}\\
\mathcal{L}^{-1}\left\{\frac{\Omega(s)}{\tau(s)}\right\} &= \frac{t}{(J+K^2C)}
\end{aligned}$$

14.3 Leadscrew with Translating Stage

When the system is moving, the lead-screw will exert a force F_1 retarding the motion of the mass. The mass will in turn exert a torque T on the lead-screw retarding it's motion. The equations of motion for the carriage and lead-screw will be

$$M\ddot{x} = F - F_1 - c_2\dot{x} - k_2 x \tag{1}$$
$$J\ddot{\theta} = T - c_1\dot{\theta} - k_1\theta \tag{2}$$

We need a relationship between x and θ. From the definition of the pitch p, each revolution of the lead-screw will advance the carriage 10^{-2} [m]. One revolution is 2π [rad], therefore

$$x = \frac{p}{2\pi}\theta = N\theta \text{ where } N = \frac{p}{2\pi} \tag{3}$$

Because the lead-screw is said to be ideal (no friction present), there is 100% efficiency in power conversion from the linear to the rotational domain. Therefore

$$F_1 x = T\theta$$
$$F_1 = \frac{T}{N} \tag{4}$$

Take the Laplace transforms of equations (1) and (2) to get

$$F(s) - F_1(s) = \left(Ms^2 + c_2 s + k_2\right) X(s) \tag{5}$$
$$T(s) = \left(Js^2 + c_1 s + k_1\right) \theta(s) \tag{6}$$

Combine equations (3) through (6) and solve for the transfer function

$$F(s) - \left(Js^2 + c_1 s + k_1\right) \frac{1}{N^2} X(s) = \left(Ms^2 + c_2 s + k_2\right) X(s)$$
$$\frac{X(s)}{F(s)} = \frac{1}{\left(m + \frac{J}{N^2}\right) s^2 + \left(c_2 + \frac{c_1}{N^2}\right) s + \left(k_2 + \frac{k_1}{N^2}\right)}$$

14.4 Mass Pulley System

(a) As the pulley rotates clockwise, we assume a no-slip interface between the string and the pulley, and therefore a length of string equal to $R\theta$ must be "unwound" on the right hand side. The is clearly the distance the pulley moves upward, so $x_2 = R\theta$.

(b) Again we assume a no-slip interface and look at a rotation of θ radians. We already established both that the center of the pulley moves upward a distance $R\theta$ and that the string length increases on the right hand side by a distance $R\theta$. That means the string length on the left hand side has decreased by $R\theta$. The position of the mass with respect to the center of the pulley is shorter (further upward) by $R\theta$, and the center of the pulley has moved upward itself by $R\theta$, so $x_1 = 2R\theta = 2x_2$.

(c) We have already derived geometric equations which show (together) that all three variables (x_1,x_2, and θ) must be linearly coupled at all time. This means we can describe the dynamics of the system with a single equation of motion. Applying apply Newtons law at each inertia and mass (there are no other nodes):

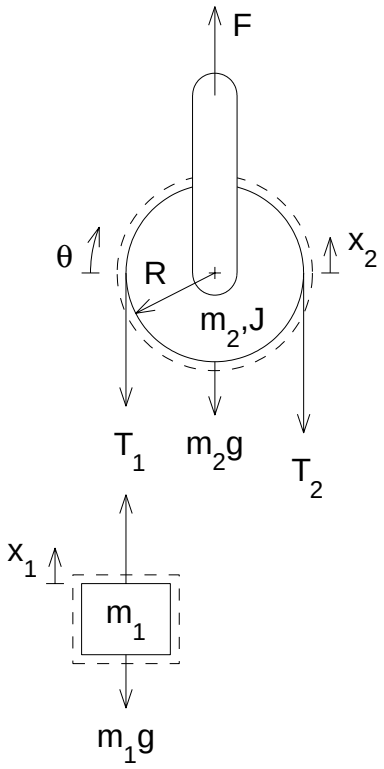


Figure 14.4.1: Pulley Free Body Diagrams

$$
\begin{aligned}
m_1\ddot{x}_1 &= T_1 - m_1 g \\
m_2\ddot{x}_2 &= F - T_1 - T_2 - m_2 g \\
J\ddot{\theta} &= (T_2 - T_1)R
\end{aligned}
$$

From the force balance at x_1 and the geometric relation between x_1 and x_2, we can get an expression for T_1 in term of x_2: $T_1 = 2m_1\ddot{x}_2 + m_1 g$. Using this new expression, the torque balance, and the geometric relation between x_2 and θ, we can write T_2 as:

$$T_2 = T_1 + \frac{J}{R^2}\ddot{x}_2 = m_1 g + \left(2m_1 + \frac{J}{R^2}\right)\ddot{x}_2$$

Combining, we get:

$$\ddot{x}_2 = \frac{F - (2m_1 + m_2)g}{4m_1 + m_2 + \frac{J}{R^2}}$$

The result is (of course) simply a constant acceleration (i.e. no damping or spring effect anywhere).

14.5 DC Motors

(a)

$$\begin{aligned} V_m &= K_m\Omega \\ V_m &= i_m R \rightarrow i_m = \frac{K_m}{R}\Omega \\ T &= K_m i_m \text{ Note: in this case the current leaving the motor thus negative} \\ T &= -\frac{K_m^2}{R}\Omega \end{aligned}$$

(b)

$$\begin{aligned} T_1 &= K_{m1} I \rightarrow I = \frac{T_1}{K_{m1}} \\ T_2 &= K_{m2} I \rightarrow T_2 = \frac{K_{m2}}{K_{m1}} T_1 \\ V &= K_{m1}\Omega_1 = IR + K_{m2}\Omega_2 \\ \Omega_2 &= \frac{1}{K_{m2}}(K_{m1}\Omega_1 - IR) \\ &= \frac{1}{K_{m2}}(K_{m1}\Omega_1 - R\frac{T_1}{K_{m1}}) = \frac{1}{K_{m2}}(K_{m1}\Omega_1 - R\frac{T_2}{K_{m2}}) \end{aligned}$$

(c) This system is just a transmission with losses. In the case of the motors, the system outputs equivalent torques and shows losses in the motor velocities. In most mechanical transmissions, the velocities will be equivalent with the losses showing up in the output torque.

14.6 Gear Train

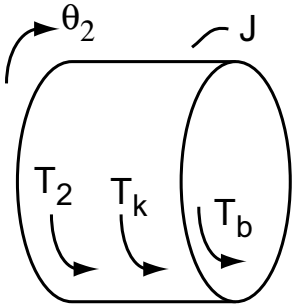


Figure 14.6.1: Freebody Diagram

Figure 14.6.1 shows the freebody diagram for the inertia of the system. For this problem we can ignore all of the dynamics of the transformer except for the actual transformer. T_2 is the transformed input torque T_1. In my case I have drawn T_2 such that $T_2 = \frac{r_2}{r_1}T_1$. If you drew T_2 acting in the positive direction you would get $T_2 = -\frac{r_2}{r_1}T_1$.

(a) The equation of motion for this system is

$$\begin{aligned} T_b &= b\dot{\theta}_2 \\ T_k &= k\theta_2 \\ T_2 &= \frac{r_2}{r_1}T_1 \\ J\ddot{\theta}_2 &= -T_2 - T_b - T_k \\ J\ddot{\theta}_2 + b\dot{\theta}_2 + k\theta_2 &= -\frac{r_2}{r_1}T_1 \\ \ddot{\theta}_2 + 1.25\dot{\theta}_2 + 10\theta_2 &= -\frac{r_2}{r_1}T_1 \end{aligned}$$

To determine ζ and ω_n, we ignore the input thus

$$\begin{aligned} \omega_n &= \sqrt{10} = 3.16 \\ 2\zeta\omega_n &= 1.25 \rightarrow \zeta = 0.198 \approx 0.2 \end{aligned}$$

(b) To determine the system equation in terms of θ_1, we only need to note the characteristics of the transformer. With the directions as defined on the picture, we know that $\theta_2 = -\frac{r_1}{r_2}\theta_1$ (Note: this is the inverse of the torque relationship). Since r_1 and r_2 are constants all the derivatives of θ_1 and θ_2 share the same relationship. Thus the characteristic equation becomes

$$\begin{aligned} -\frac{r_1}{r_2}(J\ddot{\theta}_1 + b\dot{\theta}_1 + k\theta_1) &= -\frac{r_2}{r_1}T_1 \\ (\ddot{\theta}_1 + 1.25\dot{\theta}_1 + 10\theta_1) &= \left(\frac{r_2}{r_1}\right)^2 T_1 \end{aligned}$$

You end up with the same coefficients in the characteristic equation, thus ζ and ω_n are the same as part a.

(c) Figure 14.6.2 shows a sketch of the time response for $\theta_1(t)$ and $\theta_2(t)$ (for my sketch $r_2/r_1 = 2$). Note the two responses are in opposite directions with the same phase and frequency. The magnitudes are different by r_2/r_1.

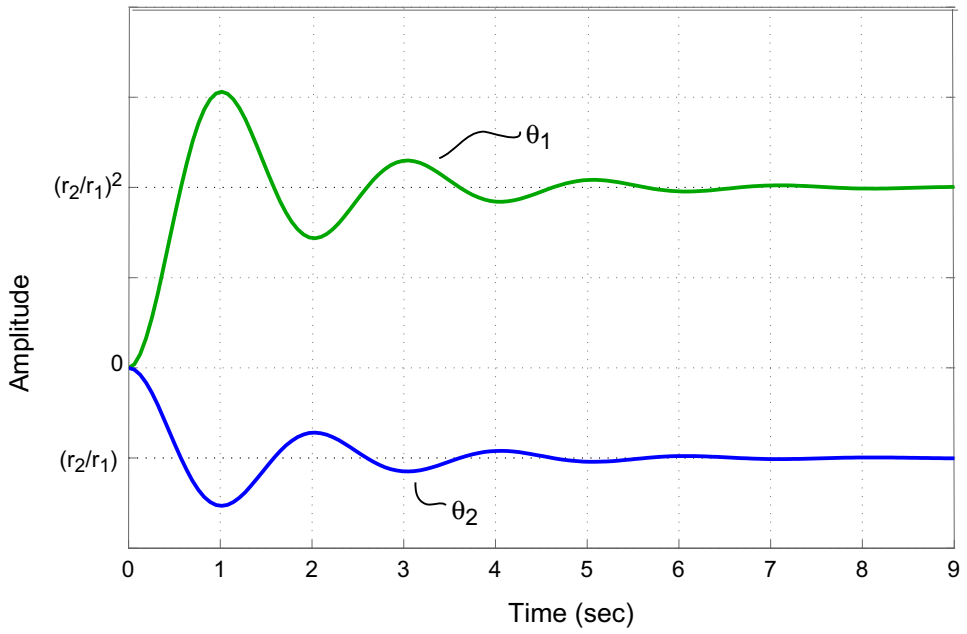


Figure 14.6.2: Sketch of $\theta_1(t)$ and $\theta_2(t)$

(d) Since the system only contains dynamic elements on one side of the transformer, the transformer only effects the magnitude and sign of the system dynamics. If like in problem 1, the system contained dynamic elements on both sides of the transformer, the transformer would effect the dynamics of the system by scaling both the magnitude of the response and the equivalent values of the inertia, damping, and stiffness (see problem 1 where the equivalent inertia of the transformer is $J_1 + \frac{r_1}{r_2} J_2$).

14.7 Non-Ideal Transformer

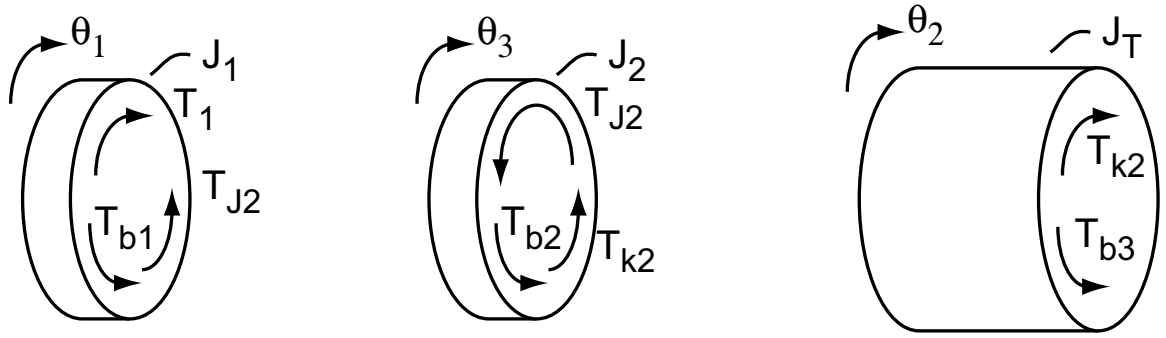


Figure 14.7.1: Freebody Diagrams

To solve this problem we need to make a few assumptions about the model elements. First, we need to consider the springs connecting the inertia elements in the model. We are told to assume that k_1 is very large, thus the input torque T_1 is assumed to act directly upon J_1. The next question we need consider is what to do with the damping caused by the bearings. In the case of the bearing in the upper left of the picture, the losses associated with this bearing clearly act directly between ground ($\dot{\theta} = 0$) and the inertia J_1. Less clear is how to handle the bearing in the center of the picture. Our first instinct would be to set this bearing to act between J_2 and J_L. We can see that this would be incorrect since the bearing drag would go to 0 when $\dot{\theta}_2 = \dot{\theta}_3$ ($\dot{\theta}_3$ is the rotational velocity of J_2). This is clearly incorrect since the bearing still generates a drag between the shaft and ground. There are three different assumptions we can make instead. One, we can have the bearing drag act between J_2 and ground. Two, we could instead have the bearing drag act between J_L and ground. Lastly, we could split the drag and have it act between ground and J_L and J_2. All are equally valid but I will solve the problem by splitting the drag between the two inertia.

(a) Figure 14.7.1 shows the free body diagrams for the three inertias in the model. Note the inclusion of the torque T_{J2} on inertias J_1 and J_2. This torque represents the torque through the gear transformer and will be solved algebraically later. All the other torques are expressed as follows (T_1 is an input thus not accounted for here):

$$\begin{aligned} T_{b1} &= b_1\dot{\theta}_1 \\ T_{b2} &= b_2\dot{\theta}_3 \\ T_{k2} &= k_2(\theta_2 - \theta_3) \\ T_{b3} &= b_3\dot{\theta}_2 \end{aligned}$$

The equations of motion then become

$$\begin{aligned} J\ddot{\theta} &= \sum T \\ J_1\ddot{\theta}_1 &= T_1 - T_{b1} - T_{J2} \\ J_1\ddot{\theta}_1 + b_1\dot{\theta}_1 &= T_1 - T_{J2} \\ J_2\ddot{\theta}_3 &= -T_{J2} - T_{b2} - T_{k2} \\ J_2\ddot{\theta}_3 + b_2\dot{\theta}_3 + k_2\theta_3 &= k_2\theta_2 - T_{J2} \\ J_L\ddot{\theta}_2 &= T_{k2} - T_{b3} \\ J_L\ddot{\theta}_2 + b_3\dot{\theta}_2 + k_2\theta_2 &= k_2\theta_3 \end{aligned}$$

(b) Now that we have written the equations of motion, we need to eliminate the unknown torque T_{J2}. To do this we apply our knowledge about the transformer which equate $\theta_1 and \theta_3$. Specifically:

$$\theta_3 = -\frac{r_1}{r_2}\theta_1$$

where r_1 is the radius of J_1 and r_2 is the radius of J_2.

$$\begin{aligned} J_1\ddot{\theta}_1 + b_1\dot{\theta}_1 &= T_1 - T_{J2} \\ J_2\ddot{\theta}_3 + b_2\dot{\theta}_3 + k_2\theta_3 &= k_2\theta_2 - T_{J2} \\ \frac{r_1}{r_2}(J_2\ddot{\theta}_1 + b_2\dot{\theta}_1 + k_2\theta_1) &= -k_2\theta_2 + T_{J2} \\ \left(J_1 + \frac{r_1}{r_2}J_2\right)\ddot{\theta}_1 + \left(b_1 + \frac{r_1}{r_2}b_2\right)\dot{\theta}_1 + \frac{r_1}{r_2}\theta_1 &= T_1 - k_2\theta_2 \\ J_L\ddot{\theta}_2 + b_3\dot{\theta}_2 + k_2\theta_2 &= k_2\theta_3 = -k_2\frac{r_1}{r_2}\theta_1 \end{aligned}$$

We can determine the transfer function by taking the Laplace transforms of remaining two equations of motion.

$$\begin{gathered} \Theta_1(s)\left[\left(J_1 + \tfrac{r_1}{r_2}J_2\right)s^2 + \left(b_1 + \tfrac{r_1}{r_2}b_2\right)s + \tfrac{r_1}{r_2}\right] = T_1(s) - k_2\Theta_2(s) \\ \Theta_1(s) = \frac{T_1(s) - k_s\Theta_2(s)}{\left(J_1 + \frac{r_1}{r_2}J_2\right)s^2 + \left(b_1 + \frac{r_1}{r_2}b_2\right)s + \frac{r_1}{r_2}k_2} \\ \Theta_2(s)\left[J_L s^2 + b_3 s + k_2\right] = -k_2\tfrac{r_1}{r_2}\Theta_1(s) \\ \Theta_2(s) = \frac{-k_2\frac{r_1}{r_2}\Theta_1(s)}{J_L s^2 + b_3 s + k_2} \end{gathered}$$

For simplicity, I am going to define the following transfer functions:

$$\begin{aligned} G_1(s) &= \frac{1}{\left(J_1 + \frac{r_1}{r_2}J_2\right)s^2 + \left(b_1 + \frac{r_1}{r_2}b_2\right)s + \frac{r_1}{r_2}k_2} \\ G_3(s) &= \frac{-k_2\frac{r_1}{r_2}}{J_L s^2 + b_3 s + k_2} \end{aligned}$$

(c) Figure 14.7.2 shows the block diagram for this system with the transfer functions as defined above.

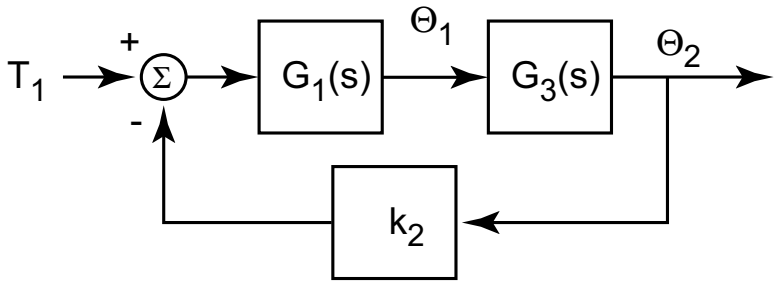


Figure 14.7.2: Block Diagram

(d) Using the block diagram, we see that

$$\begin{aligned} G(s) &= \frac{G_1(s)G_3(s)}{1 + k_2 G_1(s)G_3(s)} \\ G(s) &= \frac{-k_2 \frac{r_1}{r_2}}{[\left(J_1 + \frac{r_1}{r_2}J_2\right)s^2 + \left(b_1 + \frac{r_1}{r_2}b_2\right)s + \frac{r_1}{r_2}k_2][J_L s^2 + b_3 s + k_2] + k_2^2 \frac{r_1}{r_2}} \end{aligned}$$

14.8 Equivalent Mass and Inertia

(a) This problem is a little tricky. To solve it correctly you need to replace the mass with an equivalent force (f_m) as shown in the figure.

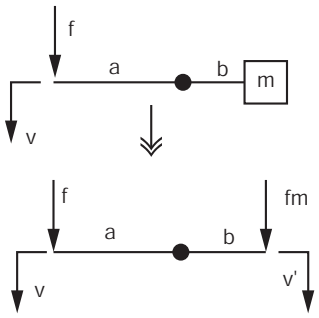


Figure 14.8.1: Free Body Diagram for 1-a

$$
\begin{aligned}
f_m &= m\dot{v}' \\
\frac{v}{a} &= -\frac{v'}{b} \\
v' &= -\frac{a}{b}v \\
f_m &= -m\frac{a}{b}\dot{v} \\
af &= -bf_m \\
f &= m\frac{a^2}{b^2}\dot{v} \\
m_{eq} &= m\frac{a^2}{b^2}
\end{aligned}
$$

(b) This problem should be solved using a free-body diagram. You should note that all of the masses travel at the same velocity, thus you only need a free body diagrams for one wheel and the combined masses.

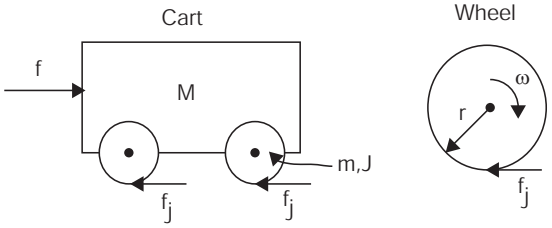


Figure 14.8.2: Free Body Diagram for 1-b

Note the force f_j, this force is not a friction force, although it is created by the friction between the tire and road surface, it is the force required to accelerate the rotational inertia

of the wheel. From the figure,

$$\begin{aligned}
\text{On the cart } \Sigma F &= (M+4m)\ddot{x} = f - 4f_j \\
\text{On one wheel } \Sigma T &= J\dot{\omega} = f_j r \\
\text{No slip means } v &= \dot{x} = r\omega \Rightarrow \dot{\omega} = \frac{\ddot{x}}{r} \\
f_j &= \frac{J}{r^2}\ddot{x} \\
\left(M + 4m + 4\frac{J}{r^2}\right)\ddot{x} &= f \\
M_{eq} &= M + 4m + 4\frac{J}{r^2}
\end{aligned}$$

(c) Once again you should be solving this problem using free body diagrams. Note that the first pulley does not have any inertia. From the FBD,

$$\begin{aligned}
\text{Pulley } \#1\ \Sigma T &= 0 = r1(f_1 - f_2) + T \\
\text{Pulley } \#2\ \Sigma T &= J\dot{\omega'}
\end{aligned}$$

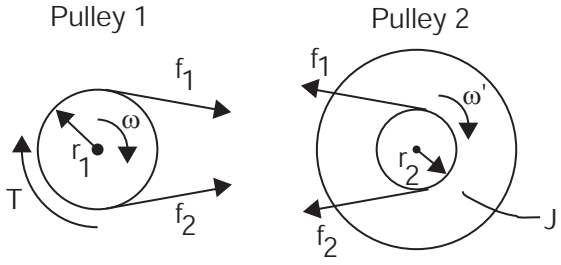


Figure 14.8.3: Free Body Diagram for 1-c

14.9 Inertia in Geartrain

The solution for this problem is not available.

14.10 Motor Driving Inertia Through Gear Train

The solution for this problem is not available.

15 MATLAB and Simulink

15.1 Matlab Spirograph

(a) and (b) A gear of radius r_2 turns inside a gear-faced hole of radius r_0. The inner gear has a pen fixed at a radius r_3 from its center. The resulting path of the pen is most easily described as that of the center of the gear (which just moves in a circular path) superimposed with the position of the pen with respect to the center of the gear. The angle between the fixed center of the outer hole and the center of the gear at any given time is θ_1, while the corresponding angle from the center of the gear to the pen is θ_3. θ_3 and θ_1 are proportional to one another. Figure 15.1.1 shows the geometry of the gears (above) and the resulting path (below):

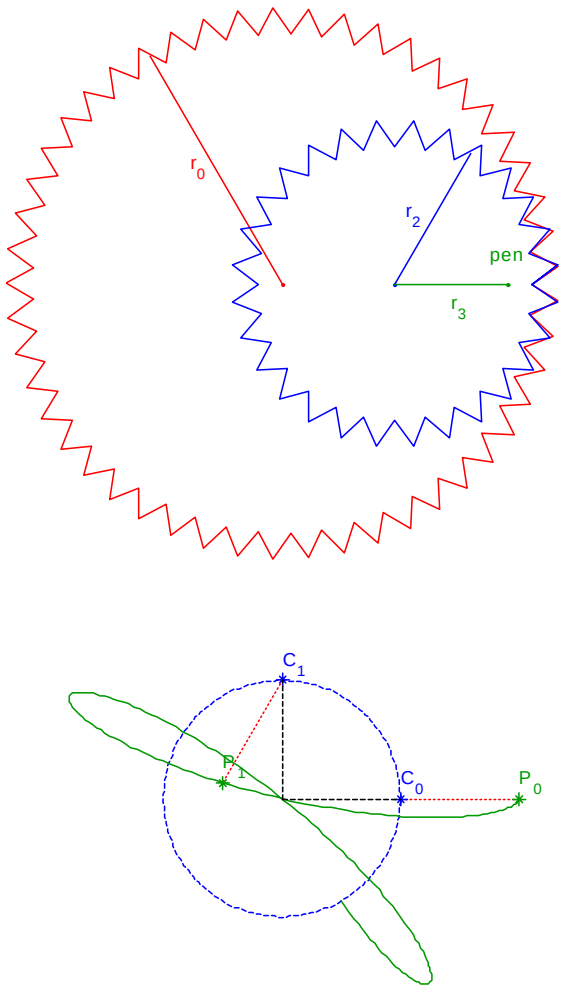


Figure 15.1.1: Geometry of gears and resulting paths

Above, θ_1 is shown as a dashed line for two locations, 0 and 90 degrees. Here, $r_0 = 7$, $r_2 = 4$, and $r_3 = 3$, so that $\theta_2 = (-4/3)\theta_1$, shown as the lines connecting gear center, c, with pen location, p, for each of two locations of the inner gear. The figure below overlays the gear geometry with the resulting spirograph:

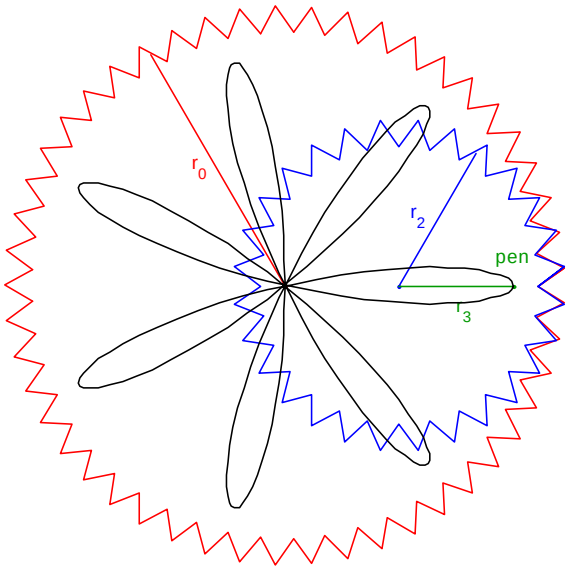


Figure 15.1.2: Overlay of geometry of gears and resulting spirograph

(c) The script `spirographsol.m` is avaliable at the end of the problem.

(d) Your plots will vary.

(c) You can convert the MATLAB script given into a MATLAB function by completing the following 2 steps:

1) change the TOP LINE of the m-file into a function declaration (beginning with the word function)
2) take out the lines in the m-file that defined the various radii (r0, r2 and r3)

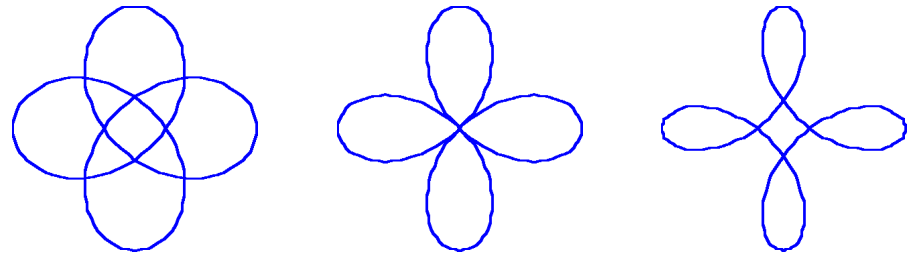
The m-file `spisol.m` below defines a spirographing function called spisol this file is found at the end of the problem. Note that the function has no output. (It will still plot the picture, however.) If we wished to define the vector of complex numbers describing the path as the output of the function, we could rewrite the top line of the function above as:

```
function [position]=spisol(r0,r2,r3)
```

The rest of the m-file would remain the same. To plot the example shown earlier, you would call the function as shown in Figure 15.1.1:

```
spi(7,4,3)
```

```
subplot(131); spi(4,3,2/3); axis off
subplot(132); spi(4,3,1); axis off
subplot(133); spi(4,3,4/3); axis off
```



```
subplot(131); spi(9,4,3); axis off
subplot(132); spi(9,5,4); axis off
subplot(133); spi(9,5,3); axis off
```

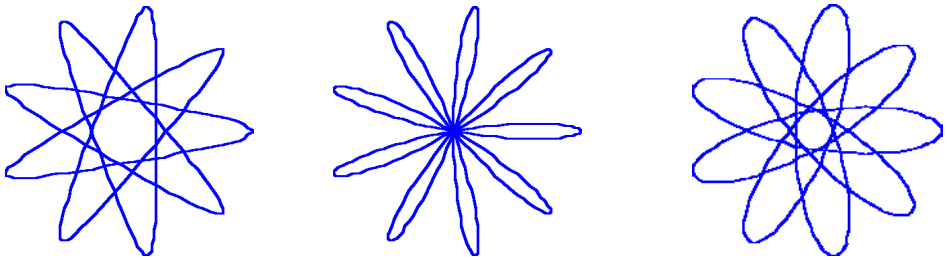


Figure 15.1.3: Spisol.m call and resulting plot

15.2 MATLAB Plotting

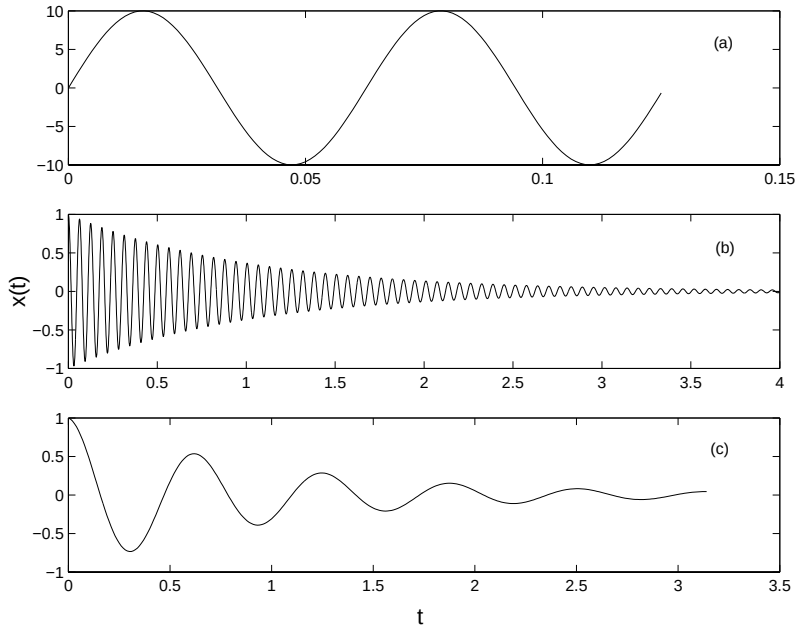


Figure 2: Problem 5: (a) $10\sin 100t$, (b) $e^{-t}\cos 100t$, (c) $Re\{e^{st}\}$; $s = -1 + 10i$.

Problem 5: The objective of this problem is to recognize the time scales involved so that proper time range and sampling rates may be selected. For sinusoidal functions or exponential functions with a complex exponent of the form $ae^{i\omega t}$, the period is given by

$$T = \frac{2\pi}{|\omega|} \tag{36}$$

For decaying exponential functions of the form ae^{-bt} the time constant is given by

$$\tau = \frac{1}{b} \tag{37}$$

The function will be within 1% of its final value in 5τ. Therefore, one must recognize the slowest timescale when plotting a function so that all significant behavior will be shown. The graphs are shown in Figures 2 and 3.

Figure 15.2.1:

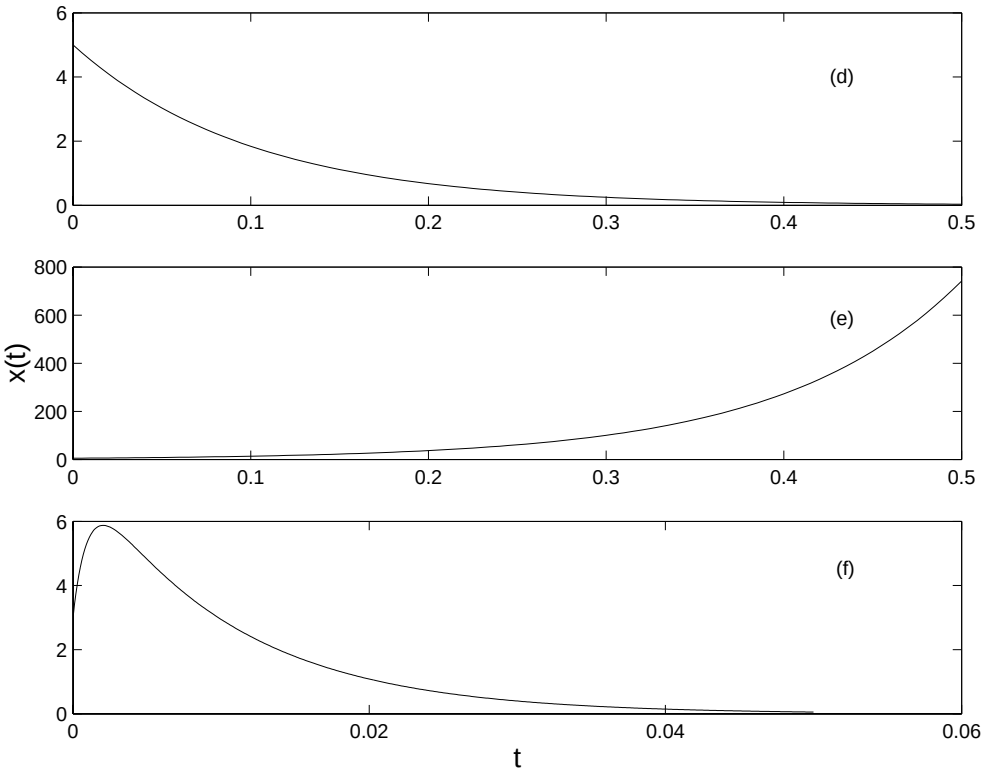


Figure 3: Problem 5: (d) $5e^{-10t}$, (e) $5e^{10t}$, (f) $8e^{-100t} - 5e^{-1000t}$.

Figure 15.2.2:

15.3 Simulink Introduction

The solution for this problem is not available.

15.4 Wackygraph

(a) The m-file wackygraph.m creates a strange looking graph with many loops (see Figure 15.4.1). We are interested in understanding the effects of each parameter on the shape of the graph. Begin by examining the function being plotted:

$$position = (r_0 + r_1)e^{j\theta_1} + r_2 e^{j\theta_3}$$

Recall that a complex exponential is a circle when plotted in the the complex plane. r_0 and r_2 are constants, but r_1 is a function of θ_1. Substituting relations given in the m-file, we obtain

$$position = 2r_0 e^{j\theta_1} + g_2 \sin(N\theta_1)e^{j\theta_1} + r_2 e^{-jg_1\theta_1} \quad (1)$$

Therefore, we recognize that the diameter of the main figure is $2r_0$. This circle of main diameter is modulated by two additional terms, $g_2 \sin(N\theta_1)e^{j\theta_1}$ and $r_2 e^{-jg_1\theta_1}$. $g_2 sin(N\theta_1)e^{j\theta_1}$ has the same angle θ_1 as the main circle but the modulation changes as the angle changes. For $N > 1$, the size of the perturbation changes faster than θ_1. The other perturbation $r_2 e^{-jg_1\theta_1}$ has a primary radius r_2. The perturbation rotates in the opposite direction from the direction of tracing out the large circle of radius r_0 and with a different speed based on the magnitude of g_1 (see Figure 15.4.2 for a diagram explaining the variables). r_1 is a dependent variable because it is calculated in the m-file.

(b) Using Equation (1) we can separate position into its real and imaginary parts using $z = re^{j\theta} = r\cos\theta + jr\sin\theta$. We find the real part is

$$Re(position) = 2r_0 \cos\theta_1 + g_2 \sin(N\theta_1)\sin\theta_1 + r_2 \cos(g_1\theta_1)$$

and the imaginary part is

$$Im(position) = 2r_0 sin\theta_1 + g_2 sin(N\theta_1) sin\theta_1 - r_2 sin(g_1\theta_1)$$

(c) To comment a MATLAB file, use the % symbol before the comments. A commented version of wackygraph.m is given below.

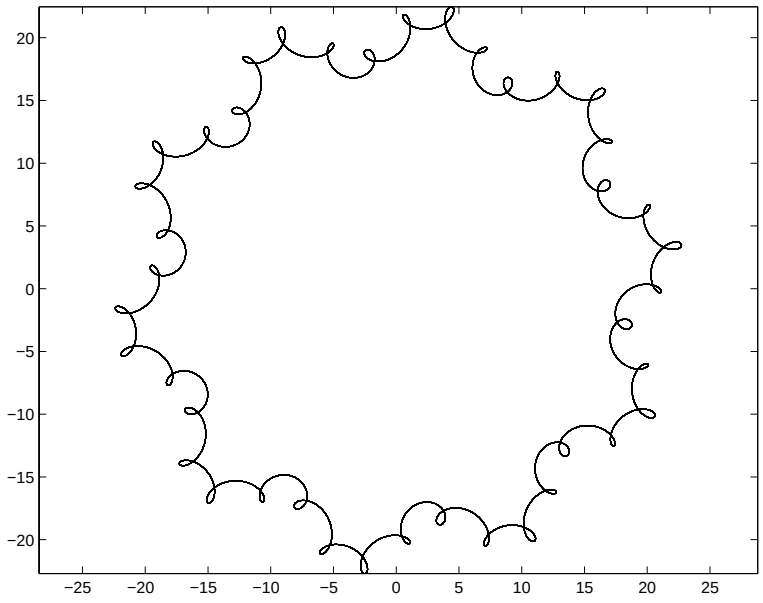


Figure 15.4.1: Wackygraph with default parameters.

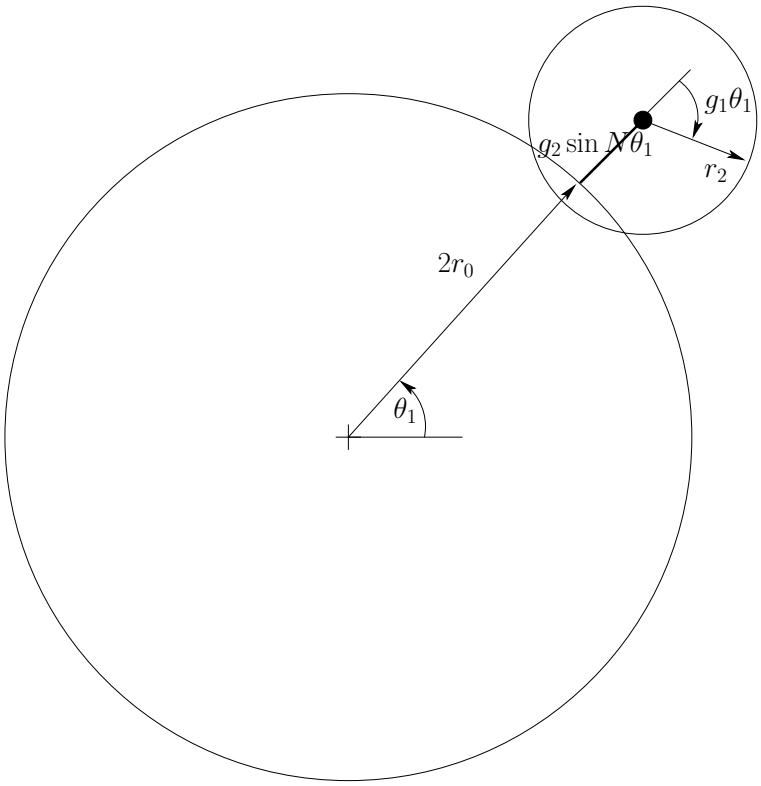


Figure 15.4.2: Diagram of the Wackygraph system.

```
% wackygraph %

DL Trumper 9/9/02

r0 = 10; %define r0
r2 = 1; %define r2
N = 10; %define N
g1 = 40; %define g1
g2 = 0.2*r0; %define g2
theta1 = [0:pi/1000:100*pi]; %define the vector theta1
theta3 = -g1*theta1; %define the vector theta3 as proportional to theta1
r1 = r0 + g2*sin(N*theta1); %calculate the vector r1
position = (r0+r1).*exp(j*theta1) + r2*exp(j*theta3); %calculate position
figure(1) %select the window Figure 1
plot(position) %plot the variable position
axis equal %force the x and y axes to be scaled equally
```

We must use .* instead of . for element-by-element multiplication. Otherwise, Matlab returns the following error:

```
??? Error using ==> *
Inner matrix dimensions must agree.
```

(d) Various plots are given in Figure 15.4.3.

(e) A possible script is given below

```
function wackyscript(r0, r2, g1, g2, N)
%wackyscript(r0, r2, g1, g2, N)
%Justin Verdirame
%2.003 Problem Set 2
%Fall 2002
%
%This function takes as input r0, r2, g1, g2, N and outputs
%a plot based on wackygraph.m written by Prof Trumper.
theta1 = [0:pi/1000:100*pi]; %define vector
theta3 = -g1*theta1; %calculate vector theta3
r1 = r0 + g2*sin(N*theta1); %calculate vector r1
position = (r0+r1).*exp(j*theta1) + r2*exp(j*theta3);
figure(1) %creates window for the plot if it does not exist
plot(position) %plot vector
axis equal %forces the x and y axes to have the same scaling
```

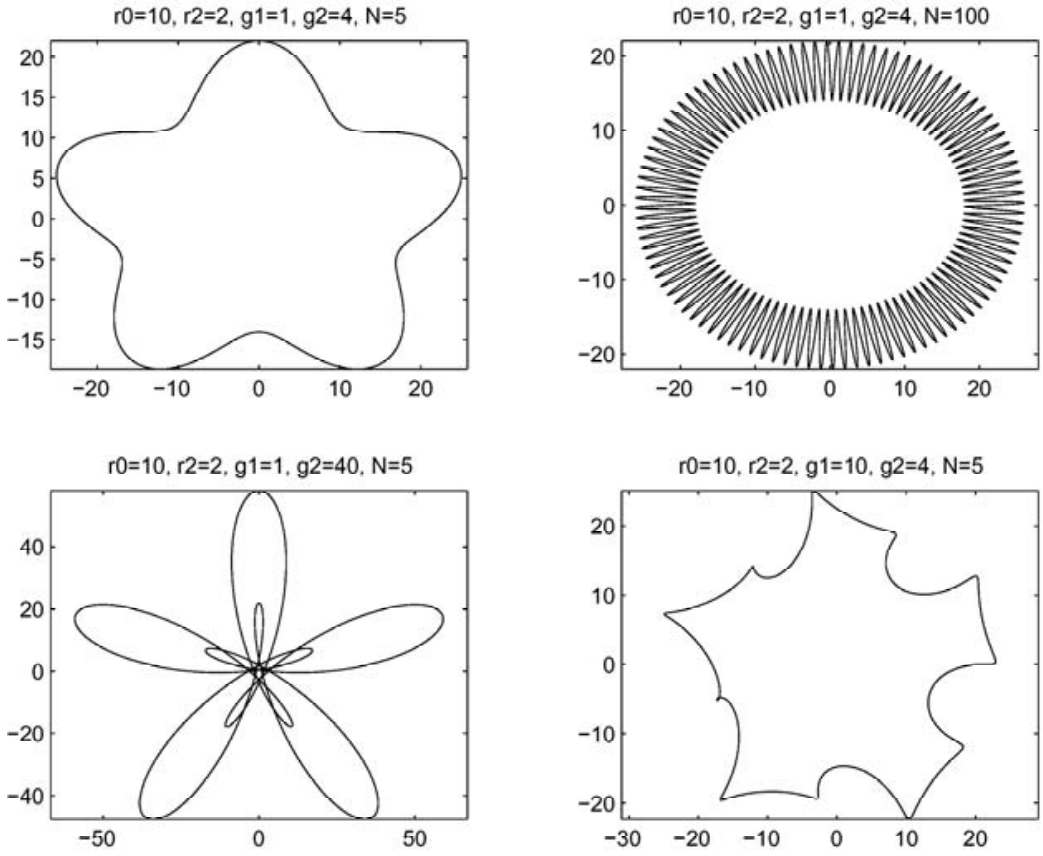


Figure 15.4.3: Wackygraph plotted with various parameters.

16 Case Studies

16.1 CD Player 1

(a) Let the outer radius of the CD be R, and let the radius of the hole be r. Let the mass density of the disc be ρ per unit area, and let the total mass of the CD be m. Then

$$m = \rho(\pi R^2) - \rho(\pi r^2) = \pi\rho(R^2 - r^2)$$

The moment of inertia is

$$I_{CD} = \rho(\pi R^2)(\frac{1}{2}R^2) - \rho(\pi r^2)(\frac{1}{2}r^2) = \frac{1}{2}\pi\rho(R^4 - r^4)$$

and

$$\frac{1}{2}\pi\rho(R^4 - r^4) = \frac{1}{2}\pi\rho(R^4 - r^4)\left[\frac{m}{\pi\rho(R^2 - r^2)}\right] = \frac{1}{2}m(R^2 + r^2)$$

Substitute $R = 0.060$ m, $r = 0.0075$ m, and $m = 0.028$ kg to get $I_{CD} = 5.12\text{e-}5$ kg-m^2.

(b) At steady state, the motor torque T_m is balanced by the frictional torque $T_{fric} = B_m\omega_{ss}$. If ω_{ss} is the same for the disc mounted and not mounted, the damping coefficient B_m must also be the same in both cases. In the transient, it is necessary to consider the complete dynamic analysis. The constitutive equations are

$$T_I = I\frac{d\omega_m}{dt} \qquad \text{and} \qquad T_{fric} = B_m\omega_m$$

where I is the total moment of inertia of the rotating parts, and the torque balance is

$$T_I = T_m - T_{fric}$$

which lead to the differential equation

$$I\frac{d\omega_m}{dt} + B_m\omega_m = T_m \qquad \text{or} \qquad \left(\frac{I}{B_m}\right)\frac{d\omega_m}{dt} + \omega_m = \frac{T_m}{B_m}$$

from which we learn that the decay time-constant is $\tau = I/B_m$. Now we are given the time it takes the spindle to reach 95% of ω_{ss} which is known to be 3 time-constants, for the spindle alone, and for spindle plus the mounted CD, so

$$\tau_{noCD} = \frac{I_r}{B_m} = \frac{0.3}{3} \qquad \text{and} \qquad \tau_{withCD} = \frac{I_r + I_{CD}}{B_m} = \frac{2.0}{3}$$

and, by subtraction to eliminate I_r, we find

$$\frac{I_{CD}}{B_m} = \frac{2.0 - 0.3}{3} \qquad \text{or} \qquad B_m = \frac{3I_{CD}}{1.7} = 9.04\text{e-}5 \text{ N-m-s/r}$$

This is the answer for both parts (i) and (ii).

(c) To estimate I_r, return to τ_{noCD} and solve for I_r

$$I_r = B_m\tau_{noCD} = 9.04\text{e-}5\frac{0.3}{3} = 9.04\text{e-}6 \text{ kg-m}^2$$

(d) The standard form for state-determined representation of a dynamic system is

$$\begin{aligned} \tfrac{d}{dt}\mathbf{x} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned}$$

where $\mathbf{x}$ is a column matrix of the *state* variables, $\mathbf{u}$ is a column matrix of the *input* variables, and $\mathbf{y}$ is a column matrix of the desired *output* variables. The parameters which govern the natural response of the system are contained in the square matrix $\mathbf{A}$. The parameters which describe how the input is delivered are contained in $\mathbf{B}$, the parameters which describe how the output depends on the state variables are contained in $\mathbf{C}$, and the parameters which describe how the output depends directly on the input are contained in $\mathbf{D}$.

In the present case there is only one state variable ω_m, one input T_m, and one output ω_m, so all the matrices reduce to scalars. The state equation for ω_m is

$$\frac{d\omega_m}{dt} = -\frac{B_m}{I_{total}}\omega_m + \frac{1}{I_{total}}T_m$$

where $I_{total} = I_r + I_{CD} = (0.904 + 5.12)$e-5 $= 6.02$e-5 kg-m^2. The matrices of the standard form reduce to the following scalars:

$$\mathbf{x} = \omega_m, \quad \mathbf{A} = -\frac{B_m}{I_{total}}, \quad \mathbf{u} = T_m, \quad \mathbf{B} = \frac{1}{I_{total}}, \quad \mathbf{y} = \omega_m, \quad \mathbf{C} = 1, \quad \mathbf{D} = 0$$

16.2 CD Player 2

(a) The dimensions of the parameter G are [torque]/[angular velocity], or [FLT], or [ML^2/T]. In the SI system, the common unit is the Newton-meter per radian per second, or N-m-s.

(b) Assume that $\omega_{sensed} = \omega_m$, and insert $T_m = G(r - \omega_m)$ in the state equation in Problem 16.1(d) to get

$$\frac{d\omega_m}{dt} = -\frac{B_m + G}{I_{total}}\omega_m + \frac{G}{I_{total}}r$$

as the state equation for the controlled system. Comparing this equation for the controlled system with the equation in 1(d) above, note that the input **u** has changed from a torque T_m to a speed r, and that, **A** has changed from $-B_m/I_{total}$ to $-(B_m + G)/I_{total}$, and **B** has changed from $1/I_{total}$ to G/I_{total}. The system is still a first-order dynamic system but its behavior can easily be modified by adjusting the gain G.

(c) The steady-state speed corresponding to 4 revolutions/second is $\omega_{ss} = 8\pi = 25.1$ radians/second.

(i) For the uncontrolled system, the motor torque required to maintain this speed is

$$T_m = B_m\omega_{ss} = (9.04\text{e-}5)(8\pi) = 2.27\text{e-}3 \text{ N-m}$$

(ii) For the controlled system, the reference speed required to maintain the same speed is

$$r = \frac{B_m + G}{G}\omega_{ss} = \frac{9.04\text{e-}5 + G}{G}25.1$$

(d) The decay time-constant for the uncontrolled system is $\tau_u = I_{total}/B_m$, while the decay time-constant for the controlled systgem is $\tau_c = I_{total}/(B_m + G)$. If the controlled system is to be five times faster than the uncontrolled system, it is necessary for $\tau_c = \tau_u/5$, or

$$\frac{I_{total}}{B_m + G} = \frac{1}{5}\frac{I_{total}}{B_m}$$

from which we obtain

$$G = 4B_m = 4(9.04\text{e-}5) = 36.2\text{e-}5 \text{ N-m-s}$$

(e) The scripts for this problem are found below. The symbol v in the script stands for the angular velocity ω_m. After the equation is integrated to obtain $\omega_m(t)$, the motor torque T_m is constructed from the control algorithm

$$T_m = G(r - \omega_m)$$

and plotted. See Fig. 16.2.1.

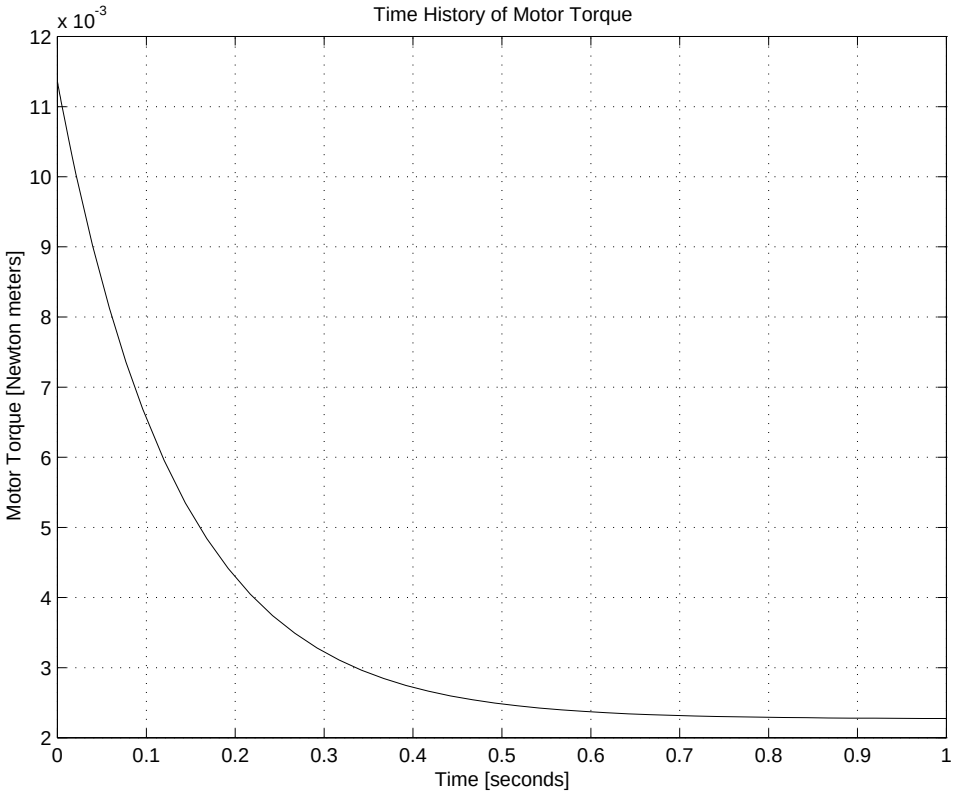


Figure 16.2.1: Transient Motor Torque under Velocity Feedback Control

Note that the maximum motor torque occurs at $t = 0$ when $\omega_m = 0$. The magnitude of the maximum torque is

$$T_m(0) = Gr = (B_m + G)\omega_{ss} = (B_m + 4B_m)\omega_{ss} = 5B_m\omega_{ss} = 11.35\text{e-}3 \text{ N-m}$$

Note that the five-fold increase in response speed is accompanied by a five-fold increase in maximum torque.

16.3 CD Player 3

The equation of motion for the CD-player is

$$I\frac{d\omega_m}{dt} + B_m\omega_m = T_m \qquad \text{or} \qquad \frac{d\omega_m}{dt} = -\frac{B_m}{I}\omega_m + \frac{1}{I}T$$

where: (i) $I = I_r$ when no disc is mounted on the spindle, or (ii) $I = I_r + I_d$ when a disc is mounted on the spindle.

(a) Using the result of Problem 16.1 with $B_m = 2B$, the magnitude $M(\Omega)$ of the angular velocity response amplitude and the phase angle $\phi(\Omega)$ for an input torque of $T_m = T_a \sin\Omega t$ are

$$M(\Omega) = \frac{T_a}{B_m}\frac{1}{\sqrt{1+(\frac{I\Omega}{B_m})^2}} \qquad \text{and} \qquad \phi(\Omega) = -\tan^{-1}\frac{I\Omega}{B_m}$$

The zero-frequency limit of the magnitude $M(\Omega)$ is

$$M(0) = \frac{T_a}{B_m}$$

Since no specific value of the input torque amplitude T_a was given, we will express the the magnitude of the response in terms of the ratio $M(\Omega)/M(0)$. For the three frequencies 1 rad/sec, 5 rad/sec, 15 rad/sec the magnitude ratios and phase angles are:

(i) for no disc on the spindle ($I = I_r$ = 9e-6e-6 kg-m^2, B_m = 9e-5 N-m/r/s),

$$\frac{M(1)}{M(0)} = \frac{1}{\sqrt{1+(\frac{(9\text{e}-6)(1)}{9\text{e}-5})^2}} = 0.995 \quad \text{and} \quad \phi = \tan^{-1}\frac{(9\text{e}-6)(1)}{9\text{e}-5} = 5.71\,\text{deg}$$

$$\frac{M(5)}{M(0)} = \frac{1}{\sqrt{1+(\frac{(9\text{e}-6)(5)}{9\text{e}-5})^2}} = 0.894 \quad \text{and} \quad \phi = \tan^{-1}\frac{(9\text{e}-6)(5)}{9\text{e}-5} = 26.6\,\text{deg}$$

$$\frac{M(15)}{M(0)} = \frac{1}{\sqrt{1+(\frac{(9\text{e}-6)(15)}{9\text{e}-5})^2}} = 0.555 \quad \text{and} \quad \phi = \tan^{-1}\frac{(9\text{e}-6)(15)}{9\text{e}-5} = 56.3\,\text{deg}$$

(ii) for a disc on the spindle ($I = I_r + I_d$ = 5.9e-5 kg-m^2, B_m = 9e-5 N-m/r/s),

$$\frac{M(1)}{M(0)} = \frac{1}{\sqrt{1+(\frac{(5.9\text{e}-5)(1)}{9\text{e}-5})^2}} = 0.647 \quad \text{and} \quad \phi = \tan^{-1}\frac{(5.9\text{e}-5)(1)}{9\text{e}-5} = 33.2\,\text{deg}$$

$$\frac{M(5)}{M(0)} = \frac{1}{\sqrt{1+(\frac{(5.9\text{e}-5)(5)}{9\text{e}-5})^2}} = 0.1671 \quad \text{and} \quad \phi = \tan^{-1}\frac{(5.9\text{e}-5)(5)}{9\text{e}-5} = 73.0\,\text{deg}$$

$$\frac{M(15)}{M(0)} = \frac{1}{\sqrt{1+(\frac{(5.9\text{e}-5)(15)}{9\text{e}-5})^2}} = 0.0564 \quad \text{and} \quad \phi = \tan^{-1}\frac{(5.9\text{e}-5)(15)}{9\text{e}-5} = 84.2\,\text{deg}$$

(b) The low-frequency asymptote for $M(\Omega)/M(0)$ is unity, and the high frequency asymptote is $M(\Omega)/M(0) = B_m/I\Omega$. These two asymptotes intersect at the break frequency $\Omega_{break} = B_m/I$. The numerical values are:

(i) Ω_{break} = 9e-5/9e-6 = 10 rad/sec when no disc is mounted on the spindle.

(ii) Ω_{break} = 9e-5/5.9e-5 = 1.525 rad/sec when a disc is mounted on the spindle.

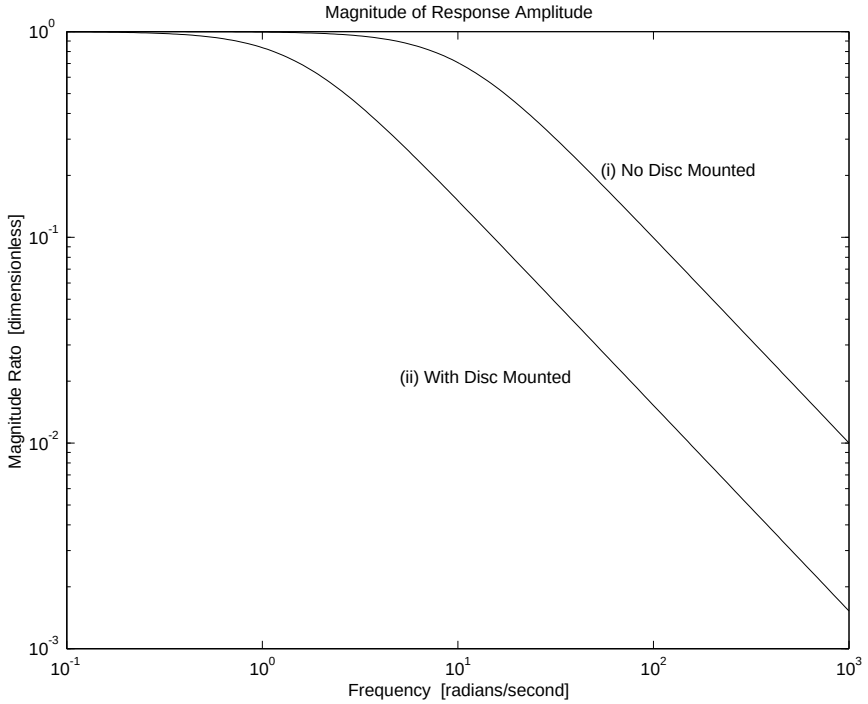


Figure 16.3.1: Magnitude Ratio for CD-Player

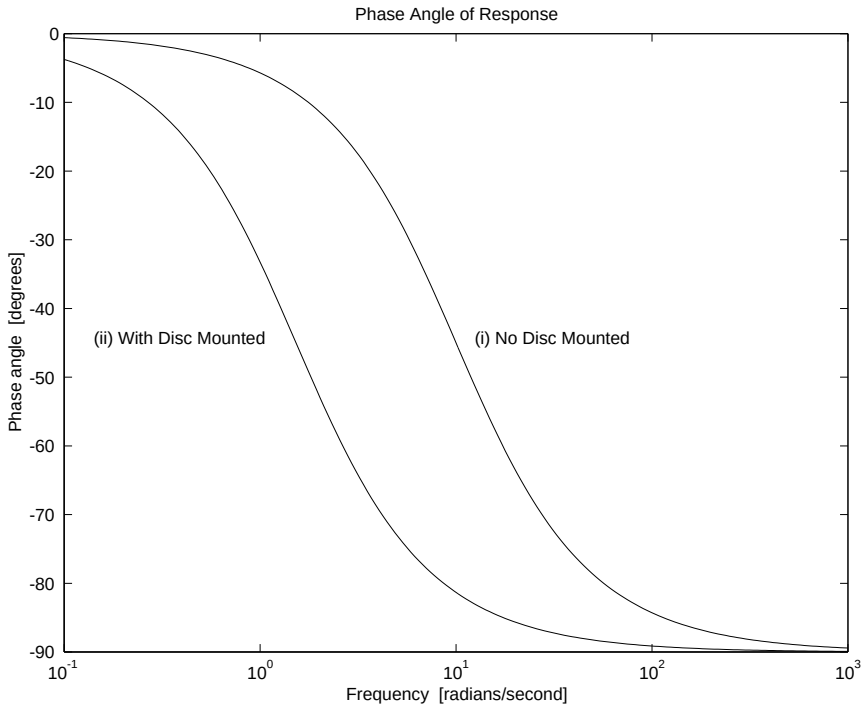


Figure 16.3.2: Phase Angle for CD-Player

(c) Bode plots for Case (i) and Case (ii) are shown in Fig. 16.3.1 for magnitude and in Fig. 16.3.2 for phase angle. These plots were obtained by running MATLAB scripts similar to the one shown in the solution to Problem 4.21.

(d) The decay time-constant for the CD-player is $\tau = I/B_m$. The time for 98% of the transient to decay is $4\tau = 4I/B_m$. This time is a natural property of the system and is independent of the frequency Ω of the applied oscillating torque T_m. This time does, however, depend on the value of I which depends on whether a disc is mounted or not.

(i) With no disc mounted the time to reach steady state is 4(9e-6)/(9e-5) = 0.4 seconds.

(ii) With a disc mounted the time to reach steady state is 4(5.9e-5)/(9e-5) = 2.62 seconds.

16.4 Pinewood Derby 1

(a) The potential energy PE of a mass m elevated a distance h in the gravity field is PE $= mgh$. In SI units the 5 ounce mass is 5/16 pounds = (5/16)(0.4536 kg/pound) = 0.1418 kg. The 3 foot elevation $h = 3(0.3048$ meters/foot$) = 0.914$ meters, and the acceleration of gravity is 9.81 m/s^2. The energy available to move the racecar is

$$mgh = (0.1418)(9.81)(0.914) = 1.271\text{kg (m/s)}^2 = 1.271\text{newton meters} = 1.271\text{Joules}$$

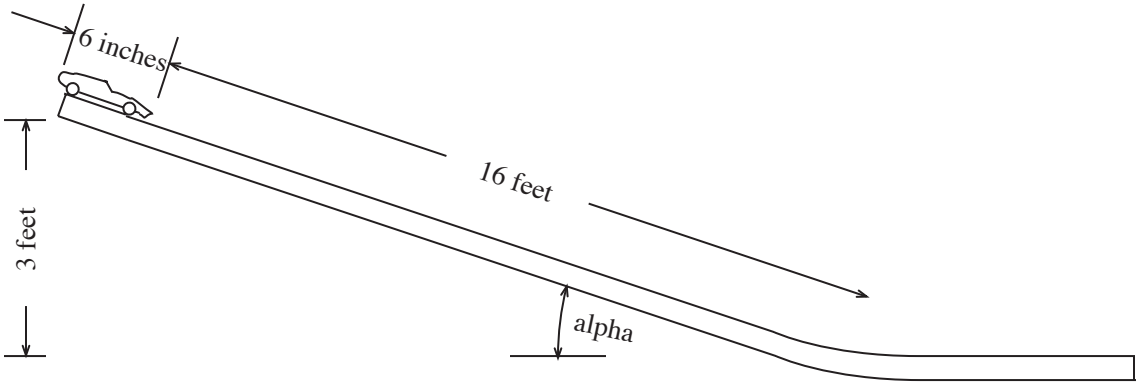


Figure 16.4.1: Pinewood Derby

(b) If the center of mass is at the front of the car, its elevation is $h = 0.914$m and the available potential energy is as calculated in (a). When the car is rolling on the level portion of the track this energy has been transformed into kinetic energy KE= $\frac{1}{2}mv^2$, so the maximum velocity is

$$v = \sqrt{2\text{PE}/m} = \sqrt{2gh} = \sqrt{2(9.81)(0.914)} = 4.24 \text{ m/s}$$

(c) If the center of mass is at the rear of the car its elevation is

$$h = 0.914 + \frac{6}{12}(0.3048)\sin(\text{alpha}) = 0.914 + \frac{6}{12}(0.3048)(\frac{3}{16}) = 0.943 \text{ m}$$

and the maximum velocity on the level portion of the track is

$$v = \sqrt{2(9.81)(0,943)} = 4.30 \text{ m/s}$$

a 1.4 % increase in speed over the case where the mass center is at the front of the car.

(d) The energy relations are silent regarding the time taken to effect the energy transformation. To introduce the time variable it is necessary to consider the equation of motion of the car. On the sloping section of track the force which accelerates the mass along the track is the component of the weight parallel to the track: $mg\sin(\text{alpha})$. With no friction, this is the only force, and the equation of motion is

$$mg\sin(\text{alpha}) = m\frac{dv}{dt}$$

Integrating this equation from the initial condition $v = 0$ at $t = 0$ yields

$$v = gt\sin(\text{alpha})$$

Let s represent distance down the track from the starting point, so that $ds/dt = v$. Then integrating the equation

$$\frac{ds}{dt} = gt\sin(\text{alpha})$$

from $s = 0$ at $t = 0$ to $s = L$ at time $t = T$ yields

$$L = \frac{1}{2}gT^2\sin(\text{alpha})$$

from which it follows that

$$T = \sqrt{\frac{2L}{g\sin(\text{alpha})}}$$

The time T to reach the bottom of the inclined track in case (c) is obtained by substituting $L = 16.5$ feet $= (16.5)(0.3048)$m $= 5.09$m and $\sin(\text{alpha}) = 3/16$ to get

$$T = \sqrt{\frac{2(5.09)}{9.81(3/16)}} = 2.35 \text{ sec}$$

16.5 Pinewood Derby 2

At the limiting velocity v_{ss}, the retarding force bv just equals the component of the weight parallel to the track, $mg\sin(\text{alpha})$, so

$$b = \frac{mg\sin(\text{alpha})}{v_{ss}}$$

If $v_{ss} = 2(4.30) = 8.60$ m/s, then

$$b = \frac{(0.1418)(9.81)(3/16)}{8.60} = 0.0303 \text{ kg/sec}$$

16.6 Pinewood Derby 3

Consider the equation of motion of a racecar with mass m on an incline of angle (alpha), acted on by the weight component parallel to the track, and by a viscous retarding force bv, as shown in Figure 16.6.1.

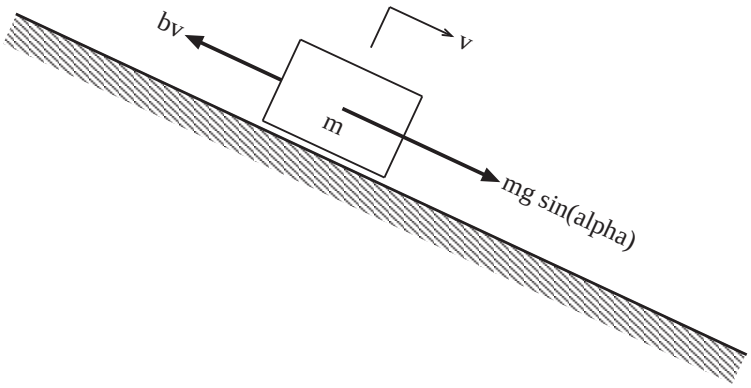


Figure 16.6.1: Racecar Dynamics Diagram

$$m\frac{dv}{dt} = mg\sin(\text{alpha}) - bv$$

or, after dividing through by m, and rearranging

$$\frac{dv}{dt} + \frac{b}{m} = g\sin(\text{alpha})$$

This is the equation treated in the MATLAB scripts 'carsol.m' and 'carviscsol.m'.

(a) For part (a) use the data $m = 0.1418$ kg, (alpha) = inv sin $3/16 = 10.81$ degrees, $b = 0.0303$ kg/sec, and $T = 2.35$ sec to get the plot for Part (a).

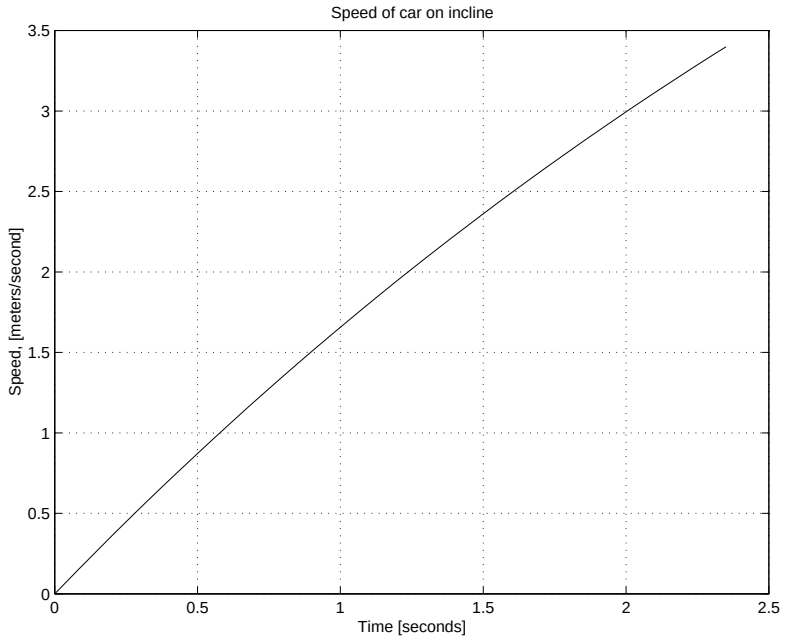


Figure 16.6.2: Plot for Part (a)

At the end of 2.35 seconds the speed of the car is 3.40 m/s. To indicate the effect of increasing the weight, you can run the program again with all the same data, except with the mass increased to 0.1701 kg (6 ounces) to get a final velocity of 3.53 m/s at the end of 2.35 seconds.

(b) The limiting speed or 'terminal velocity' established in Problem 16.5 is $v_{ss} = 8.60$ m/s. The speed which is 99.9 % of this is 8.591 m/s. To determine the time it takes to reach this speed, fix the values of $m = 0.1418$ kg, (alpha) = 10.81 degrees, and $b = 0.0303$ kg/sec, and run the program for a sequence of time intervals, iterating toward a final velocity of 8.591 m/s. For example,

Try T = 10 secs, get V = 7.59 m/s
Try T = 20 secs, get V = 8.49 m/s
Try T = 30 secs, get V = 8.596 m/s
Try T = 29 secs, get V = 8.593 m/s
Try T = 28 secs, get V = 8.589 m/s
Try T = 28.5 secs, get V = 8.591 m/s

The time history of velocity in the 28.5 second interval is displayed in the plot for Part (b).

The previous result is not very accurate because of round-off error. If we test the MATLAB progams with the given data, we find that for the very large times, T = 70 to T = 100, the final velocity is essentially constant at $v = 8.61045$ m/s; *i.e.,* the program believes that the terminal velocity is not 8.60 m/s, but 8.61045 m/s. If we then iterate to a final velocity of 8.602 m/s, which is 99.9% of 8.61045 m/s, we find that the time required is T = 32.5 sec, which is 14% larger than our previous result of 28.5 sec.

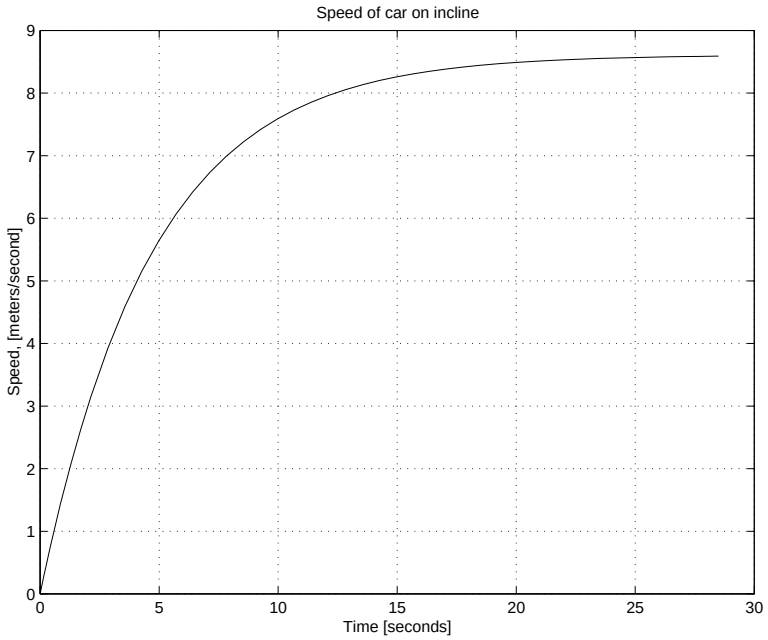


Figure 16.6.3: Plot for Part (b)

16.7 Pinewood Derby 4

(a) Replace $f_{viscous}$ by f_{drag} to get the free body diagram of the car mass acted on by the component of the weight, and the retarding *form-drag* force, shown below.

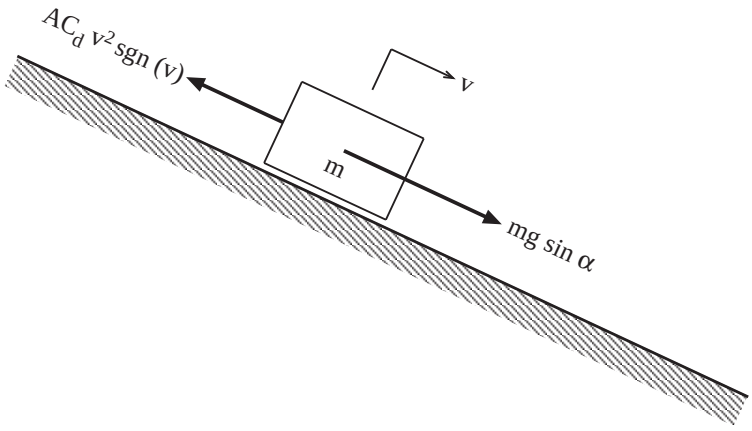


Figure 16.7.1: Model Diagram

The equation of motion for the model is

$$m\frac{dv}{dt} = mg\sin\alpha - AC_d v^2 \text{sgn}\, v$$

(b) The dimensions of the force f_{drag} are $[F] = [M][L]/[T^2]$; the dimensions of the area A are $[L^2]$; the dimensions of the velocity squared $v^2 = [L^2]/[T^2]$; and the *signum* function is dimensionless. The dimensions of the parameter C_d are

$$[C_d] = \left[\frac{f_{drag}}{Av^2 \text{sgn} v}\right] = \left[\frac{ML/T^2}{(L^2)(L^2/T^2)}\right] = [M/L^3]$$

In SI units, the dimensions of C_d are kg/m^3.

(c) On a horizontal track $\alpha = 0$, and the only horizontal force acting on the mass is the retarding form-drag force. Starting with a positive velocity v_o at $t = 0$, the mass will slow down, but will not reverse direction; *i.e.,* $v(t)$ will be non-negative, so the equation of motion can be written

$$m\frac{dv}{dt} = -AC_d v^2$$

The initial condition is $v = v_o$ at $t = 0$. The differential equation can be integrated by separating the variables v and t

$$\int_{v_o}^{v} \frac{dv}{v^2} = -\int_0^t \frac{AC_d}{m} dt$$

$$\frac{-1}{v} + \frac{1}{v_o} = -\frac{AC_d}{m} t$$

The time history of the speed of the car is obtained by solving the above equation for $v(t)$.

$$v(t) = \frac{v_o}{1 + \frac{AC_d v_o}{m} t}$$

Note that $v(t)$ decreases smoothly with time, and asymptotically approaces the limit $v = 0$ as t approaches infinity.

16.8 Pinewood Derby 5

The differential equation for the model in which a car of mass m descends a very long inclined track with angle α under the influence of gravity and form drag is

$$m\frac{dv}{dt} = mg\sin\alpha - AC_d v^2 \operatorname{sgn} v$$

as given at the beginning of Problem 16.7.

(a) At the terminal velocity v_{ss} there is no longer any acceleration ($dv/dt = 0$), and the drag force is equal and opposite to the accelerating force component, so that

$$v_{ss}^2 = \frac{mg\sin\alpha}{AC_d}$$

(b) If $m = 0.1418$ kg, $\alpha = 10.81$ degrees $= \sin^{-1}(3/16)$, and v_{ss} =8.60 m/s, then the product of the frontal area A and the drag coefficient C_d is

$$AC_d = \frac{mg\sin\alpha}{v^2} = \frac{(0.1418)(9.81)(3/16)}{(8.60)^2} = 0.00353 \text{ kg/m}$$

(c) To use the scripts 'car2sol.m' and 'cardragsol.m' to determine the time T for V to reach $(0.999)(8.60) = 8.5914$ m/s fix the inputs of $m = 0.1418$ kg, $\alpha = 10.81$ deg, and $AC_d = 0.00353$ kg/m^{3}n and then call 'car2-sol' and input various guesses for T, iterating toward "Final speed = 8.5914 m/s". For example:

Try T = 10 secs, get V = 8.362 m/s
Try T = 20 secs, get V = 8.594 m/s
Try T = 19 secs, get V = 8.5919 m/s
Try T = 18.9 secs, get V = 8.5917 m/s
Try T = 18.8 secs, get V = 8.5915 m/s
Try T = 18.7 secs, get V = 8.5913 m/s
Try T = 18.75 secs, get V = 8.5914 m/s (close enough)

This result is not very accurate, because of round-off error. The value 0.00353 kg/m obtained in (b) above for the parameter aC_d was rounded off to three significant figures. If we assume that AC_d is exactly equal to 0.00353 kg/m, and work backwards we find that v_{ss} is not exactly equal to 8.60 m/s but is actually 8.5958 m/s correct to five significant figures. If we repeat the exercise we find that at T = 17.45 sec, the final speed is 8.5872 m/s which is 99.9 % of 8.5958 m/s. Both of the previous results assume that there are no errors in the integration carried out by MATLAB. MATLAB is pretty good, but it does make small errors which can accumulate over a long integration. If we let it run for long times T, we find that throughout the range from $T = 35$ secs to $T = 70$ secs, the final speed is essentially constant (to five significant figures) at 8.5970 m/s; *i.e.,* MATLAB disagrees, in the fourth significant figure, with both $v_{ss} = 8.6000$ m/s and $v_{ss} = 8.5958$ m/s.

(d) In Problem 4(b) of Problem 16.7, it was found that with viscous friction the time to reach 99.9% of the terminal velocity was in the neighborhood of 28.5 to 32.5 seconds. Here, with form drag, the same speed is reached in about 17.45 to 18.75 seconds. With form drag (quadratic speed dependence) the terminal velocity is approached more quickly than is the case for viscous friction (linear speed dependence). The time histories of velocity for the two models are shown in the Fig. 2(d)-1.

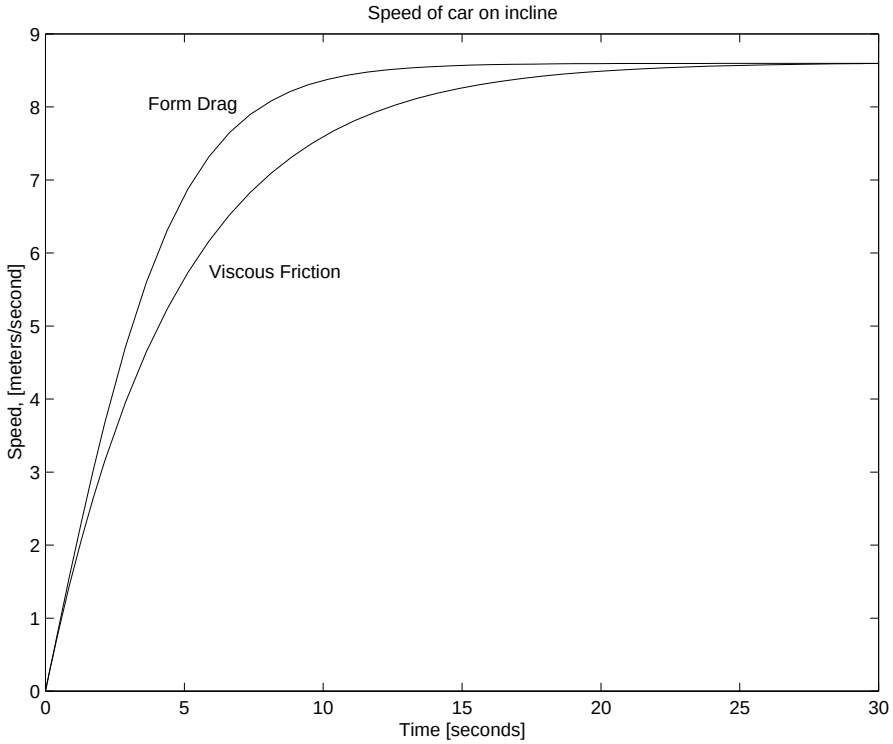


Figure 16.8.1: Fig. 2(d)-1

To explain why a car with quadratic drag approaches the terminal velocity more rapidly than a car with linear drag, compare the constitutive equations for the retarding forces as shown in Fig. 16.8.2. At all speeds below the common terminal velocity there is less retardation with form drag than there is for viscous friction. This means that, when both cars have the same velocity, the car with form drag accelerates faster than the car with viscous friction. Now the the slopes of the velocity curves in Fig. 16.8.1 represent the accelerations of the cars. Thus at each speed, the form drag curve will have a steeper slope than the viscous friction curve. As a consequence the form drag curve must approach the terminal velocity more quickly than the viscous friction curve.

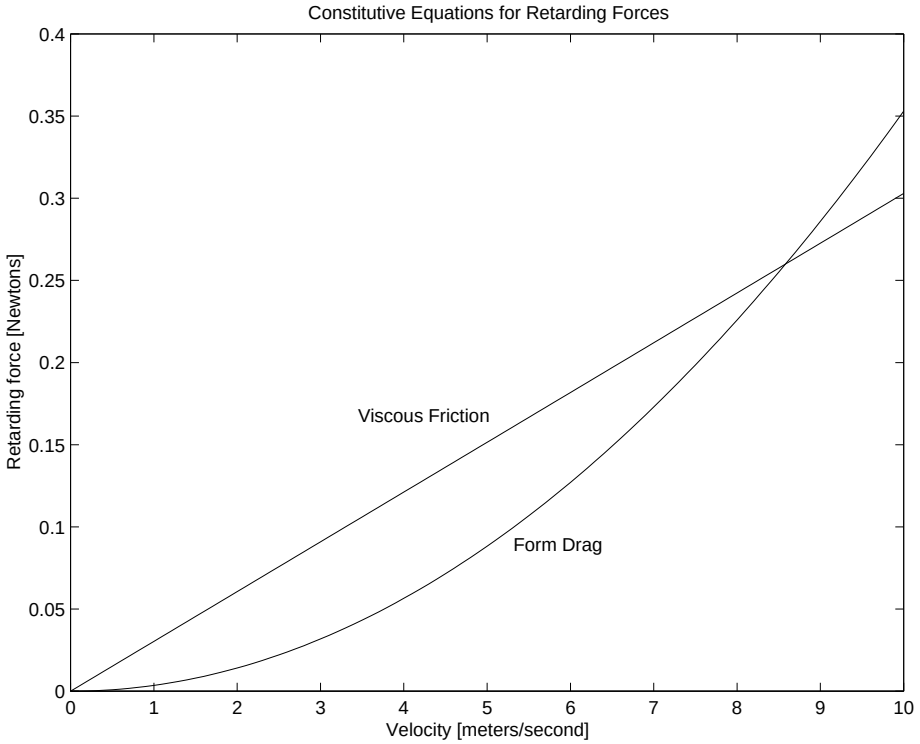


Figure 16.8.2: Fig. 2(d)-2

NOTE re MATLAB technique: To plot two graphs in the same Figure, as in Fig. 2(d)-1. Run 'car2sol' to obtain a plot of viscous friction case. Then, in the MATLAB Command Window, type "hold on". (this keeps the plot in the Figure screen). Next run 'car2sol' (with the same time interval T). The plot for the form-drag case then appears on top of the existing plot. To print captions such as "Viscous Friction" on the interior of the plot, type, in the MATLAB Command Window, "gtext('Viscous Friction')". When you hit 'return' the cursor on the Figure turns into a giant cross which can be moved by the mouse. Then when you click the mouse the caption appears on the Figure starting from the intersection point of the cross. You can make as many additional plots as you wish, as long as 'hold' is 'on'. To return to normal one-at-a-time plots, type "hold off" in the MATLAB Command Window.

16.9 Engine Block Vibration 1

Weight of engine block $W = 200$ pounds, stiffness $k = 18,000$ pounds/inch, damping coefficient $b = 2$ pounds /inch/second. Input force is $f(t) = f_a \sin \Omega t$ with $f_a = 2$ pounds.

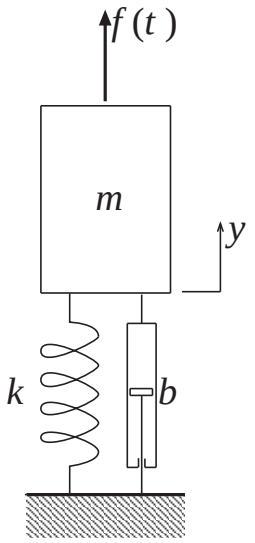


Figure 16.9.1: Engine Block Subjected to Sinusoidally Varying Force

Let $y(t)$ be displacement of engine block from equilibrium position. Geometric compatibility requires that the block velocity $v(t)$ satisfy $v = dy/dt$. Restoring force is $f_k = ky$, damping force is $f_{fric} = bv$, input force is $f(t) = f_a \sin \Omega t$, and Newton's second law is $f_m = mdv/dt$ with $m = W/g$. When these constitutive equations are substituted in the force-balance requirement $f_m = f(t) - f_k - f_{fric}$ and v is replaced by dy/dt the result is the equation of motion

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f_a \sin \Omega t$$

(a) If the input force is considered to be the imaginary part of the complex excitation $f_a \exp(i\Omega t)$, it is appropriate to consider the steady-state response to be the imaginary part of the complex response $A\exp(i\Omega t)$ where A is the complex amplitude of the response , which is to be found by requiring the complex response to satisfy the equation of motion. Substitution of $y = A\exp(i\Omega t)$ into

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f_a \exp(i\Omega t)$$

yields

$$A = \frac{f_a}{k + i\Omega b - m\Omega^2}$$

(b) (i) At very low frequencies $A \to f_a/k$.

(ii) At very high frequencies $A \to -f_a/m\Omega^2$.

(iii) The magnitudes of (b) and (c) become equal when $\Omega^2 \to \Omega^2_{break} = k/m = \omega_o^2$.

(c) In terms of the behavioral parameters ω_o and ζ defined by

$$\omega_o^2 = \frac{k}{m} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{b}{m}$$

the ratio Z of $A(\Omega)$ to $A(0)$,

$$Z = \frac{A(\Omega)}{A(0)} = \frac{1}{(1 - \frac{\Omega^2}{\omega_o^2}) + i2\zeta\frac{\Omega}{\omega_o}}$$

is a convenient dimensionless form of the complex amplitude A. The engine displacement amplitude has its peak magnitude when $|Z|^2$ is a maximum. Now

$$|Z|^2 = \frac{1}{(1 - \frac{\Omega^2}{\omega_o^2})^2 + 4\zeta^2\frac{\Omega^2}{\omega_o^2}}$$

and its maximum occurs when its derivative with respect to Ω^2/ω_o^2 vanishes; *i.e.,* when

$$\frac{d|Z|^2}{d\Omega^2/\omega_o^2} = 2(1 - \frac{\Omega^2}{\omega_o^2}) + 4\zeta^2 = 0$$

The solution of this equation for Ω yields

$$\Omega_{peak} = \omega_o\sqrt{1 - 2\zeta^2}$$

(d) To evaluate $|Z(\Omega_{peak})|$, insert $\Omega_{peak}^2 = \omega_o^2(1 - 2\zeta^2)$ in the preceding expression for $|Z|^2$ to get

$$|Z(\Omega_{peak})|^2 = \frac{1}{4\zeta^4 + 4\zeta^2(1 - 2\zeta^2)} = \frac{1}{4\zeta^2(1 - \zeta)^2}$$

so that

$$|Z(\Omega_{peak})| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

For comparison, note that $|Z(\omega_o)| = 1/2\zeta$. The actual peak occurs at a frequency that is smaller than the undamped natural frequency ω_o by a factor of $\sqrt{1 - 2\zeta^2}$ and the magnitude of the peak is greater than the magnitude of the response at $\Omega = \omega_o$ by the factor $1/\sqrt{1 - \zeta^2}$.

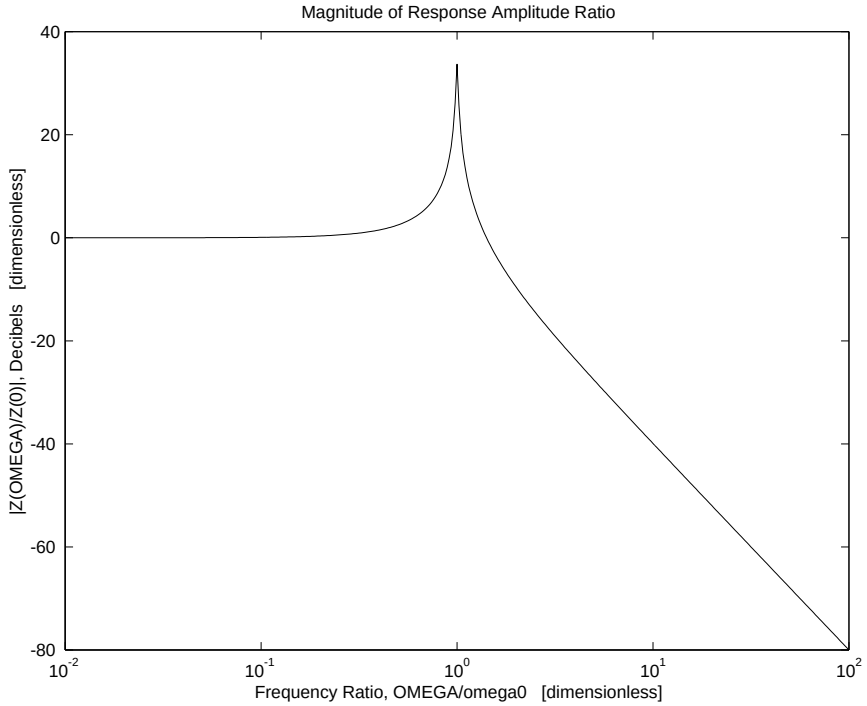


Figure 16.9.2: Bode Plot of Magnitude of $A(\Omega)/A(0)$ in dB vs. Frequency Ratio Ω/ω_o

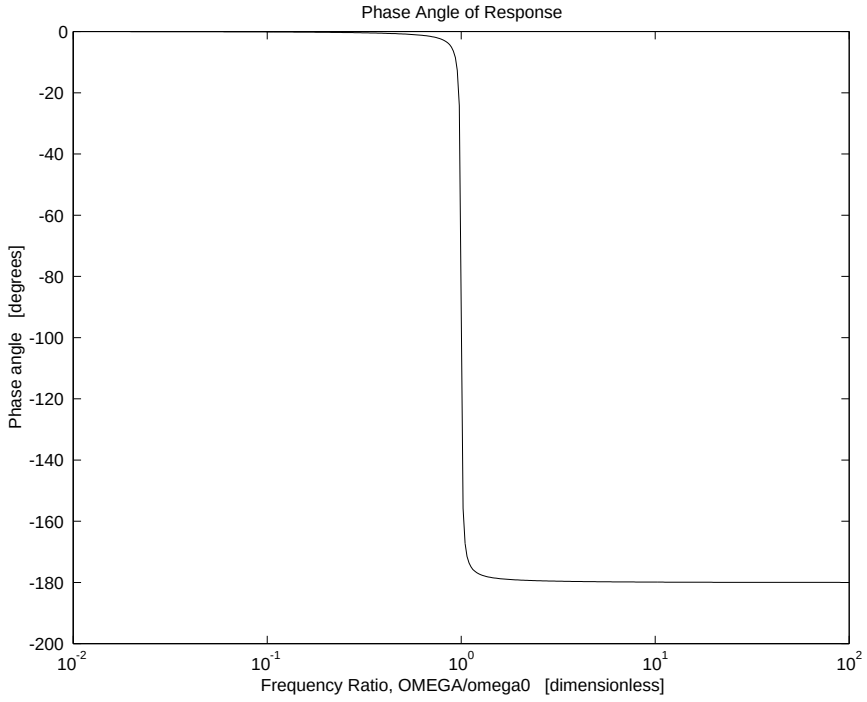


Figure 16.9.3: Bode Plot of Phase Angle vs. Frequency Ratio Ω/ω_o

(e) The Bode plots shown in Fig.2 and Fig.3 were obtained by running the script enginebodesol.m.

16.10 Engine Block Vibration 2

Repeat Problem 16.9, but with $f(t) = C\Omega^2 \sin \Omega t$ with $C = 0.005$ pound-sec^2.

(a) Substitution of $y = A\exp(i\Omega t)$ into

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = C\Omega^2\exp(i\Omega t)$$

yields

$$A = \frac{C\Omega^2}{k + i\Omega b - m\Omega^2}$$

(b) (i) At very low frequencies $A \to C\Omega^2/k$.

(ii) At very high frequencies $A \to -C/m$.

(iii) The magnitudes of (b) and (c) become equal when $\Omega^2 \to \Omega^2_{break} = k/m = \omega_o^2$.

(c) In terms of the behavioral parameters ω_o and ζ defined by

$$\omega_o^2 = \frac{k}{m} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{b}{m}$$

the ratio Z of $A(\Omega)$ to $A(\infty)$,

$$Z = \frac{A(\Omega)}{A(\infty)} = \frac{\frac{\Omega^2}{\omega_o^2}}{(1 - \frac{\Omega^2}{\omega_o^2}) + i2\zeta\frac{\Omega}{\omega_o}}$$

is a convenient dimensionless form of the complex amplitude A. The engine displacement amplitude has its peak magnitude when $|Z|^2$ is a maximum. Now

$$|Z|^2 = \frac{\frac{\Omega^4}{\omega_o^4}}{(1 - \frac{\Omega^2}{\omega_o^2})^2 + 4\zeta^2\frac{\Omega^2}{\omega_o^2}}$$

and its maximum occurs when its derivative with respect to Ω^2/ω_o^2 vanishes; *i.e.,* when

$$2\frac{\Omega^2}{\omega_o^2}[(1 - \frac{\Omega^2}{\omega_o^2})^2 + 4\zeta^2\frac{\Omega^2}{\omega_o^2}] - \frac{\Omega^4}{\omega_o^4}[-2(1 - \frac{\Omega^2}{\omega_o^2}) + 4\zeta^2] = 0$$

The solution of this equation for Ω^2/ω_o^2 yields

$$\frac{\Omega^2}{\omega_o^2} = \frac{1}{1 - 2\zeta^2}$$

so that

$$\Omega_{peak} = \frac{\omega_o}{\sqrt{1 - 2\zeta^2}}$$

(d) To evaluate $|Z(\Omega_{peak})|$, insert $\Omega^2_{peak}/\omega_o^2 = 1/(1 - 2\zeta^2)$ in the preceding expression for $|Z|^2$ to get

$$|Z(\Omega_{peak})|^2 = \frac{(\frac{1}{1-2\zeta^2})^2}{(1 - \frac{1}{1-2\zeta^2})^2 + \frac{4\zeta^2}{1-2\zeta^2}} = \frac{1}{4\zeta^2(1 - \zeta)^2}$$

so that

$$|Z(\Omega_{peak})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Comparing this result with that of Problem 16.9, we note that the actual peak here occurs at a frequency that is *greater* than the undamped natural frequency ω_o by a factor of $1/\sqrt{1-2\zeta^2}$, but the magnitude of the peak is the same. It is greater than the magnitude of the response at $\Omega = \omega_o$ by the same factor $1/\quad\overline{1-\zeta^2}$.

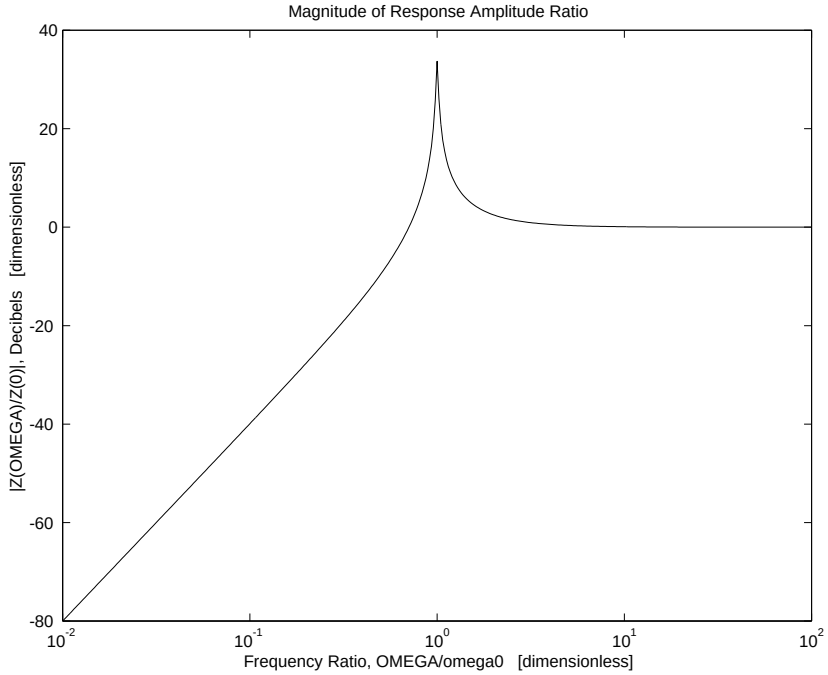


Figure 16.10.1: Bode Plot of Magnitude of $A(\Omega)/A(0)$ in dB vs. Frequency Ratio Ω/ω_o

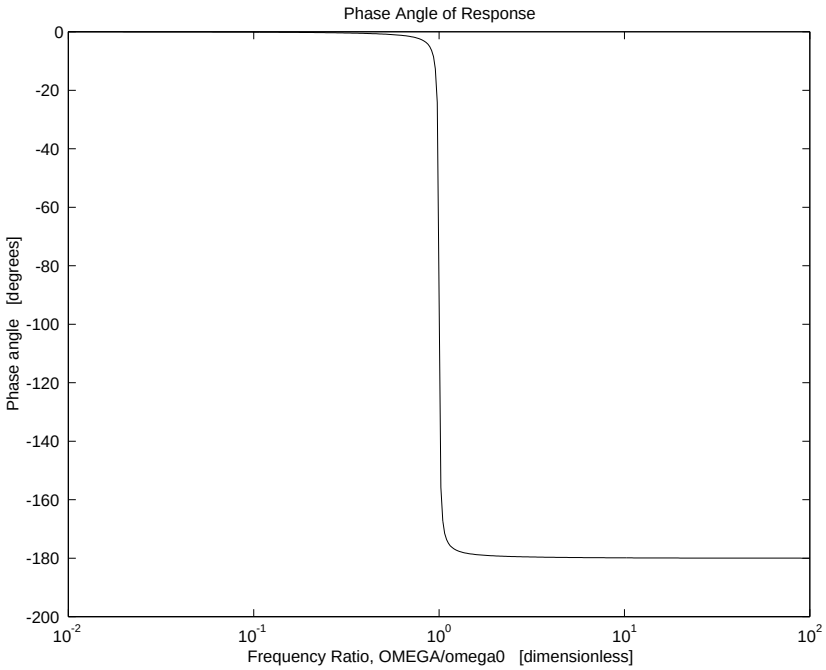


Figure 16.10.2: Bode Plot of Phase Angle vs. Frequency Ratio Ω/ω_o

(e) The Bode plots shown in Fig. 16.10.1 and Fig. 16.10.2 were obtained by running the following script (adapted from the Script for Problem 16.9):

16.11 Engine Block Vibration 3

The equation of motion

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t) = f_a \sin \Omega t$$

obtained in Problem 16.9 can be rewritten in the standard form for state-determined systems by taking $y(t)$ and $v(t) = dy/dt$ as state varibles and writing the state equations in matrix form

$$\frac{d}{dt}\left\{ \begin{array}{c} y \\ v \end{array} \right\} = \left[\begin{array}{cc} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{array} \right] \left\{ \begin{array}{c} y \\ v \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} \frac{f_a}{m} \sin \Omega t$$

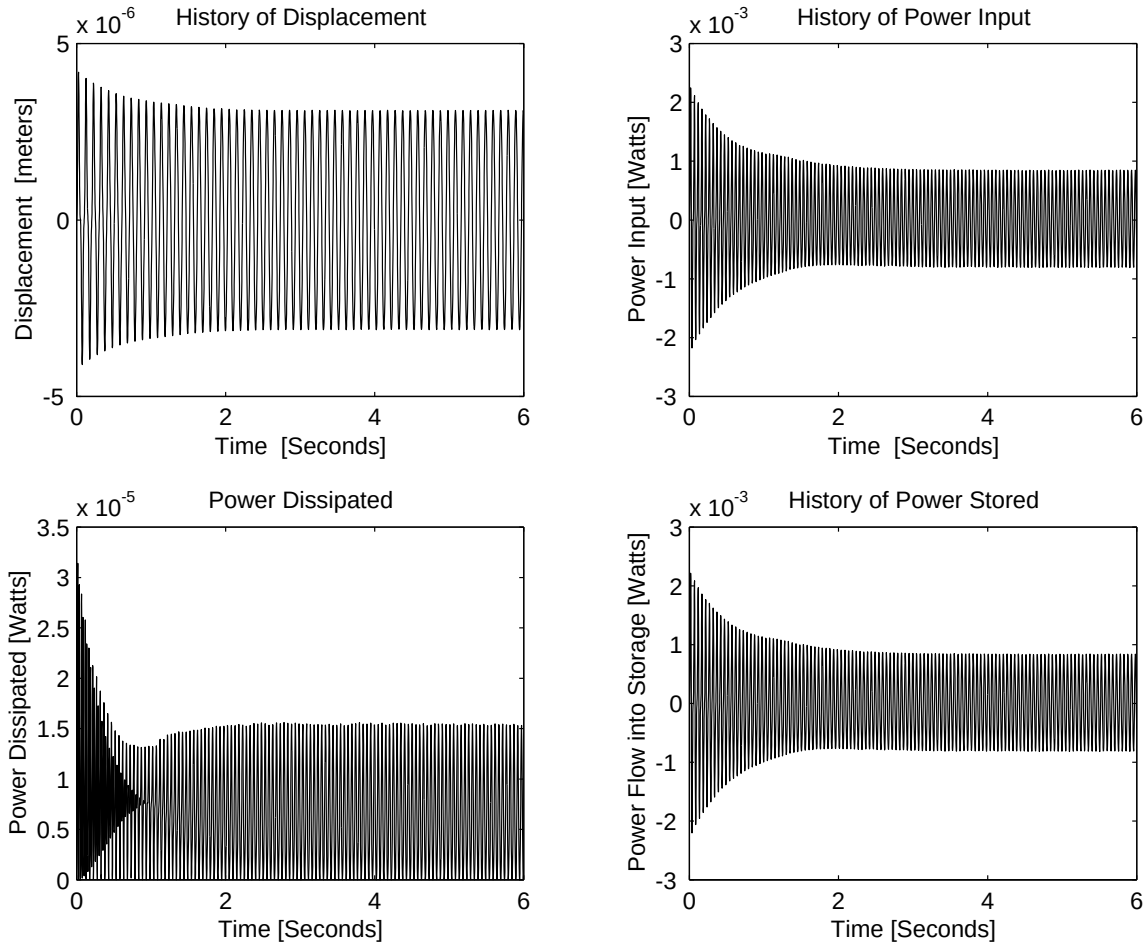


Figure 16.11.1: Engine Start-Up when $\Omega = \Omega_{break}/3$

One of the desired outputs is the state variable $y(t)$. The other outputs desired are $P_{in} = f(t) * v(t)$;

$P_{diss} = bv * v$; and $P_{stored} = P_{in} - P_{diss}$. The values of the physical parameters are:

$m = 200$ pounds	=	90.7 kg
$k = 18,000$pounds/inch	=	3.152e6 Newton/meter
$b = 2$ pounds/inch/second	=	350 Newtons/meter/second
$f_a = 2$ pounds	=	8.90 Newtons
$\Omega_{break}/3 = 62.1$ rad/sec	=	62.1 rad/sec
$\Omega_{break} = 186.4$ rad/sec	=	186.4 rad/sec
$3\Omega_{break} = 559$ rad/sec	=	559 rad/sec

The plots shown in Figs. 6-8 were obtained by running the following MATLAB scripts. The input script is called 'engstartsol.m' and the script called by 'engstartsol.m' is called 'eqengstartsol.m'.

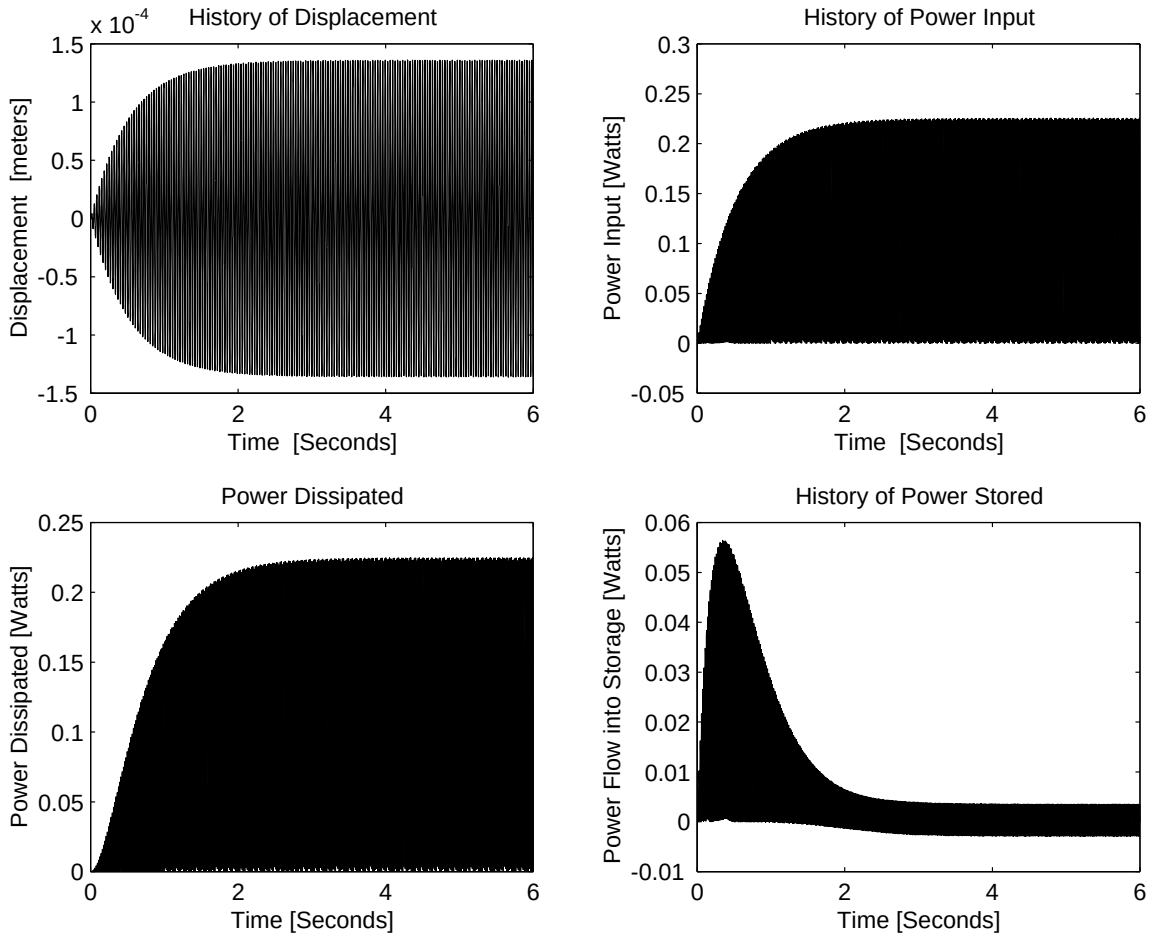


Figure 16.11.2: Engine Start-Up when $\Omega = \Omega_{break}$

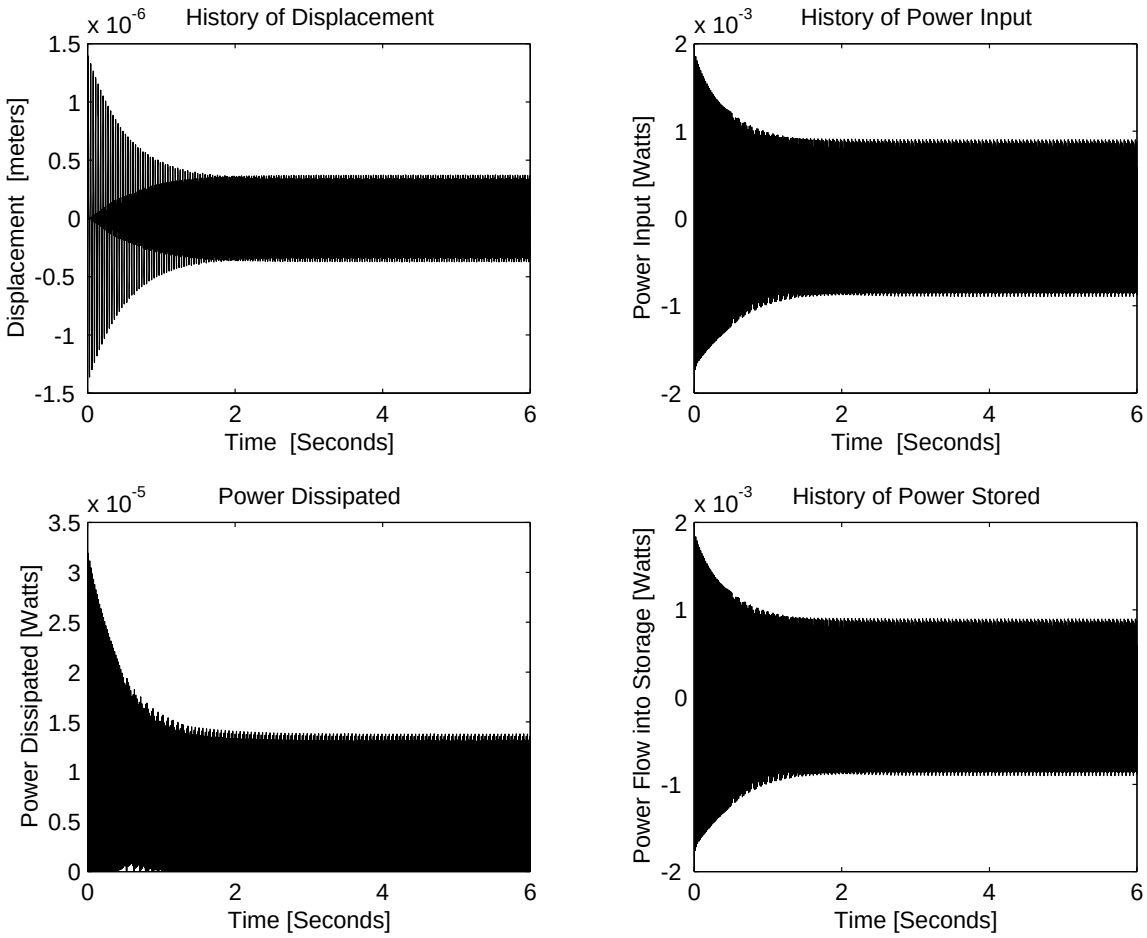


Figure 16.11.3: Engine Start-Up when $\Omega = 3\Omega_{break}$

16.12 Plate On Springs 1

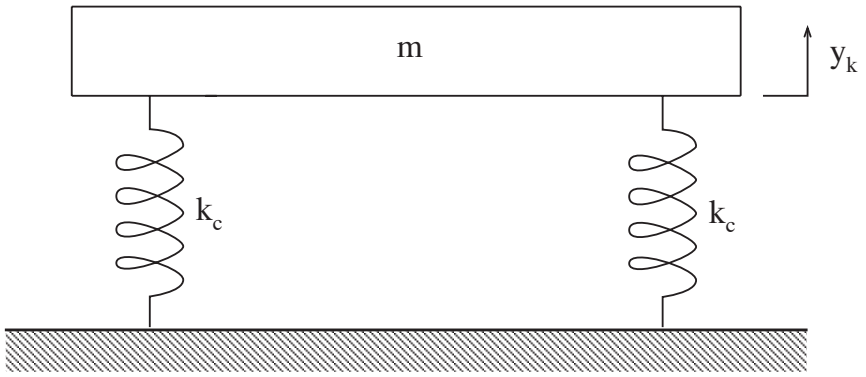


Figure 16.12.1: Steel Plate on Springs

The displacement y_k measures the vertical extension of the four springs from the equilibrium configuration where the weight of the plate mg is balanced by initial compressive forces in the springs. The excess tensile force in each spring due to a displacement y_k is $f_{ki} = k_c y_k \quad (i = 1, \cdots, 4)$. The total vertical spring force is $f_k = 4f_{ki} = 4k_c y_k = ky_k$. The constitutive equation for the mass is Newton's law $f_m = mdv_m/dt$. When friction is absent and there is no external load, the force balance equation is simply $-f_k = f_m$, or

$$-ky_k = m\frac{dv_m}{dt}, \quad \text{or} \quad -ky_k = m\frac{dv_k}{dt}$$

on using the geometric compatibility condition $v_m = v_k$. The standard form of the differential equation (See Lecture notes of 9/15/99) is

$$\frac{d^2y_k}{dt^2} + (\frac{k}{m})y_k = 0$$

Guess a solution of the form $y_k = A\exp(\lambda t)$, which leads to the characteristic equation

$$\lambda^2 + \omega_o^2 = 0$$

where $\omega^2 = k/m$. The roots of the characteristic equation are $+i\omega_o$ and $-i\omega_o$, and the general solution of the differential erquation is

$$y_k = A\exp(i\omega_o t) + B\exp(-i\omega t)$$

An alternative form of the general solution can be obtained by introducing *Euler's formula* $\exp(i\omega_o t) = \cos\omega_o t + i\sin\omega_o t$.

$$y_k = C_1 cos\omega_o t + C_2 \sin\omega_o t$$

These two forms of solution are equivalent if the constants of integration are related as follows:

$$\begin{aligned} C_1 &= A + B \qquad A = \frac{1}{2}(C_1 + C_2) \\ C_2 &= A - B \qquad B = \frac{1}{2}(C_1 - C_2) \end{aligned}$$

The given initial conditions are $y_k(0) = 0$ and $dy_k/dt(0) = v_o$. Using the trigonometric form of the general solution, we have

$$y_k = C_1 \cos \omega_o t + C_2 \sin \omega_o t \qquad \text{and} \qquad \frac{dy_k}{dt} = -C_1 \omega_o \sin \omega_o t + C_2 \omega_o \cos \omega_o t$$

Setting $t = 0$ yields

$$C_1 = 0 \qquad \text{and} \qquad C_2 = \frac{v_o}{\omega_o}$$

so the displacement history of the plate, starting from rest at the equilibrium position with the initial velocity v_o is

$$y_k(t) = \frac{v_o}{\omega_o} \sin \omega_o t$$

16.13 Plate On Springs 2

The unloaded plate system here is the same as that considered in Problem 16.12. The mass of the plate is m and the effective stiffness of the four springs is k. The natural frequency of the unloaded system is ω_o where $\omega^2 = k/m$. When the book of mass $m/2$ is attached to the plate, the resulting system has mass $3m/2$ and stiffness k, and its natural frequency of free oscillation is ω_1 where $\omega_1^2 = 2k/3m$. The oscillations in both Case I and Case II involve motions with the book attached to the plate. The difference between Cases I and II lies in the initial conditions, not in the basic system.

(a) The ratio of frequencies of Case I oscillations to unloaded plate oscillations is $\omega_1/\omega_o = \overline{2/3} = 0.816$.

(b) The ratio of frequencies of Case II oscillations to unloaded plate oscillations is $\omega_1/\omega_o = \overline{2/3} = 0.816$.

(c) In the (book + plate) system there is an additional gravity load of $mg/2$, so the equilibrium position of the (book + plate) system is beneath the equilibrium position of the unloaded plate by a distance of $\Delta = mg/2k$.

(d) In Case I the system starts from rest at the equilibrium position of the unloaded plate. Viewed from the equilibrium position of the (book + plate) system, the initial displacement $y_k(0) = \Delta$, and the initial velocity is $dy_k/dt(0) = 0$.

(e) In Case II there is an initial velocity $dy_k/dt(0) = -v_1$ imparted by the impact of the book upon the plate. This initial velocity can be determined by applying conservation of linear momentum to the impact. Immediately before the impact the plate has no momentum while the book has a downward momentum of $1/2mv_o$. Immediately after the impact the (book + plate) has the downward momentum $3mv_1/2$. For these two momenta to be equal it is necessary that $v_1 = v_o/3$. The initial conditions for Case II are thus: $y_k(0) = \Delta$ and $dy_k/dt(0) = -v_o/3$.

(f) The general solution for both Case I and Case II is the same as the general solution for Problem 16.12, except that ω_o is everywhere replaced by ω_1. For Case I, the initial conditions require

$$y_k(0) = \Delta = C_1 \qquad \text{and} \qquad \frac{dy_k}{dt}(0) = 0 = C_2\omega_1$$

so that

$$y_k(t) = \Delta \cos \omega_1 t$$

where $\Delta = mg/2k$, and $\omega_1^2 = 2k/3m$.
For Case II, the initial conditions require

$$y_k(0) = \Delta = C_1 \qquad \text{and} \qquad \frac{dy_k}{dt}(0) = -v_1 = C_2\omega_1$$

so that

$$y_k(t) = \Delta \cos \omega_1 t - \frac{v_1}{\omega_1} \sin \omega_1 t$$

where $\Delta = mg/2k$, $v_1 = v_o/3$, and $\omega_1^2 = 2k/3m$

16.14 Mousetrap Dynamics 1

The solution for this problem is not available.

16.15 Mousetrap Dynamics 2

The solution for this problem is not available.

16.16 Mousetrap Dynamics 3

The solution for this problem is not available.

16.17 Mousetrap Dynamics 4

The solution for this problem is not available.

16.18 Hydraulic Elevator Design

(a) As the elevator falls, there is both the constant force due to the acceleration of gravity and a damping force which is proportional to (and in a direction opposing) the velocity.

$$\begin{aligned} \Sigma F &= ma \\ -mg - cv &= ma \\ c\dot{x} + m\ddot{x} &= -mg \end{aligned}$$

Note that the x coordinate is defined to be positive in the up direction. Therefore, mg is in the negative x direction. The cv term must always act in opposition to the direction of the velocity. (If the mass is moving in the negative x direction, velocity is negative, and the force acts in the positive direction to slow the fall, for instance.)

When the elevator suddenly fails, whatever force has been compensating for the force of gravity (from tension in elevator cables or force applied by the hydraulic lift, etc) suddenly disappears, resulting in a net step in force. Thus, the fall of the elevator can be well-modeled as a 1st order system with a step input:

$$\begin{aligned} c\dot{x} + m\ddot{x} &= -mg \cdot u_s(t) \\ (cs + ms^2)X(s) = -mgU(s) &= -mg\frac{1}{s} \\ X(s) &= -\frac{mg}{s^2(c+ms)} \end{aligned}$$

Note that the velocity and acceleration are just the first and second derivatives of x(t):

$$\begin{aligned} V(s) = sX(s) &= -\frac{mg}{s(c+ms)} \\ A(s) = s^2X(s) &= -\frac{mg}{c+ms} \end{aligned}$$

The acceleration, $A(s)$, should look familiar; it is just the Laplace domain representation of a 1st order system. We know the response will start at some value (which we can find by using the I.V.T.) and go to some steady state value (which we can find by using the F.V.T.) via an exponential decay with some time constant. Rewriting $A(s)$,

$$A(s) = s^2X(s) = -\frac{mg/c}{\frac{m}{c}s + 1}$$

it should be clear to you that the time constant is:

$$\tau = \frac{m}{c} = \frac{2000\text{kg}}{2000\text{kg/sec}} = 1 \text{ [s]}$$

Using the Initial Value Theorem:

$$A(0) = \lim_{s\to\infty} s\frac{mg/c}{\frac{m}{c}s+1} = \frac{-gm/c}{m/c} = -g = -9.8 \text{ [m/s}^2\text{]}$$

This is what wed expect from the basic physics. Using the Final Value Theorem:

$$A(\infty) = \lim_{s\to 0} s\frac{mg/c}{\frac{m}{c}s+1} = 0 \text{ [m/s}^2\text{]}$$

therefore

$$a(t) = -9.8e^{-t} \text{ [m/s}^2\text{]}$$
$$v(t) = 9.8e^{-t} + C_1 \text{ [m/s]}$$

To match the initial condition that v(0)=0,

$$v(t) = 9.8e^{-t} - 9.8 \text{ [m/s]}$$

Position can be obtain by taking another integral:

$$x(t) = -9.8e^{-t} - 9.8t + C_2$$

Then we need to match i.c.s again:

$$x(t) = -9.8e^{-t} - 9.8t + 9.8$$

You can use the step() function in MATLAB to plot the step response of a system (given its transfer function), rather than deriving it by hand (as above).

Figure 16.18.1 are plots of position, velocity and acceleration. We can first solve to find the time at which the elevator has reached 10 meters. *This happens at about* $t = 1.87$ *seconds*

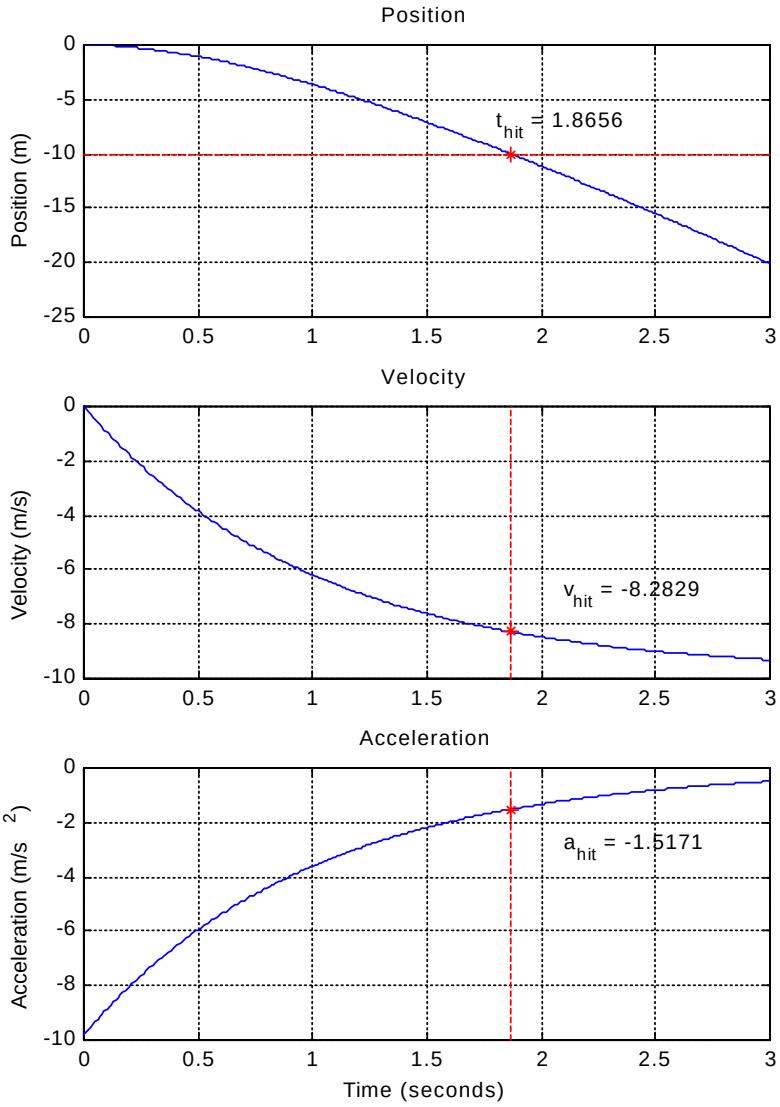


Figure 16.18.1: Plots of Position, Velocity and Acceleration

The total force on a passenger must satisfy $F = ma$, where the acceleration is shown (over time) in the plot above, and the mass is 100 kg.

(b) We can calculate the response of the elevator hitting the spring-damping system at the bottom of the shaft by calculating a FREE RESPONSE from the initial conditions and a FORCED RESPONSE due to a step in force (of the mass of the elevator times gravity). For convenience, well redefine time such that t=0 when the elevator hits and x=0 where the elevator hits.

To get the velocity to settle within .1 m/s as quickly as possible, well aim for a response which overshoots by just under 0.1 m/s. This would be close to critically damped. (The velocity at impact is about 8.3 m/s, so a response which slows down and then reverse to move upward at no more than 0.1 m/s would have an overshoot of about 1.2%.)

Its fine to have a response which overshoots by more than this amount, but note that this will generally increase the time needed for the system to settle.

Figure 16.18.2 shows the response with the following parameters:

$$k = 6990 \text{ [N/m]}$$
$$c = 2000 + 5300 \text{ [Ns/m]}$$

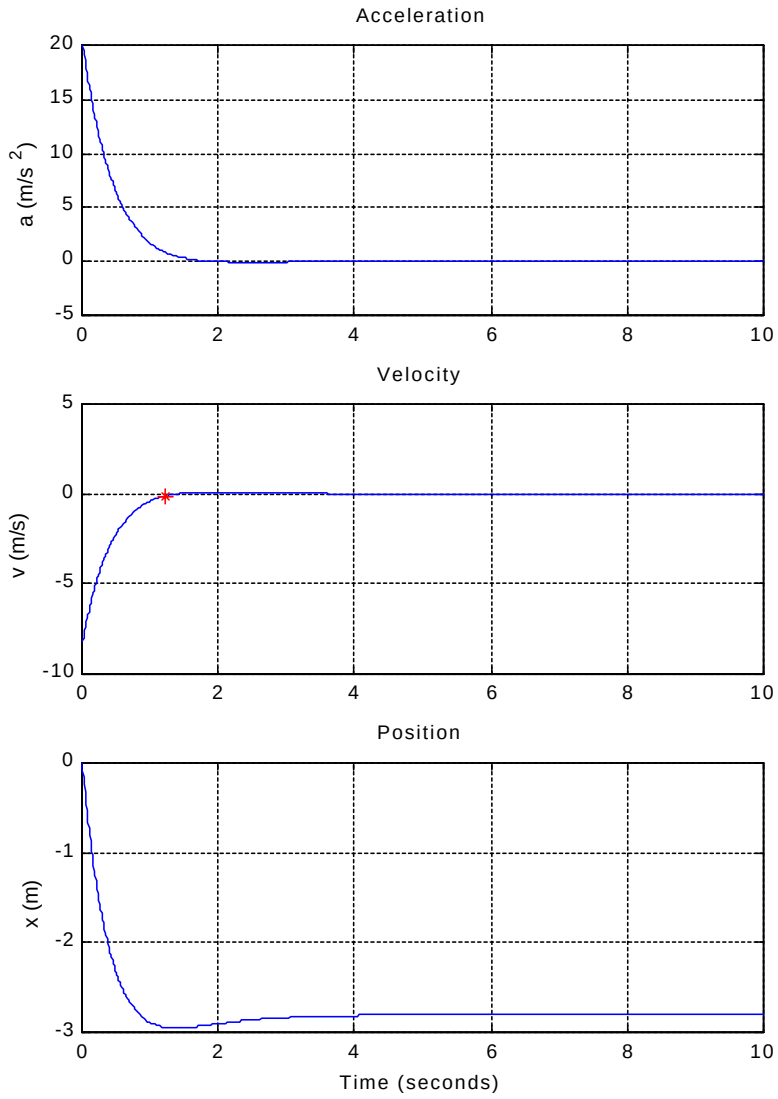


Figure 16.18.2: Plot of Position, Velocity and Acceleration with $k = 6990$ [N/m] and $c = 2000+5300$ [Ns/m]

Note that the acceleration requirement is barely met (under 20 m/s^2) at t=0, and that the position goes down to a minimum value of almost 3 meters!! (Thats a large of deflection in the system...)

The system settles to within 0.1 m/s in about 1.24 seconds.

For a problem like this, you may find it convenient to write a MATLAB script or function, so you can iterate through possible solutions and develop at intuition about the dynamics of the system. (Here, since the system is allowed to overshoot, you can calculate the free and forced responses for an underdamped second order system as in 2.27, given the initial conditions (v=vhit) and step in force (m*g).

Below is a second alternative which puts more importance on minimizing the maximum deflection. The settling time is a bit longer (2.4 sec; about 2x that of the previous solution). The maximum deflection is cut by about 40% (from 3m to 1.8m). Figure 16.18.3 shows the responses when $k = 22,500$ [N/m] and $c = 2000$ [Ns/m].

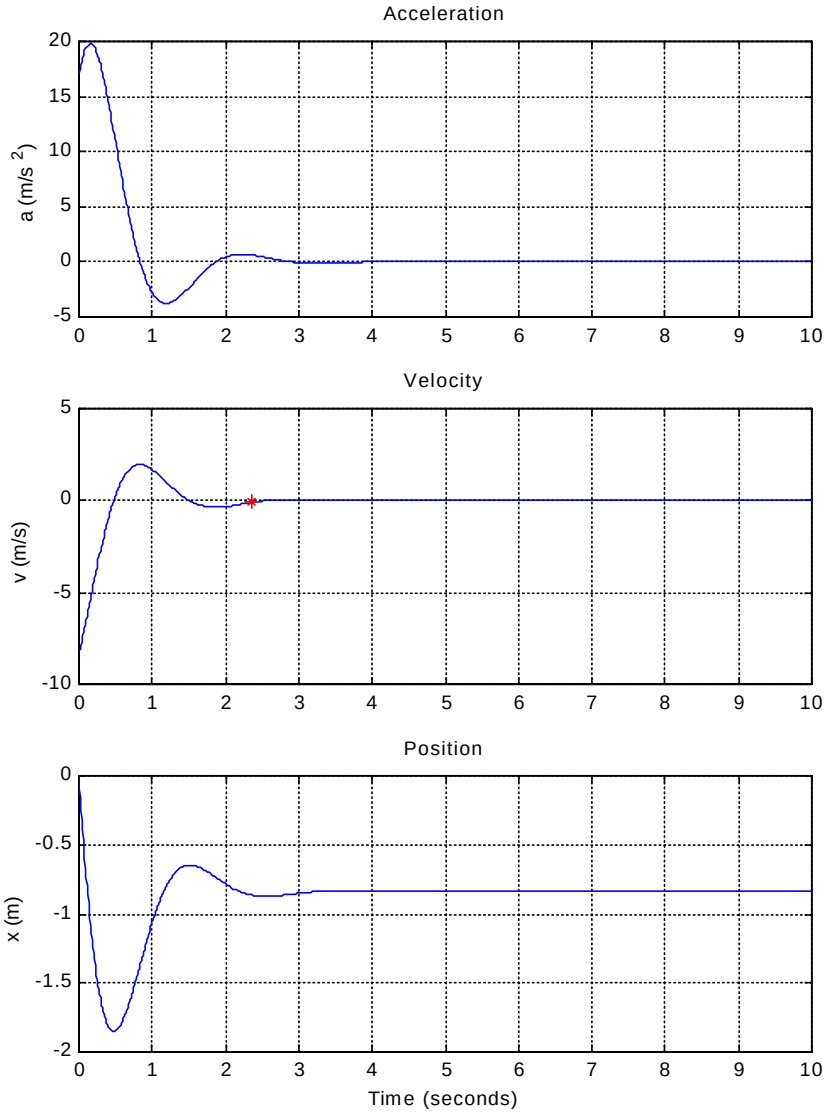


Figure 16.18.3: Plot of Position, Velocity and Acceleration with $k = 22,500$ [N/m] and $c = 2000$ [Ns/m]

16.19 Servo Position Control

The solution for this problem is not available.

16.20 Servo Frequency Compensation

The solution for this problem is not available.

16.21 Servo Torque Disturbance

The solution for this problem is not available.

16.22 Plate On Springs Damped 1

The rigid plate of mass m is supported by four springs, each of stiffness k_c and damping parameter b_c. If y_k is measured from the equilibrium position where

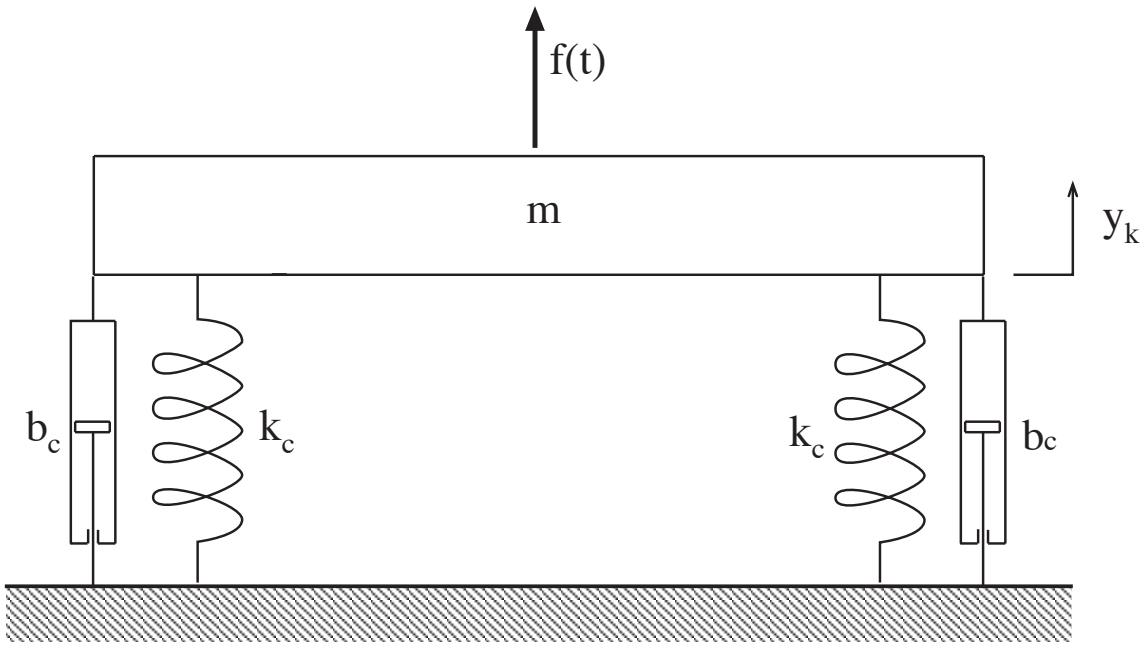


Figure 16.22.1: Plate on Springs with Damper

the weight of the plate is balanced by the initial compression of the springs, the constitutive equations are

$$f_m = m\frac{dv_m}{dt} \qquad f_{ki} = k_c y_{ki} \qquad f_{bi} = b_c v_{bi} \qquad (i = 1, \cdots, 4)$$

Geometric compatibility requires that, for $(i = 1, \cdots, 4)$,

$$y_{ki} = y_k \qquad v_m = v_{bi} = \frac{dy_{ki}}{dt} = \frac{dy_k}{dt}$$

Because there are four springs,

$$f_k = \sum_1^4 f_{ki} = 4k_c y_k = k y_k \qquad \text{and} f_b = \sum_1^4 f_{bi} = 4b_c \frac{dy_k}{dt} = b\frac{dy_k}{dt}$$

Finally force balance requires that

$$f(t) - f_k - f_b = f_m$$

which leads to the differential equation

$$m\frac{d^2 y_k}{dt^2} + b\frac{dy_k}{dt} + k y_k = f(t)$$

When the behavioral parameters ω_o and ζ defined by

$$\omega_o^2 = \frac{k}{m} \qquad 2\zeta\omega_o = \frac{b}{m} \quad \text{or} \quad \zeta = \frac{b}{\sqrt{4mk}}$$

are introduced, the equation takes the form

$$\frac{d^2y_k}{dt^2} + 2\zeta\omega_o\frac{dy_k}{dt} + \omega_o^2 y_k = \frac{f(t)}{m}$$

(a) The desired behavioral parameters are $\omega_o = 5Hz(2\pi \text{ rad/cycle}) = 31.4$ rads/sec and (i) $\zeta = 0.1$, (ii) $\zeta = 0.5$, (iii) $\zeta = 1.0$, and (iv) $\zeta = 1.5$. With the plate mass fixed at 5 pounds, or $5(0.4536) = 2.268$ kilograms, the spring constant k must also be fixed at

$$k = m\omega_o^2 = 2.268(31.4)^2 = 2236 Newtons/meter$$

in order to keep the undamped natural frequency, given by $\omega_o^2 = k/m$, equal to 5 Hz. To provide the desired values of ζ, the damping parameter b must be chosen to satisfy

$$\zeta = \frac{b}{2\sqrt{km}} = \frac{b}{2m\omega_o} \qquad \text{or} \qquad b = 2\zeta m\omega_o = 2(2.268)(31.4)\zeta = 142.4\zeta$$

The b-values for the four cases are: (i) 14.24 kg/sec; (ii) 71,2 kg/sec; (iii) 142.4 kg/sec; (iv) 213.6 kg/sec. The decay time constants, $\tau = 1/\zeta\omega_0$, for the first three cases are: (i) 0.318 secs: (ii) 0.0637 secs; (iii) 0.0318 secs. In Case (iv) the roots of the characteristic equation are $\lambda = \omega_o(-\zeta \pm \sqrt{\zeta^2 - 1})$. The time constants are the negative reciprocals of the λ-values, so the longest time constant is the negative reciprocal of the smallest λ

$$\tau_{longest} = -\left[31.4(-1.5 + \sqrt{(1.5)^2 - 1})\right]^{-1} = 0.0834 secs$$

The duration of the time histories in the four cases are 5 times the corresponding time constants: (i) 1.59 secs; (ii) 0.319 secs; (iii) 0.159 secs; (iv) 0.417 secs.
The damped natural frequency ω_d, for cases with $0 < \zeta < 1$, is given by

$$\omega_d = \omega_o\sqrt{1 - \zeta^2}$$

For Case (i) $\omega_d = 31.26$ rad/sec for $\zeta = 0.1$, and for case (ii) $\omega_d = 27.21$ rad/sec. For Cases (iii) and (iv), the eigenvalues are purely real, and $\omega_d = 0$.

The data for the four cases in Part (a) are assembled in the Table below.

Case	ζ	k [N/m]	b [N/m/sec]	τ [sec]	5τ [sec]	ω_d [r/s]	$\zeta\omega_o$
(i)	0.1	2239	14.25	0.3183	1.592	31.26	3.124
(ii)	0.5	2239	71.25	0.06365	0.3183	27.21	15.62
(iii)	1.0	2239	142.5	0.03183	0.1592	0.0	31.42
(iv)	1.5	2239	213.8	0.08332	0.4166	0.0	47.13

In addition to these data, every case has $m = 2.268$ kg, $f_a = (5)(4.448) = 22.24$ Newtons, and the initial conditions, $y_k(0) = 0$ and $v_k(0) = 0$.

The time histories are obtained by inputting the above data into the MATLAB scripts. When $0 < \zeta < 1$, Cases (i) and (ii), the script to be used is 'MassSprgDmpr1.m'. When $\zeta = 1$, Case (iii), the script to be used is 'MassSprgDmpr2.m', and when $1 < \zeta$, Case (iv), the script to be used is 'MassSprgDmpr3.m'.

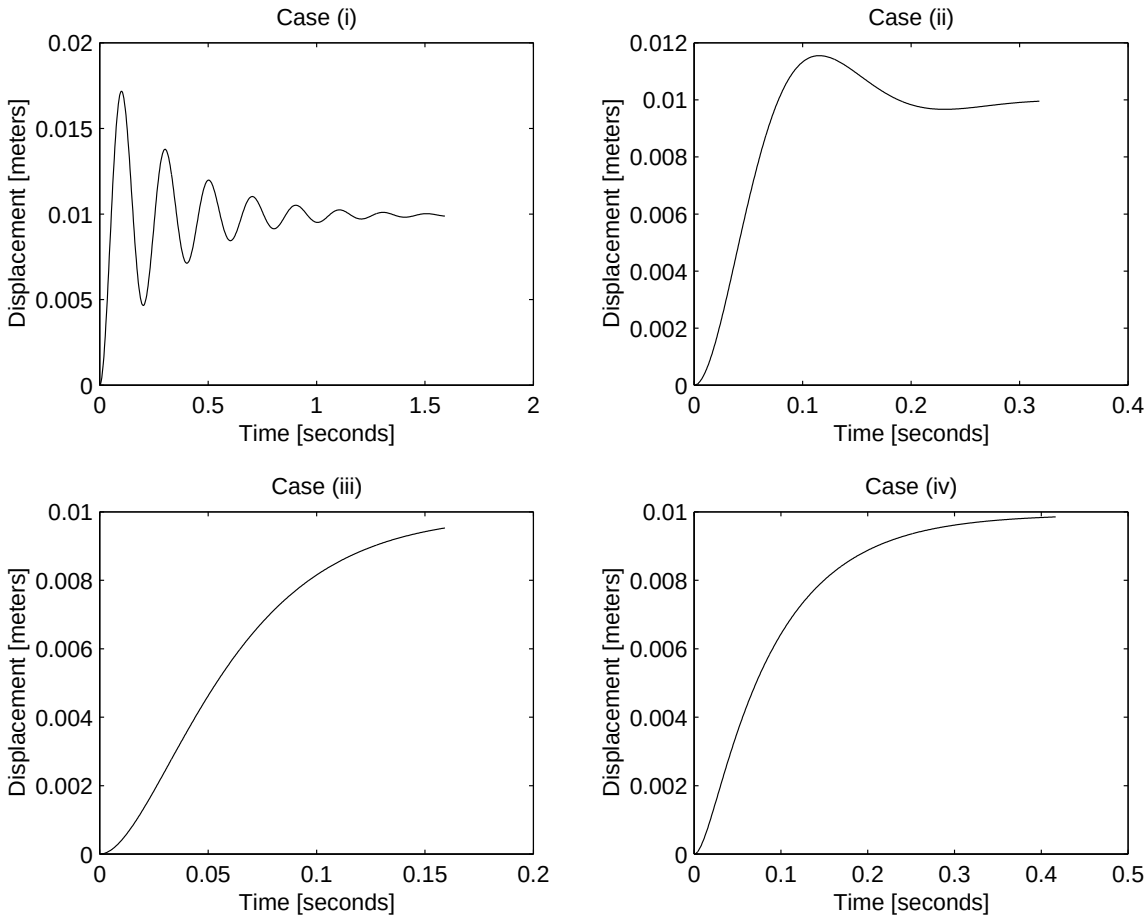


Figure 16.22.2: Part (a)

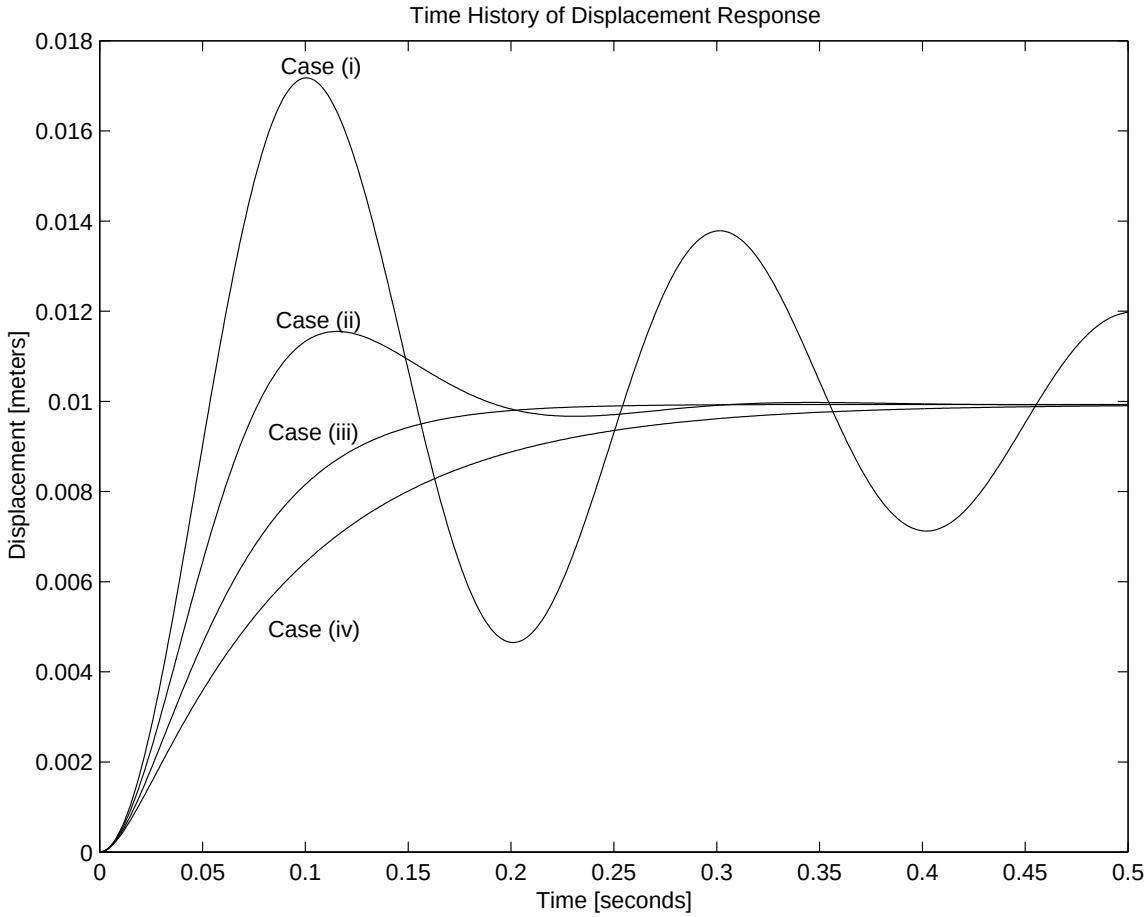


Figure 16.22.3: Part (a)

The results are displayed in Fig. 16.22.2. In Fig. 16.22.2 the duration of each response was 5 times the decay time constant of each case. An alternative comparison of the four Cases in Part (a) is displayed in Fig. 16.22.3 where all responses are plotted on the same time axis for a duration of 0.5 seconds.

(b) The desired behavioral parameters are $\omega_d = 31.42$ rad/sec (5 Hz) with the following values of damping ratio: Case (i) has $\zeta = 0.1$; Case (ii) has $\zeta = 0.3$; Case (iii) has $\zeta = 0.5$; Case (iv) has $\zeta = 0.7$. From the definitions of ω_d and ζ follow the formulas

$$k = \frac{m\omega_d^2}{1-\zeta^2} \qquad \text{and} \qquad b = \frac{2\zeta m\omega_d}{\sqrt{1-\zeta^2}}$$

The data for Part (b) are assembled in the following Table.

Case	ζ	k [N/m]	b [N/m/sec]	τ [sec]	5τ [sec]	ω_d [r/s]	$\zeta\omega_o$
(i)	0.1	2262	14.32	0.3167	1.584	31.42	0.3167
(ii)	0.3	2460	44.02	0.1012	0.5060	31.42	0.1012
(iii)	0.5	2985	82.28	0.05513	0.2757	31.42	0.0551
(iv)	0.7	4390	139.7	0.03247	0.1624	31.24	0.0325

In Part (b) all four Cases are handled by the MATLAB script 'MassSprgDmpr1.m'. The plots for durations of 5 times the decay time constant are displayed in Fig. 16.22.4.

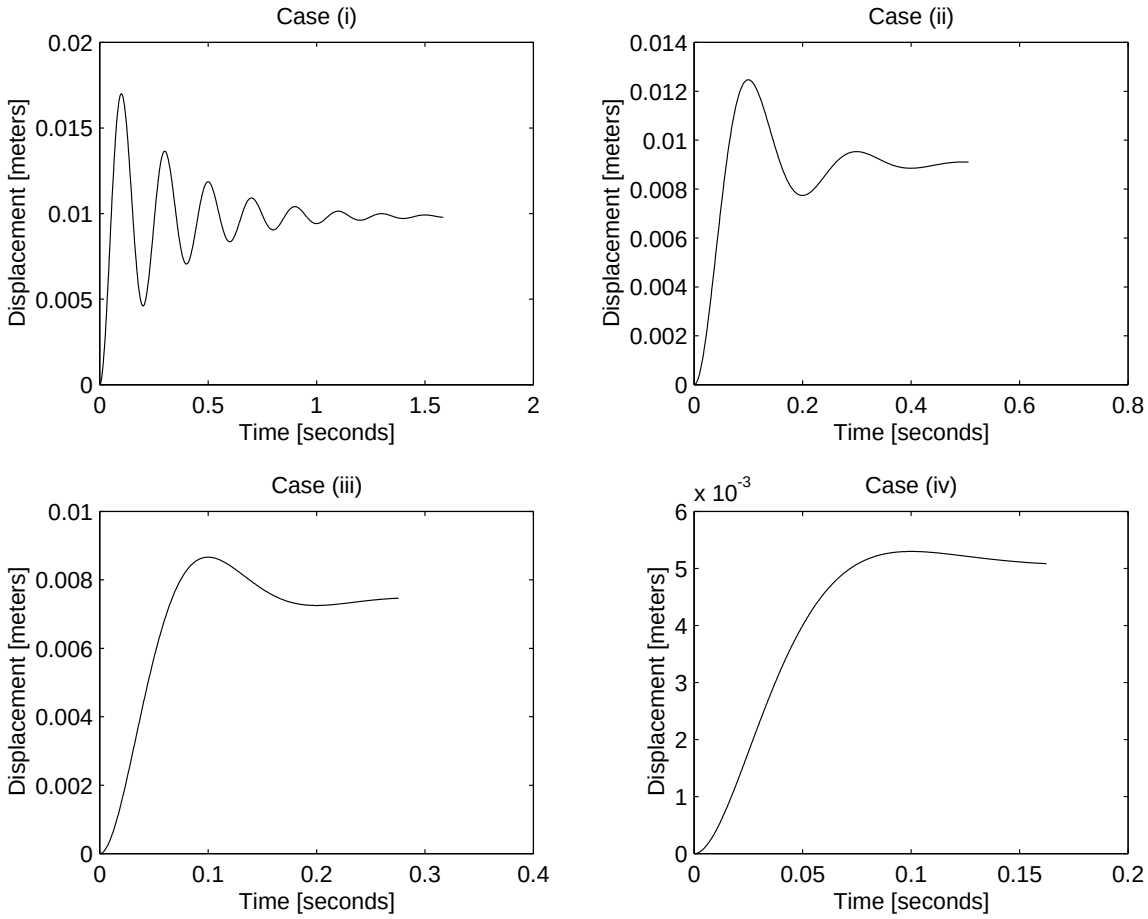


Figure 16.22.4: Part (b)

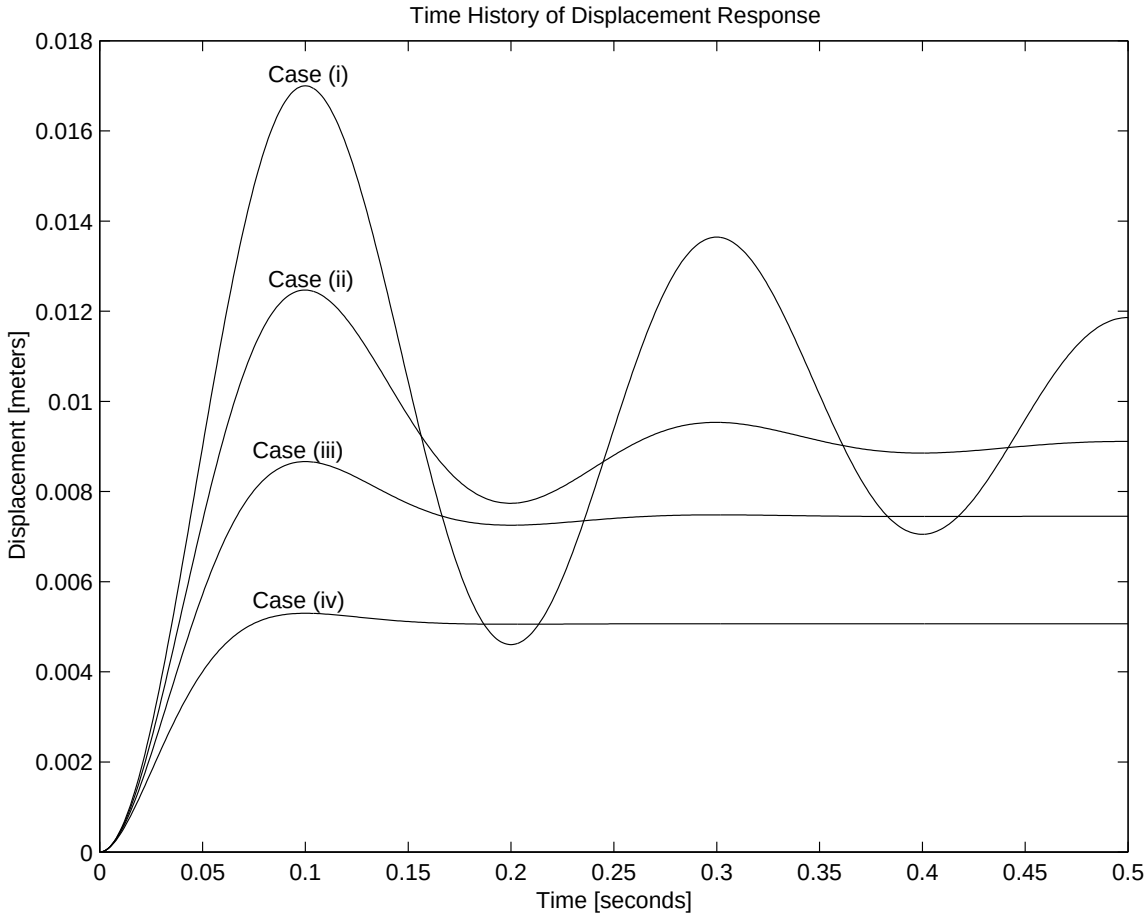


Figure 16.22.5: Part (b)

An alternative plot in which all four cases are plotted against the same time scale for a duration of 0.5 seconds is displayed in Fig. 16.22.5.

(c) The desired behavioral parameters are $\tau = 0.10$ seconds with (i) $\zeta = 0.3$, (ii) $\zeta = 0.5$, (iii) $\zeta = 0.7$, and (iv) $\zeta = 0.9$. From the definitions of τ and ζ

$$\tau = \frac{1}{\zeta\omega_o} \qquad\qquad 2\zeta\omega_o = \frac{b}{m}$$

follow the formulas

$$k = \frac{m}{\zeta^2\tau^2} \qquad \text{and} \qquad b = \frac{2m}{\tau}$$

The data for Part (c) are assembled in the following Table

Case	ζ	k [N/m]	b [N/m/sec]	τ [sec]	5τ [sec]	ω_d [r/s]	$\zeta\omega_o$
(i)	0.3	2520	45.36	0.1	0.5	31.80	10
(ii)	0.5	907.2	45.36	0.1	0.5	17.32	10
(iii)	0.7	462.9	45.36	0.1	0.5	10.20	10
(iv)	0.9	280.0	45.36	0.1	0,5	4.843	10

In Part (c) all four cases are handled by the MATLAB script 'MassSprgDmpr1.m'. The responses for durations of 5 times the decay time constant are plotted against the same time axis in Fig. 16.22.6.

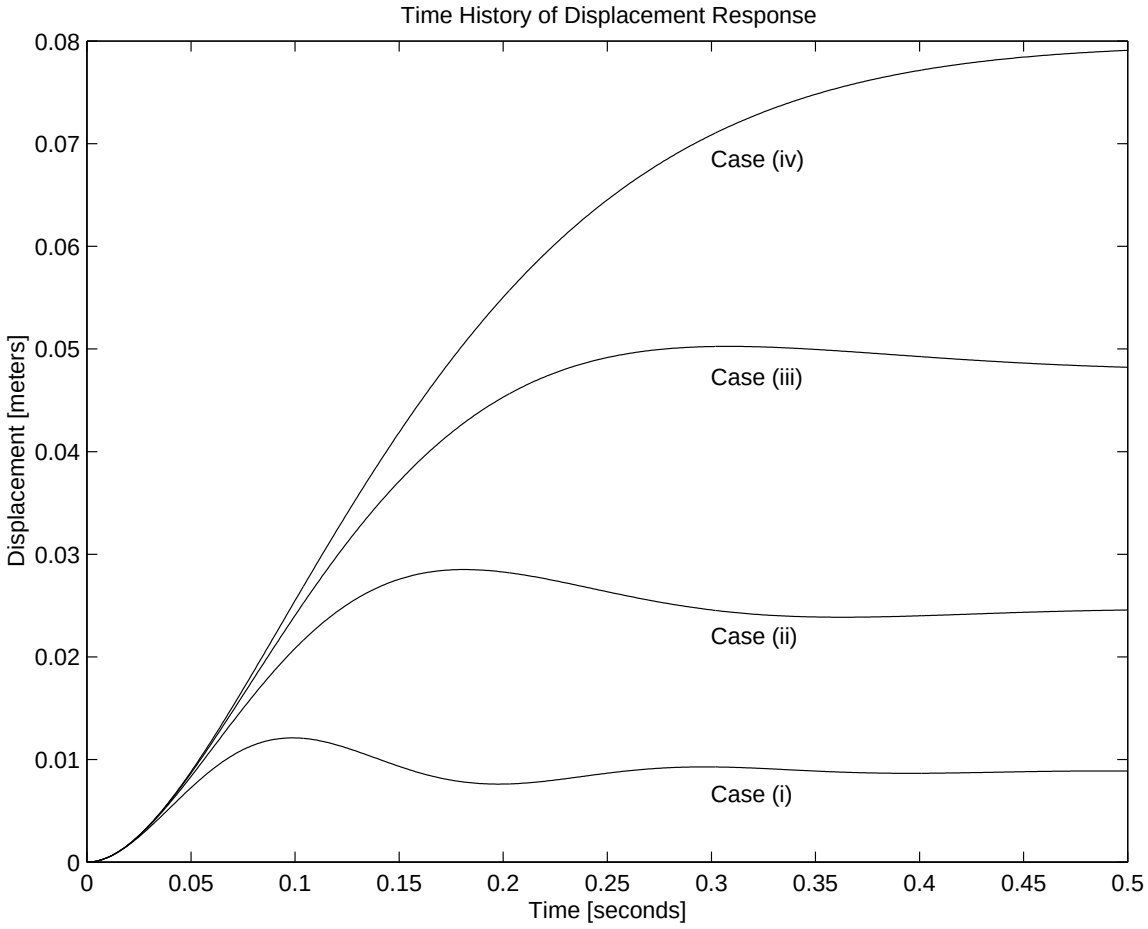


Figure 16.22.6: Part (c)

16.23 Plate On Springs Damped 2

The eigenvalues λ for the cases handled by 'MassSprgDmpr1.m' ($0 < \zeta < 1$) are complex conjugates of the form

$$\lambda = -\zeta\omega_o + i\omega_d \qquad \text{and} \qquad \lambda = -\zeta\omega_o - i\omega_d$$

The eigenvalues for the case handled by 'MassSprgDmpr2.m' ($\zeta = 1$) are a pair of repeated real roots: $\lambda = -\omega_o$ and $\lambda = -\omega_o$. The eigenvalues for the case handled by 'MassSprgDmpr3.m' ($1 < \zeta$) are $\lambda_1 = \omega_o(-\zeta + \sqrt{\zeta^2 - 1})$ and $\lambda_2 = \omega_o(-\zeta - \sqrt{\zeta^2 - 1})$. The values of $\zeta\omega_o$ and ω_d for the various cases are listed in the Tables in Problem 16.22. The eigenvalues in the upper half of the complex plane for Part (a) are plotted in Fig. 16.23.1. The complex roots for Cases (i) and (ii) have mirror image roots in the lower half-plane.

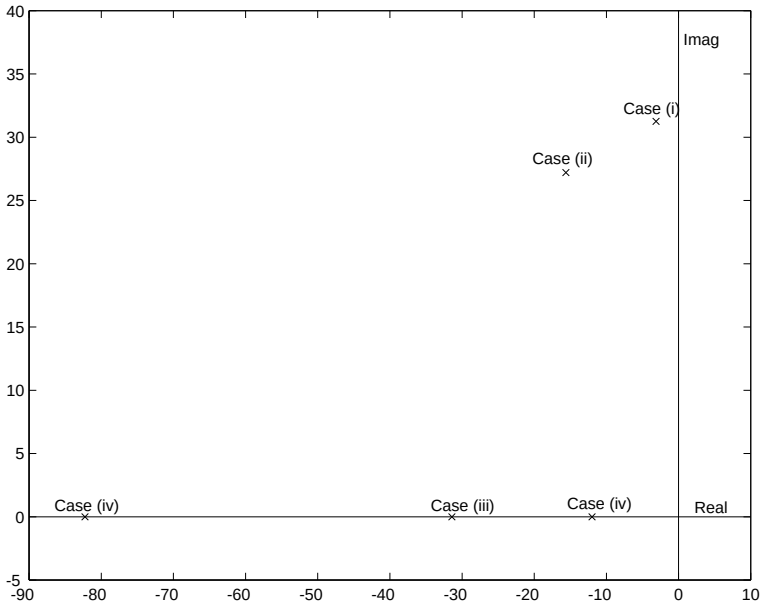


Figure 16.23.1: Eigenvalues for Part (a)

The eigenvalues in the upper half of the complex plane for Part (b) are plotted in Fig. 16.23.2, and the eigenvalues in the upper half of the complex plane for Part (c) are plotted in Fig. 16.23.3.

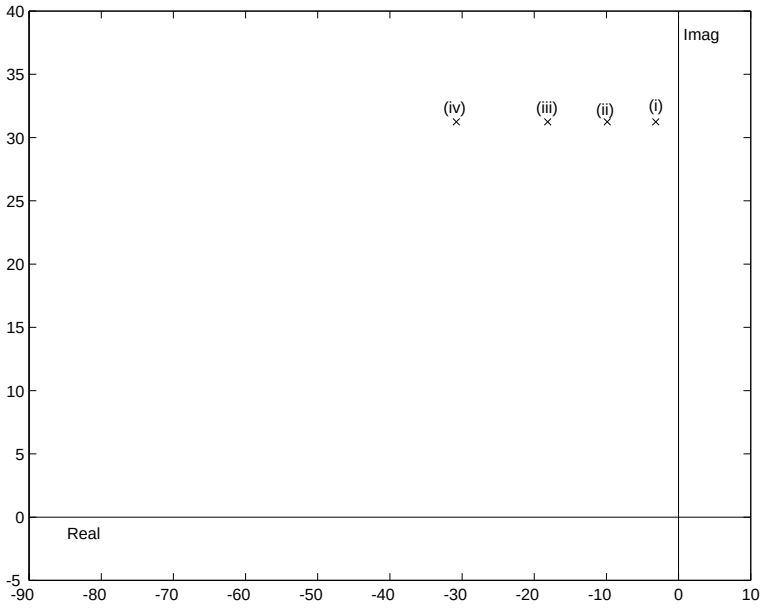


Figure 16.23.2: Eigenvalues for Part (b)

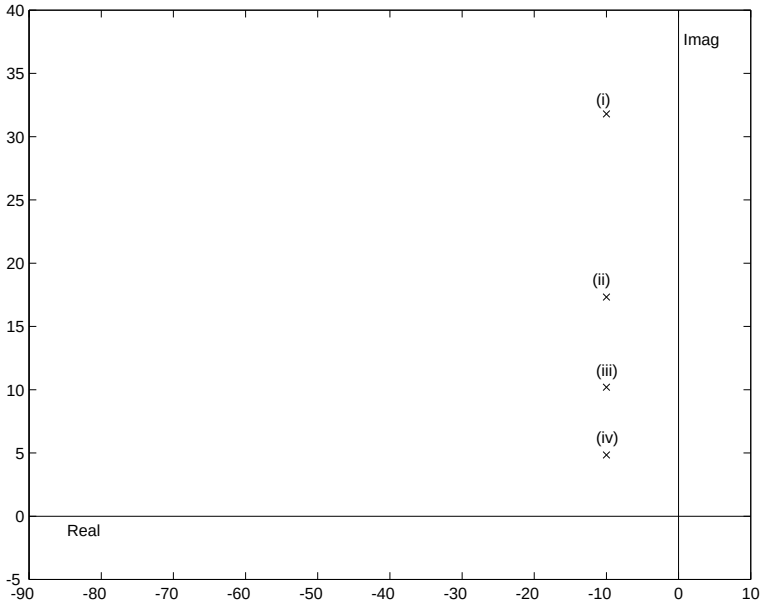


Figure 16.23.3: Eigenvalues for Part (c)

It is instructive to compare the responses from Problem 16.22 with the locations of the eigenvalues for Parts (a)-(c).

16.24 Plate On Springs Damped 3

Here the system of Problem 16.22 is excited, not by a suddenly applied force, but by the impact of a rubber ball at $t = 0$, which produces an initial velocity $v(0) = v_o$.

(a) The initial velocity is obtained by applying conservation of linear momentum to the impact.

$$\begin{aligned} \text{Downward momentum before impact} &= (2\text{pounds})(10 \text{ ft/sec}) \\ \text{Downward momentum after impact} &= (5\text{pounds})(v_o) + (2\text{pounds})(\text{-}3 \text{ ft/sec}) \end{aligned}$$

Equating these, yields $v_o = 5.2$ ft/sec $= 1.5850$ m/s (for convenience, it has been assumed that the positive direction for y_k and $v = dy_k/dt$ is downwards).

(b) A damped natural frequency of 5 Hz with $\zeta = 0.3$ is obtained with the parameters found for Case (ii) of Part (b) in Problem 16.22: $k = 2460$ N/m, and $b = 44.02$ N/m/s.

(c) When these parameters, plus the conditions $f_a = 0$, $y_k(0) = 0$, and $v(0) = 1.5850$ m/s, are input to the MATLAB script 'MassSprgDmpr1.m' the response history labeled "Response to impact of ball" in Fig. 16.24.1 is obtained.

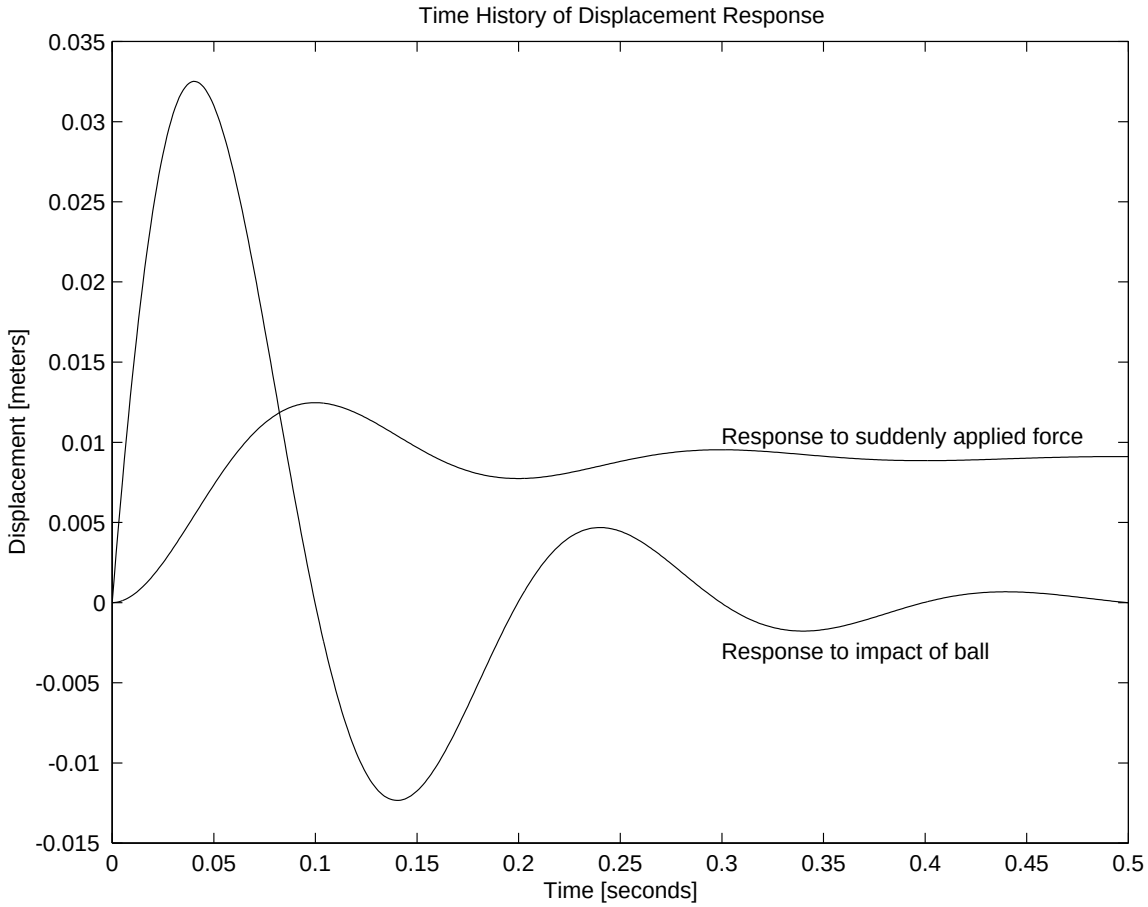


Figure 16.24.1: Comparison of responses

(d) In Fig. 16.24.1 the response to the impact of the ball is compared to the response of the same system to a step-function force of 5 pounds (Case (ii) of Part (b) in Problem 16.22). In both cases the motion starts from the original equilibrium position of the system. In both cases the mass oscillates for a few cycles and comes to rest. The damped natural frequency and damping ratio is the same in both oscillations. The final rest position is the same as the original equilibrium position in the ball-impact case, while the final rest position in the step-function force case is the new equilibrium position under a constant load of 5 pounds. The initial velocity in the step-function force case is zero (displacement response starts with zero slope), while the initial velocity in the ball-impact case is large due to the impact (displacement response starts with a steep slope). For the particular magnitudes of the excitations given the oscillation in the ball-impact case is more intense than the oscillation due to the suddenly applied force.

16.25 Plate On Springs Damped 4

Due to the application of a 1.0 Newton step-force, the system oscillates at its damped natural frequency ω_d about the new equilibrium position $y_k = \Delta$. From the given Displacement Response plot, the equilibrium offset Δ can be estimated to be $\Delta = 4.4$ x 10^{-4} meters. The damped natural frequency can be estimated by counting the cycles, 10, in 2.0 seconds, which gives a damped period $T_d = 0.2$ seconds, and $\omega_d = 2\pi/T_d = 31.4$ rad/sec. The logarithmic decrement LDR can be estimated by measuring the ratios of successive peak amplitudes. This is generally a difficult measurement to make accurately. One way to increase the accuracy is to make a large number of measurements based on different pairs of successive peaks, and average the results. Draw a line across the plot at $y = \Delta$ and measure the amplitudes of several peaks. Since only the ratio of successive peaks will be used, the measurements can be in terms of any convenient length unit. The measurements below are in millimeters.

Station	*Amplitude*	*Ratio*
0	6.22	–
1	4.86	0.781
2	4.00	0.823
3	3.08	0.770
4	2.56	0.831
5	1.94	0.758
6	1.68	0.866

The average of these first six ratios is 0.805, so the estimate for the LDR is $ln\, 0.805 = -0.217$.

(a) The estimated stiffness is

$$k = \frac{f_a}{\Delta} = \frac{1.0}{4.4 \times 10^{-4}} = 2270 \text{ Newtons/meter}$$

(b) To estimate the mass m, when the stiffness k is known, it is necessary to know the *undamped* natural frequency ω_o. At this point we know only the *damped* natural frequency ω_d. We can postpone estimation of the mass m until the damping ratio ζ is estimated in (c) below, or we can assume, since more than ten cycles of oscillation are visible, that the damping is sufficiently light to permit us to make the approximation that $\omega_o \approx \omega_d$, in which case

$$m \approx \frac{k}{\omega_d^2} = \frac{2270}{(31.4)^2} = 2.30 \text{ kg}$$

(c) To estimate the damping parameter b, we need to know the damping ratio ζ. The damping ratio follows from the log decrement ratio, LDR, according to the formula:

$$\zeta^2 = \frac{LDR^2}{\pi^2 + LDR^2}$$

Insertion of the estimation LDR $= -0.217$ yields $\zeta \approx 0.069$. (Note: With this value of ζ the undamped natural frequency is estimated as follows

$$\omega_o = \frac{\omega_d}{\sqrt{1-\zeta^2}} \approx \frac{31.4}{\sqrt{1-(0.069)^2}} = 31.5 \text{ rad/sec}$$

With this estimate for ω_o the mass m, previously estimated in (b) above at 2.30 kg, would now be estimated to be 2.29 kg. The damping parameter b is then estimated as

$$b = 2\zeta\omega_o m \approx 2(0.069)(31.5)(2.29) = 9.95 \text{ kg/sec, or } 9.95 \text{ N/m/sec}$$

16.26 Plate On Springs Damped 5

With two plate-on-springs units face-to-face, the effective stiffness of the combined system is $k = 1000$ N/m, and the effective damping parameter is $b = 10$ N/m/sec. For free vertical motion of the mass $m = 2.0$ kg, the displacement y_k from the equilibrium position of the model satisfies the equation

$$m\frac{d^2y_k}{dt^2} + b\frac{dy_k}{dt} + ky_k = 0$$

(a) The undamped natural frequency ω_o for the model is

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{2}} = 22.4 \text{ rad/sec, or } f_o = \frac{\omega_o}{2\pi} = 3.56 \text{ Hz}$$

(b) The damping ratio ζ for the model is

$$\zeta = \frac{b}{2\omega_o m} = \frac{10}{2(22.4)(2)} = 0.1116$$

(c) The decay time constant for the model is

$$\tau = \frac{1}{\zeta\omega_o} = \frac{1}{(0.1116)(22.4)} = 0.400sec$$

16.27 Plate On Springs Damped 6

To obtain a state-determined system, introduce the vertical velocity v of the plate. The *geometrical compatibility* requirement is

$$\frac{dy}{dt} = v$$

The *constitutive* equations are

$$f_{springs} = ky \qquad \text{and} \qquad f_{damping} = bv$$

and the *force balance* requirement is

$$f_m = -f_{springs} - f_{damping}$$

The dynamic state equations are two first-order differential equations for y and v

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= \frac{1}{m}f(t) - \frac{k}{m}y - \frac{b}{m}v \end{aligned}$$

(a) The deflection Δ under the weight mg is $\Delta = mg/k$ so the mass m is given by

$$m = \frac{k\Delta}{g} = \frac{(3000)(0.007)}{9.81} = 2.14 \text{ kg}$$

The undamped natural frequency is $\omega_o = \sqrt{k/m} = \sqrt{3000/2.14} = 37.4$ rad/sec, and the damping parameter is

$$b = 2\zeta\omega_o m = 2(0.5)(37.4)(2.14) = 80.0 \text{ N/m/s}$$

(b) The dynamic state equations can be rewritten in matrix form

$$\frac{d}{dt}\left\{\begin{array}{c} y \\ v \end{array}\right\} = \left[\begin{array}{cc} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{array}\right]\left\{\begin{array}{c} y \\ v \end{array}\right\} + \left\{\begin{array}{c} 0 \\ \frac{f_a}{m} \end{array}\right\}$$

or

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}f_a$$

where

$$\mathbf{x} = \left\{\begin{array}{c} y \\ v \end{array}\right\} \qquad \mathbf{A} = \left[\begin{array}{cc} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{array}\right] \qquad \mathbf{B} = \left\{\begin{array}{c} 0 \\ \frac{1}{m} \end{array}\right\}$$

where f_a is the magnitude of the constant applied force. These equations can be integrated, from prescribed initial values of y and v at $t = 0$ by the MATLAB command "ode45". A possible script for doing this is displayed at the end of the problem.

(c) When the scripts above are run with the following inputs

$m = 2.14$ kg

$k = 3000$ N/m

$b = 80.0$ N/m/s

$f_a = mg = (2.14)(9.81) = 21.0$ Newtons

the following three plots are obtained for a time interval of T = 0.35 seconds. There is very little activity in the interval between $t = 0.35$ and $t = 2.0$ seconds.

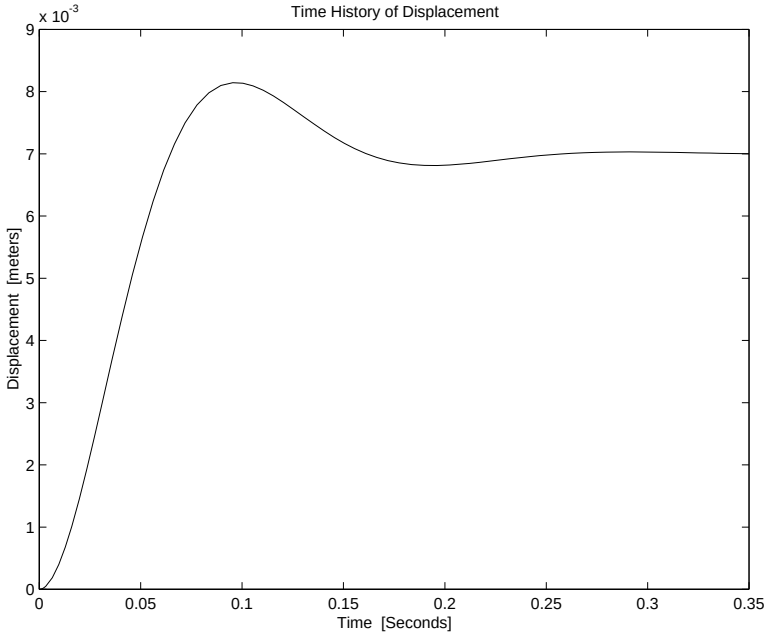


Figure 16.27.1: Plot of Position vs. Time

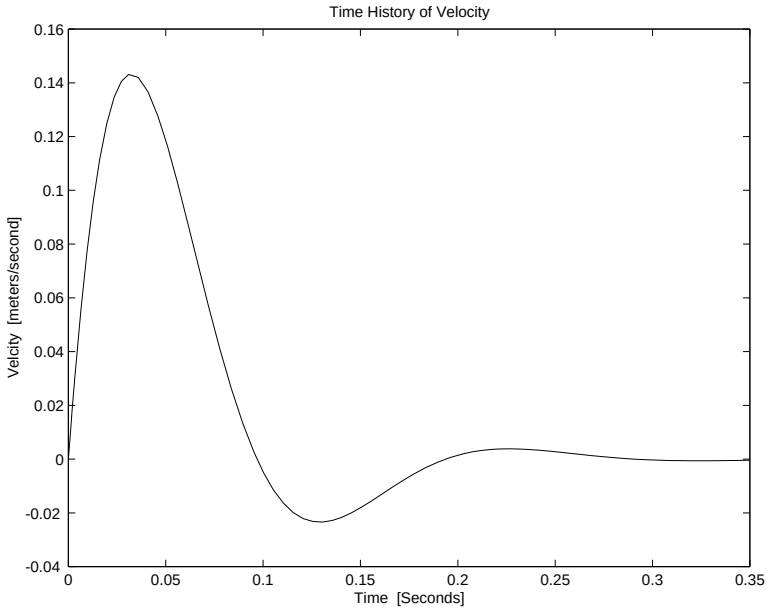


Figure 16.27.2: Plot of Velocity vs. Time

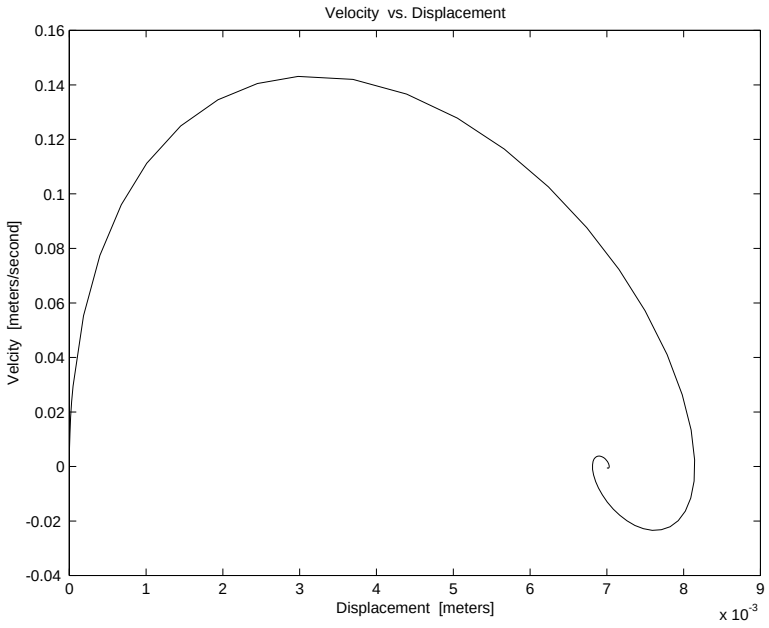


Figure 16.27.3: Plot of Velocity vs. Position

16.28 Shipping Crate 1

Let x be the displacement of the engine with respect to the stationary crate. The effective stiffness of the two end elements is $2k$, and the effective damping parameter is $2b$.

(a) The equation of motion for the engine is

$$m\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + 2kx = 0$$

(b) The engine weight is $W = 500$ pounds. Its mass is $m = W/g$. The values of k and b can be deduced from the given values of damped natural frequency $\omega_d = 2\pi$ and damping ratio $\zeta = 0.707$ from the equations

$$\omega_o^2 = \frac{\omega_d^2}{1-\zeta^2} = \frac{2k}{W/g} \qquad \text{and} \qquad 2b = 2\zeta\omega_o W/g$$

The undamped natural frequency is

$$\omega_o = \frac{2\pi}{\sqrt{1-(0,707)^2}} = 8.88 \text{ rad/sec}$$

and

$$k = \frac{W\omega_o^2}{2g} = \frac{(500)(8.88)^2}{2(386)} = 51.2 \text{ pounds/inch}$$

The damping parameter is

$$b = \zeta\omega_o\frac{W}{g} = (0.707)(8.88)\frac{500}{386} = 8.13 \text{ pounds/in/sec}$$

16.29 Shipping Crate 2

Fig. 16.29.1 shows an idealized model of the engine of mass $m = W/g$ with shock-absorbing packaging, represented by the springs with stiffness k and damping parameter b, in a crate which is fixed to a truck. The velocity of the truck, with respect to an inertial reference frame, is v_{truck}, and the velocity of the engine, with respect to an inertial reference frame, is v_{engine}. The relative displacement of the engine with respect to the crate is x_{rel}. The origin for x_{rel} is taken to be the equilibrium position of the engine in the motionless crate. In problems like this, where relative motion is involved, it is important to remember that Newton's law only applies to motions with respect to an *inertial* reference frame

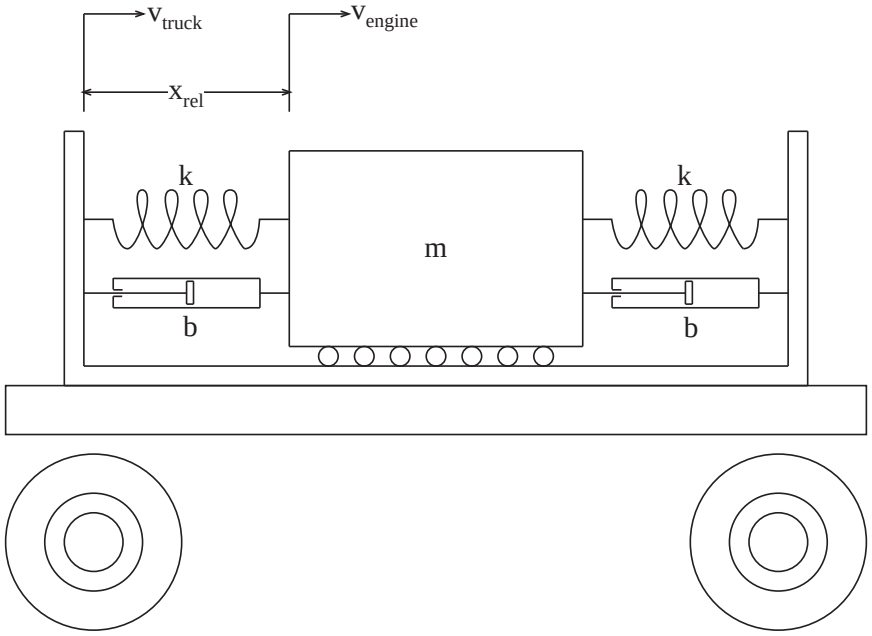


Figure 16.29.1: Model of Engine in Crate on Moving Truck

(a) The *geometric compatibility* conditions are

$$\frac{dx_{rel}}{dt} = v_{rel} \qquad \text{and} \qquad v_{engine} = v_{truck} + v_{rel}$$

The *constitutive equations* are

$$f_m = m\frac{dv_{engine}}{dt}, \qquad f_{spring} = kx_{rel}, \qquad \text{and} \qquad f_{damping} = bv_{rel}$$

and the *force balance* condition is

$$f_m = -2f_{spring} - 2f_{damping}$$

The six preceding equations constitute a mathematical model for the longitudinal motion of the engine with respect to the crate.

(b) A single differential equation in terms of x_{rel} can be derived by inserting the spring force $f_{spring} = kx_{rel}$ and the damping force $f_{damping} = bdx_{rel}/dt$ in the force-balance equation,

along with f_m expressed as $f_m = m(dv_{truck}/dt + d^2x_{rel}/dt^2)$. The result is

$$m\frac{d^2x_{rel}}{dt^2} + 2b\frac{dx_{rel}}{dt} + 2kx_{rel} = -m\frac{dv_{truck}}{dt}$$

The form of this equation is similar to that for the equation for the displacement of the steel plate on springs. Here the unknown response is a *relative* displacement and the driving force is the negative of the force that would be required to give the engine the same acceleration as the truck. A single differential equation for the relative *velocity* of the engine with respect to the crate is obtained by differentiating every term in the preceding equation.

$$m\frac{d^2v_{rel}}{dt^2} + 2b\frac{dv_{rel}}{dt} + 2kv_{rel} = -m\frac{d^2v_{truck}}{dt^2}$$

(c) A state-determined system for the state variables x_{rel} and v_{eng} can be obtained by expressing the forces in the force-balance equation of (a) in terms of x_{rel} and $v_{rel} = v_{eng} - v_{truck}$. The two first-order differential equations for the state-determined system are

$$\begin{aligned} \frac{dx_{rel}}{dt} &= v_{eng} - v_{truck} \\ \frac{dv_{eng}}{dt} &= -\frac{k}{m}x_{rel} - \frac{b}{m}v_{eng} + \frac{b}{m}v_{truck} \end{aligned}$$

and the desired output is obtained from

$$v_{rel} = v_{eng} - v_{truck}$$

In matrix form

$$\frac{d\mathbf{X}}{dt} = \mathbf{AX} + \mathbf{Bu} \qquad \text{with} \qquad y = \mathbf{CX} + \mathbf{Du}$$

with

$$\mathbf{X} = \begin{Bmatrix} x_{rel} \\ v_{eng} \end{Bmatrix} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \quad \mathbf{B} = \begin{Bmatrix} -1 \\ \frac{b}{m} \end{Bmatrix} \quad \mathbf{C} = \{\, 0 \;\; 1 \,\} \quad \mathbf{D} = [-1]$$

and $y = v_{rel}$ and $u = v_{truck}$.

17 Quiz Problems

17.1 Fun with Block Diagrams

(a)

$$\begin{aligned} T(s) &= \frac{K}{s^2 + 20s + K} \\ \omega_n &= \sqrt{K} \\ 2\zeta\omega_n &= 2\sqrt{K} = 20 \\ K &= 100 \end{aligned}$$

(b)

$$\begin{aligned} \frac{V_{out}}{V_1} &= \frac{(s+3)(6s+1)}{(s+3)(6s+1) + (8s+7)} \\ \frac{V_{out}}{V_2} &= \frac{(6s+1)}{(s+3)(6s+1) + (8s+7)} \end{aligned}$$

(c) Solve using superposition

$$\begin{aligned} V_{out}(6s^2 + 27s + 10) &= V_1(6s^2 + 19s + 3) + V_2(6s+1) \\ 6\ddot{V}_{out} + 27\dot{V}_{out} + 10V_{out} &= 6\ddot{V}_1 + 19\dot{V}_1 + 3V_1 + 6\dot{(V)}_2 + V_2 \end{aligned}$$

17.2 Complex Translation

The solution for this problem is not available.

17.3 Cylinder Step Response

The transfer function for this system is

$$\frac{x(s)}{f(s)} = \frac{1}{m_{eq}s^2 + cs + k}$$
$$\omega_n = \sqrt{\frac{k}{m_{eq}}}$$
$$2\zeta\omega_n = \frac{c}{m_{eq}}$$

(a) From graph, we measure the following

$$T \approx 1.0s \Rightarrow \omega_d = \frac{2\pi}{T} = 6.28r/s$$
$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$
$$M_p \approx 100\frac{0.75 - 0.5}{0.5} = 50$$
$$\zeta = \frac{A}{\sqrt{\pi^2 + A^2}} = 0.215$$
$$A = ln\frac{100}{M_p} = 0.693$$
$$\omega_n = 6.43r/s$$
$$m_{eq} = \frac{\omega_n^2}{k} = 4.8 \approx 5kg$$
$$c = 2\zeta\omega_n m_{eq} = 13.8Ns/m \approx 14Ns/m$$

Alternately, you could determine ζ using the log decrement method.

(b)

$$m_{eq} = m + \frac{I}{r^2}$$
$$I = 0.5\ kg\, m^2$$

17.4 JKC Frequency Response 1

The transfer function for this system is

$$\frac{\omega(s)}{\phi(s)} = \frac{k}{Js^2 + cs + k}$$
$$\text{Thus } \omega_n = \sqrt{\frac{k}{J}}$$
$$2\zeta\omega_n = \frac{c}{J}$$

(a) There are a couple of ways to solve this part of the problem. First, you can read $\omega_r = 9\, r/s$ and $M_p = 5\, dB$ from the bode plot and use the following relationships

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$
$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

to find $\zeta \approx 0.3$ and $\omega_n \approx 10\, r/s$. Or you can read $\omega_n = 10\, r/s$ directly from the phase plot ($\theta = -90^\circ$)

(b) $k = 1500$ Nm/r, $c = 90$ Nms/r

(c)

$$\omega = 1.1\, r/s \;\; \theta(t) \approx \sin(1.1t + 0)$$
$$\omega = 10\, r/s \;\; \theta(t) \approx 1.58\sin(10t - \pi/2)$$
$$\omega = 20\, r/s \;\; \theta(t) \approx 0.3\sin(20t - 2.75)$$

17.5 Mass Spring Damper Dynamics

The solution for this problem is not available.

17.6 Op-Amp Analysis

(a)

$$\frac{v_o}{v_i} = \frac{R_2 C_1 s}{(R_2 C_2 s + 1)(R_1 C_1 s + 1)}$$

(b)

$$M(\omega) = \frac{R_1 C_1 \omega}{\sqrt{(1 - R_1 R_2 C_1 C_2 \omega^2)^2 + ((R_1 C_1 + R_2 C_2)\omega)^2}}$$
$$\phi(\omega) = 90^\circ - \tan^{-1}\left(\frac{(R_1 C_1 + R_2 C_2)\omega}{(1 - R_1 R_2 C_1 C_2 \omega^2)}\right)$$

(c) Figure 17.6.1 shows the Bode plot for this system.

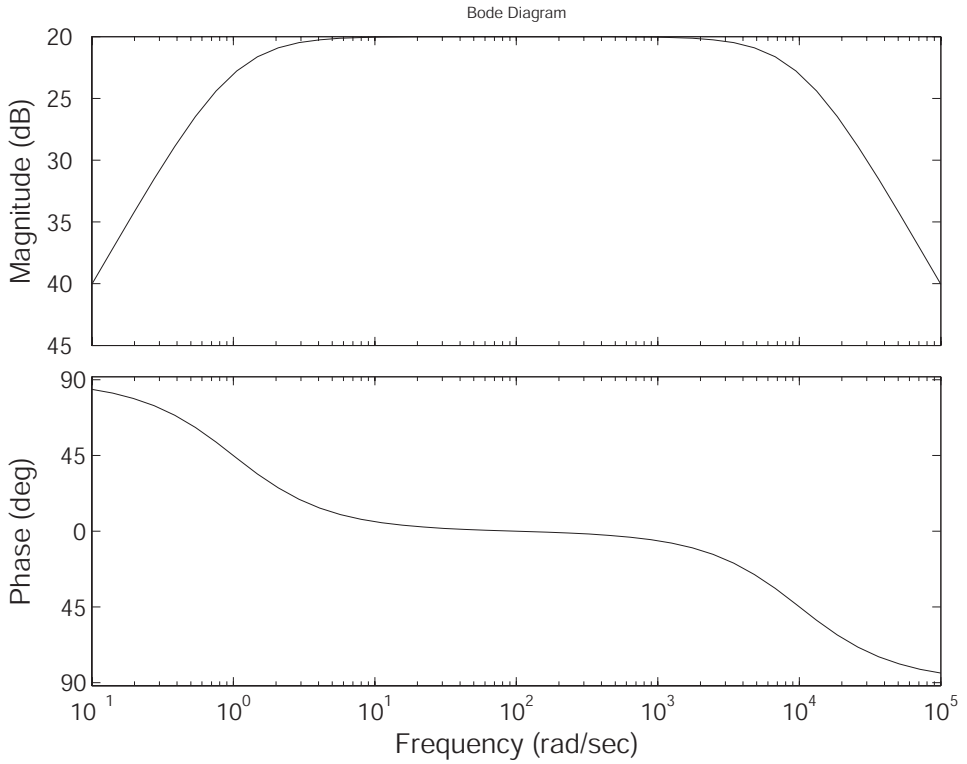


Figure 17.6.1: Bode Diagram

(d) Figure 17.6.2 shows the pole-zero plot for this system.

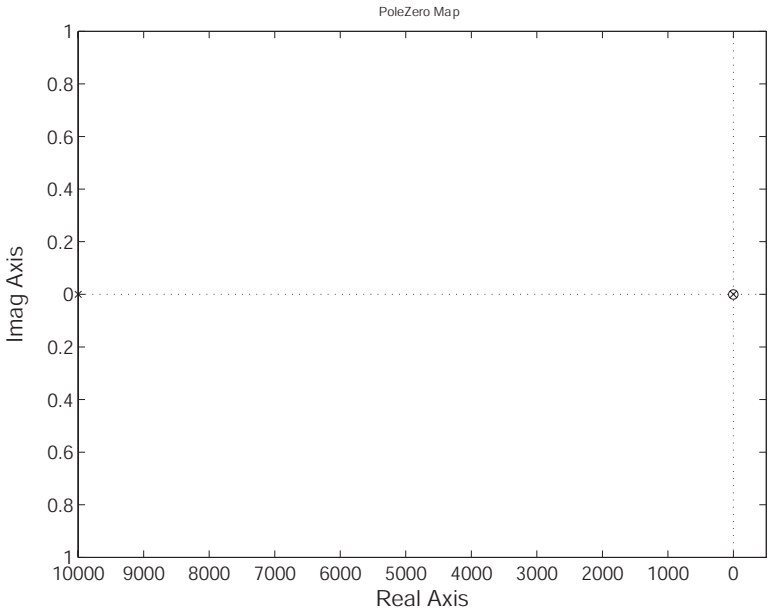


Figure 17.6.2: Pole-Zero Plot

17.7 RLC Circuit Analysis

(a)

$$\frac{V_o}{V_i} = \frac{1}{LCs^2 + RCs + 1}$$

(b)

$$\omega_n = 2 * \pi * 5000 = 31,400\, r/s = \frac{1}{\sqrt{LC}}$$
$$L = \frac{1}{\omega_n^2 C} = 0.001\, H = 1\, mH$$
$$\frac{R}{L} = 2\zeta\omega_n = 2 * 0.707 * 31,400$$
$$R = 44.4\, \Omega$$

(c) There are no zeros, poles at roots of

$$s^2 + \frac{1000}{0.001}s + \frac{1}{1e-6 * 1e-3} = 0$$
$$s_1 \approx -1e6$$
$$s_2 \approx -1e3 \text{ dominant pole}$$
$$x_{ss} = 1$$

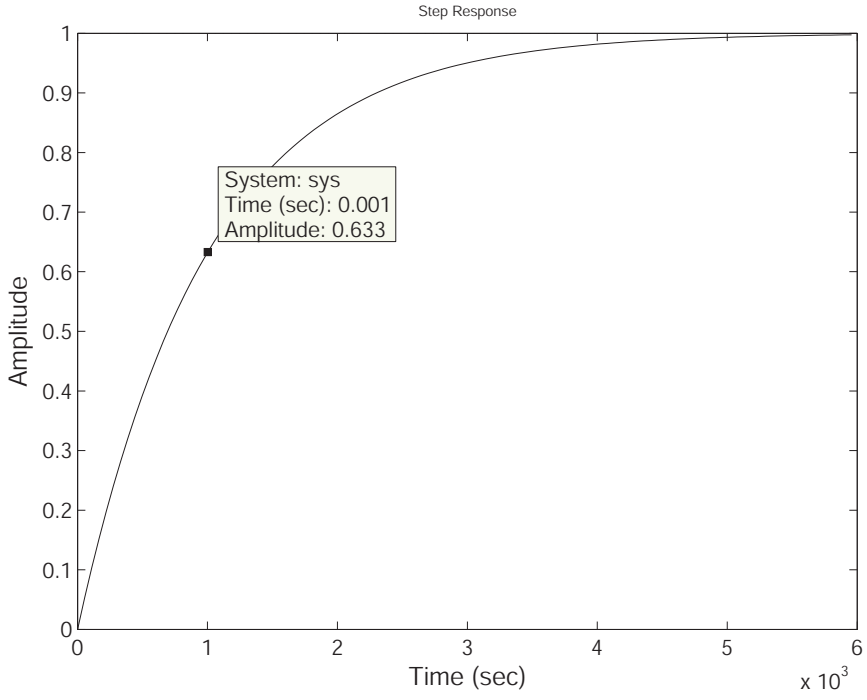


Figure 17.7.1: Step Response for part (c)

(d) The bode plot is shown in Figure 17.7.2

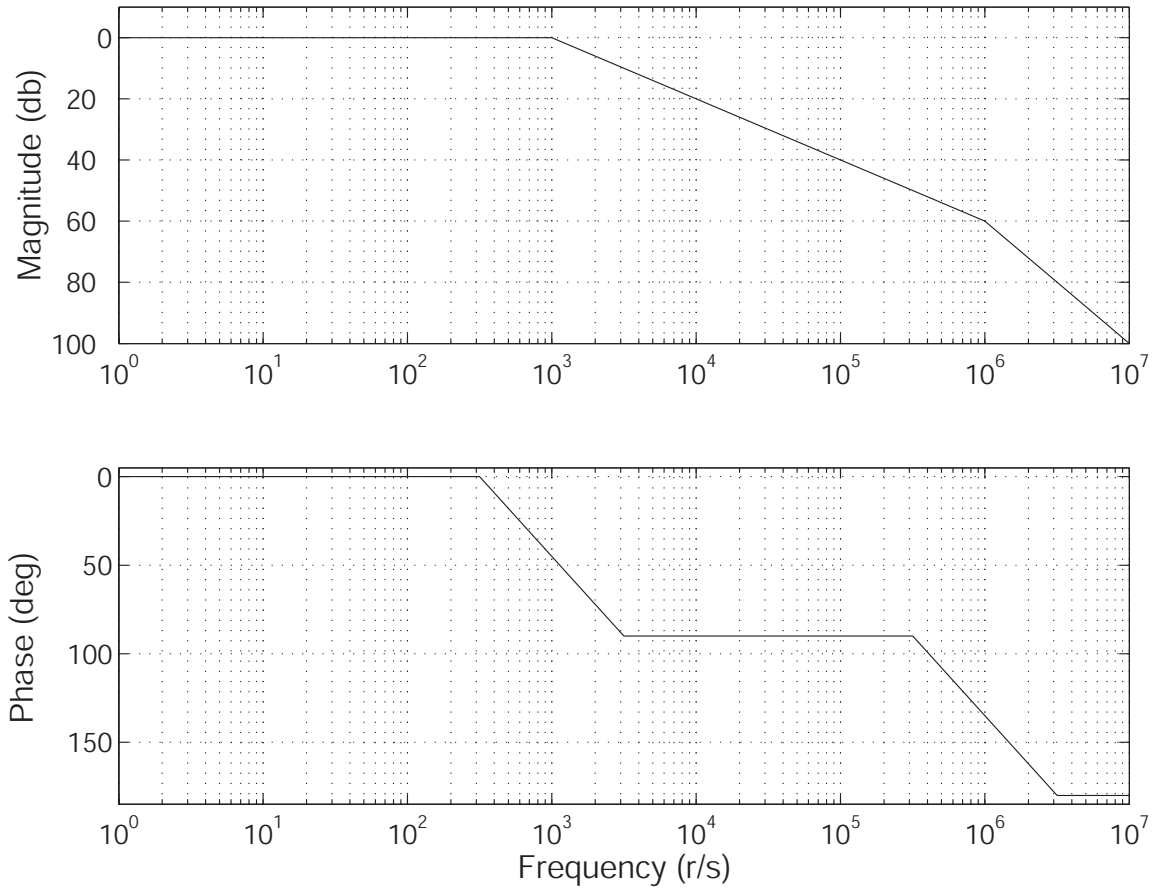


Figure 17.7.2: Bode plot for part (d)

(e) $s = -1e3 = s_2$

(f) (i)

$$\frac{v_o}{v_i} = \frac{R_2}{R_2LCs^2 + (R_1R_2C + L)s + R_1 + R_2}$$

(ii)

$$\frac{v_o}{v_i} = \frac{R_2Cs + 1}{LCs^2 + (R_1 + R_2)Cs + 1}$$

17.8 Sailing for Engineers

The solution for this problem is not available.

17.9 Second Order Step Response

The solution for this problem is not available.

17.10 Spring Damper Dynamics

The solution for this problem is not available.

17.11 Derive Blocks for Op-amp Circuit

The solution for this problem is not available.

17.12 Automobile Fender Spring/Damper System

The solution for this problem is not available.

17.13 Cu Flywheel with Eddy Current Damper

The solution for this problem is not available.

17.14 Current Driven RC Circuit

The solution for this problem is not available.

17.15 Driven Mass Spring System

The solution for this problem is not available.

17.16 Flywheel Driven by Hanging Mass

The solution for this problem is not available.

17.17 Homogeneous Second Order DE

The solution for this problem is not available.

17.18 Match Pole/Zero Plots with Step Response

The solution for this problem is not available.

17.19 Opamp Block Transfer Functions

The solution for this problem is not available.

17.20 Piston with 2nd Order Translation

The solution for this problem is not available.

17.21 Pole Zero Bode Matching

The solution for this problem is not available.

17.22 Power Semiconductor Thermal Problem

The solution for this problem is not available.

17.23 Submersible Capsule Hoist System

The solution for this problem is not available.

17.24 Transfer Function from Pole Zero Plot

The solution for this problem is not available.

17.25 Vaccine Cooler

The solution for this problem is not available.

17.26 Voltage Driven RRC Circuit

The solution for this problem is not available.

17.27 Linear Mechanical System with Position Input

The solution for this problem is not available.

17.28 Tank with Pump Inlet Lower than Outlet

The solution for this problem is not available.

17.29 Thermal Power Chip Analysis

The solution for this problem is not available.

17.30 Current Divider in Simple Network

The solution for this problem is not available.

17.31 Voltage and Current Source Driving network

The solution for this problem is not available.

17.32 Rotary Motor with Damper

The solution for this problem is not available.

17.33 Patient Rehabilitation

The solution for this problem is not available.

17.34 Delay Transfer Function

The solution for this problem is not available.

17.35 Laplace and Complex Numbers

The solution for this problem is not available.

17.36 Mechanical Stiffness

The solution for this problem is not available.

18 Math Techniques

18.1 Complex Expression Reduction

(a)

$$(2+3i)(4-5i) = 23+2i \tag{1}$$
$$\frac{2+3i}{4-5i} = -0.1707+0.5366i \tag{2}$$
$$(-6+7i)*(3+7i) = -67.0000-21.0000i \tag{3}$$
$$\frac{-6+7i}{3+7i} = 0.5345+1.0862i \tag{4}$$

(b) First write the complex numbers in

$$2+3i = \sqrt{13}e^{i\arctan 1.5} \approx \sqrt{13}e^{0.9828i} \tag{5}$$
$$4-5i = \sqrt{41}e^{i\arctan -1.25} \approx \sqrt{41}e^{-0.8961i} \tag{6}$$
$$-6+7i = \sqrt{85}e^{i(\arctan(-7/6)+\pi)} \approx \sqrt{85}e^{2.2794i} \tag{7}$$
$$3+7i = \sqrt{58}e^{i\arctan(7/3)} \approx \sqrt{58}e^{1.1659i} \tag{8}$$

To compute products of complex numbers in polar form, we can use the mnemonic **MMAA**, Moduli Multiply Arguments Add, where the modulus of $z = re^{i\phi}$ is r and the argument is ϕ. We find

$$(2+3i)(4-5i) = 23.0868e^{0.0867i} \tag{9}$$
$$\frac{2+3i}{4-5i} = 0.5631e^{i1.8788} \tag{10}$$
$$(-6+7i)*(3+7i) = 70.2140e^{-2.8379i} \tag{11}$$
$$\frac{-6+7i}{3+7i} = 1.2106e^{1.1135i} \tag{12}$$

18.2 Complex Expressions

$$(1+2i)(3-4i) = 11+2i \tag{22}$$

$$\frac{1+2i}{3-4i} = \frac{1+2i}{3-4i}\frac{3+4i}{3+4i} = -\frac{1}{5}+\frac{2}{5}i \tag{23}$$

$$(-6+4i)(5+7i) = -58-22i \tag{24}$$

$$\frac{-6+4i}{5+7i} = -\frac{1}{37}+\frac{31}{37}i \tag{25}$$

Now we wish to write the complex numbers $a+bi$ in polar form $re^{i\phi}$ using

$$r = \sqrt{a^2+b^2} \tag{26}$$

$$\phi = \arctan\frac{b}{a} \tag{27}$$

When computing the angle ϕ, one must make sure that proper angle is calculated based on which quadrant of the complex plane contains the complex number (In Matlab, one may prefer to use the command `atan2` instead of `atan`.).

$$1+2i = \sqrt{5}e^{i\arctan 2} \approx \sqrt{5}e^{1.1071i} \tag{28}$$

$$3-4i = 5e^{i\arctan(-4/3)} \approx 5e^{-0.9273i} \tag{29}$$

$$-6+4i = 2\sqrt{13}e^{i\arctan(-2/3)} \approx 2\sqrt{13}e^{2.5536} \tag{30}$$

$$5-7i = \sqrt{74}e^{i\arctan(-7/5)} \approx \sqrt{74}e^{-0.9505i} \tag{31}$$

To compute products of complex numbers in polar form, we can use the mnemonic **MMAA**, Moduli Multiply Arguments Add, where the modulus of $z = re^{i\phi}$ is r and the argument is ϕ. We find

$$(1+2i)(3-4i) = 5\sqrt{5}e^{0.1798i} \tag{32}$$

$$\frac{1+2i}{3-4i} = \frac{\sqrt{5}}{5}e^{2.0344i} \tag{33}$$

$$(-6+4i)(5+7i) = 2\sqrt{962}e^{1.4931i} \tag{34}$$

$$\frac{-6+4i}{5+7i} = 2\sqrt{\frac{13}{74}}e^{-2.7791i} \tag{35}$$

Figure 18.2.1:

18.3 Matrix Operation Practice

Part (a)

$$-5\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}-5\\-10\\-15\end{pmatrix}$$

Part (b)

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}9\\8\\0\end{pmatrix} = 25$$

Part (c)

$$\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}9 & 8 & 0\end{pmatrix} = \begin{pmatrix}9 & 8 & 0\\18 & 16 & 0\\27 & 24 & 0\end{pmatrix}$$

Part (d), the roots of $s^2 + 2s + 3$ are

$$s = -1 \pm \sqrt{2}i$$

Figure 18.3.1:

19 Recitation Problems

19.1 Recitation 1 Problem

19.2 Recitation 2 Problem

(a) From class notes the characteristic equation is:

$$m\ddot{x} + b\dot{x} + kx = 0$$

The roots/eigenvalues of this equation are at (using a shorter notation than class):

$$s_{1,2} = -\frac{b}{2m} \pm \frac{\sqrt{b^2 - 4mk}}{2m}$$

A simple check shows that the roots are distinct and imaginary ($b^2 - 4mk = -4 \times 10^6$), so we can write the solution in the form:

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} = 2e^{-\sigma t}\left(\alpha \cos \omega_d t - \beta \sin \omega_d t\right)$$

Evaluating the homogeneous response at $t = 0$ we have:

$$x(0) = x_0 = 2\alpha \qquad \text{or} \qquad \alpha = \frac{1}{2}x_0$$

Differentiating the homogeneous response and evaluating at $t = 0$ (see Chapter 1 for all the details):

$$\dot{x}(0) = \dot{x}_0 = 2(-\sigma\alpha - \beta\omega_d) \qquad \text{or} \qquad \beta = -\frac{\dot{x}_0 + \sigma x_0}{2\omega_d}$$

Case 1: $x_0 = 0$ and $\dot{x}_0 = 1$ Plugging in values we get $\alpha = 0$ and $\beta = -\frac{1}{2\omega_d}$ which leads to:

$$x(t) = \frac{1}{\omega_d} e^{-\sigma t} \sin \omega_d t$$

Case 1: $x_0 = 1$ and $\dot{x}_0 = 0$ Plugging in values we get $\alpha = .5$ and $\beta = -\frac{\zeta}{2\sqrt{1-\zeta^2}}$ which leads to:

$$x(t) = e^{-\sigma t}\left(\cos \omega_d t + \frac{\zeta}{2\sqrt{1-\zeta^2}} \sin \omega_d t\right)$$

(b)

$$\begin{aligned}
\zeta &= \frac{b}{2\sqrt{km}} = \frac{1}{\sqrt{2}} = .707 \\
\omega_n &= \sqrt{\frac{k}{m}} = 2\sqrt{2} = 2.8242 \text{ [rad/sec]} \\
\omega_d &= \omega_n\sqrt{1-\zeta^2} = 2\sqrt{2}(\sqrt{1-.5}) = 2 \text{ [rad/sec]} \\
\sigma &= \zeta\omega_n = \frac{1}{\sqrt{2}}2\sqrt{2} = 2 \text{ [rad/sec]} \\
\tau &= \frac{1}{\sigma} = .5 \text{ [sec]}
\end{aligned}$$

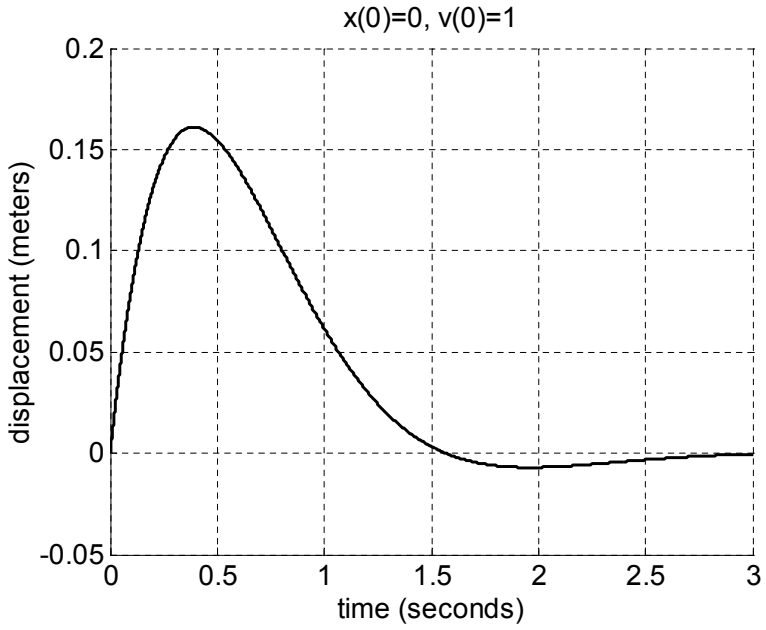


Figure 19.2.1: Nominal case, $x_0 = 0$, $\dot{x}_0 = 1$, $\zeta = 0.707$

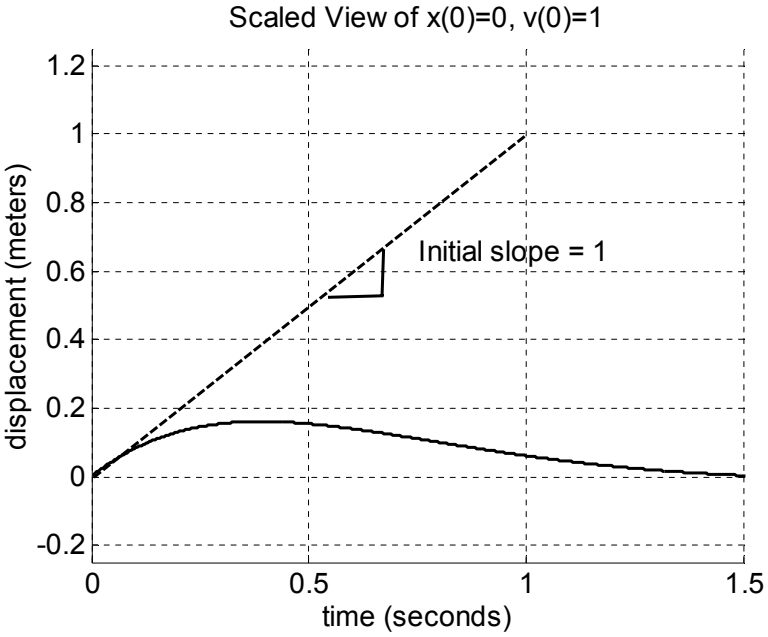


Figure 19.2.2: Close up Nominal case, $x_0 = 0$, $\dot{x}_0 = 1$, $\zeta = 0.707$ show that the initial velocity is equal to the slope

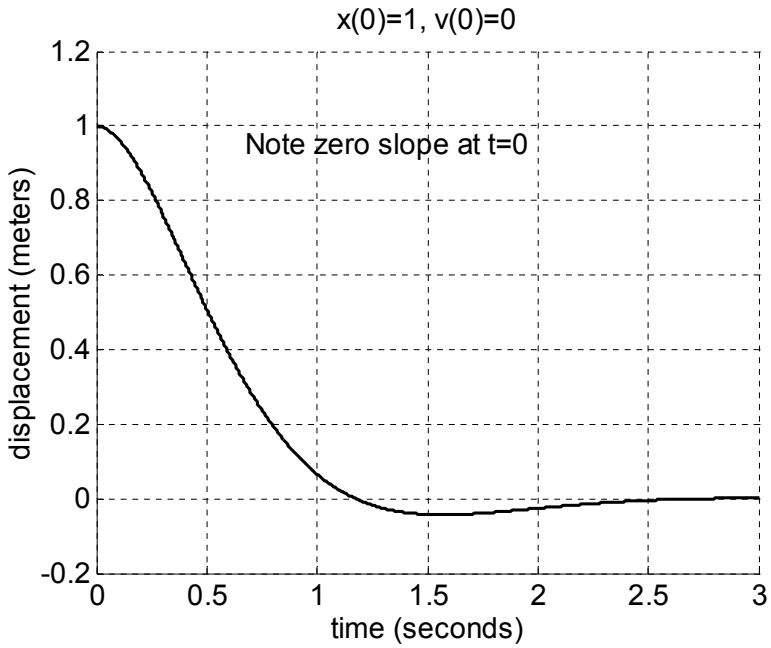


Figure 19.2.3: Close up Nominal case, $x_0 = 1$, $\dot{x}_0 = 0$, $\zeta = 0.707$ showing non-zero initial position, zero initial velocity

(c) The damping ratio $\zeta = .707$ is between zero and one so by definition the system is underdamped. Professor Dubowsky jokes that the value of .707 is "just right" because it is the best compromise between speed of response and overshoot.

(d) The exponential envelope term is $e^{-\sigma t}$. The attenuation σ is always greater than zero, so this decaying exponential is stable and will not increase without bounds, which matches our physical intuition that a shock absorber's purpose is to decrease motion.

(e) See the following plots, which all have the same mass and spring constant, but different dampers b. Initial conditions are $x_0 = 0$, $\dot{x}_0 = 1$. All the following curves are drawn to the same scale.

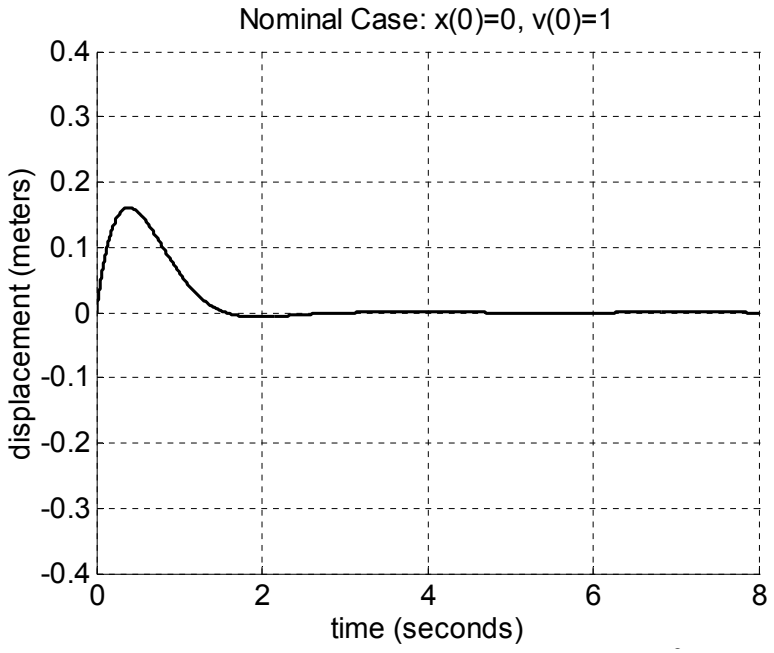


Figure 19.2.4: Nominal case, $x_0 = 0$, $\dot{x}_0 = 1$, $\zeta = 0.707$, $b = 2 \times 10^3$ [Ns/m]. Replotted to show the same scale as the other plots with varying damper b.

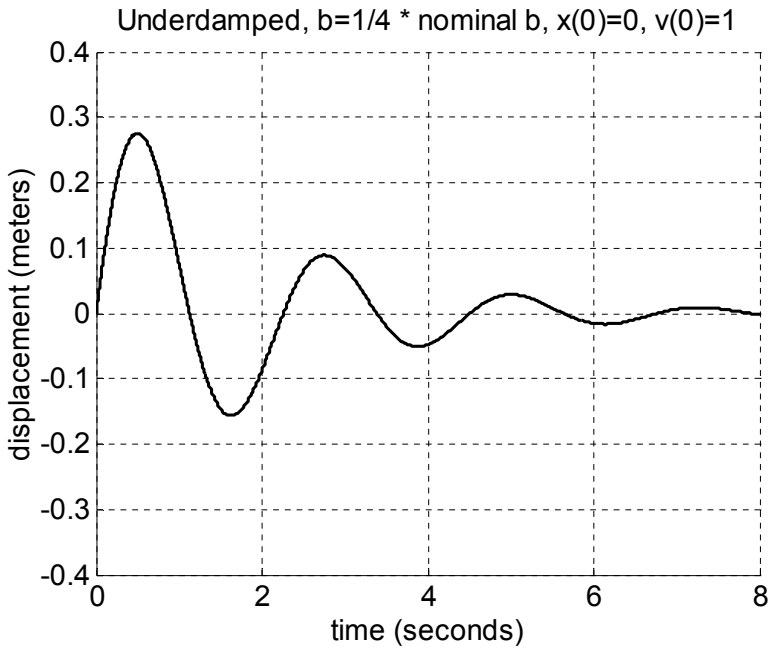


Figure 19.2.5: Smaller damping than nominal case, $x_0 = 0$, $\dot{x}_0 = 1$, $\zeta = 0.18$, $b_{new} = .25b = 500$ [Ns/m].

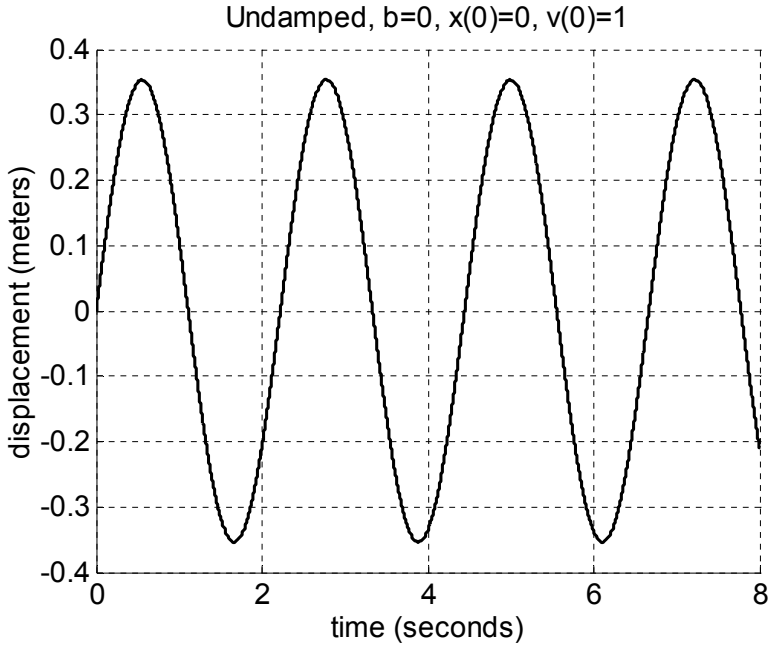


Figure 19.2.6: Undamped case, $x_0 = 0$, $\dot{x}_0 = 1$, $\zeta = 0$, $b_{new} = 0$ [Ns/m].

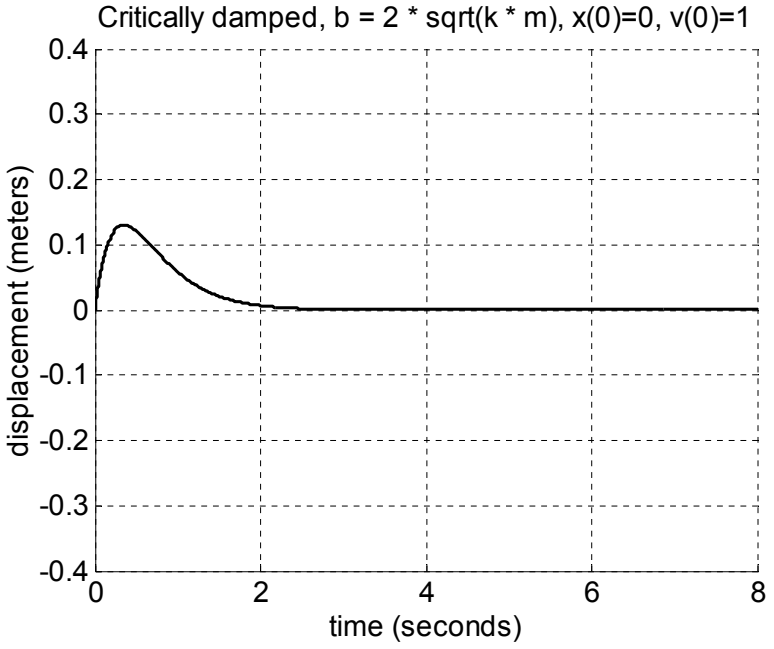


Figure 19.2.7: Critically damped case, $x_0 = 0$, $\dot{x}_0 = 1$, $\zeta = 1$, $b_{critical} = 2\sqrt{km} = 2.83 \times 10^3$ [Ns/m].

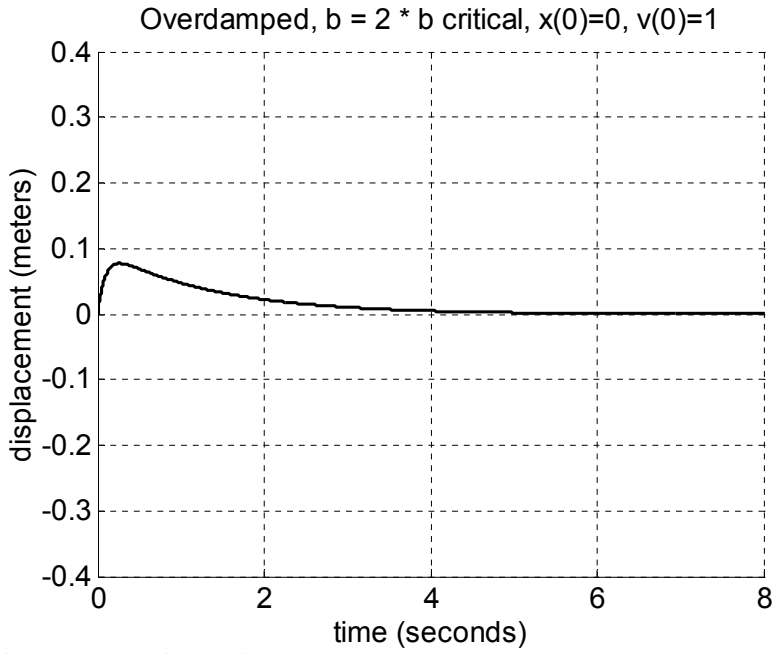


Figure 19.2.8: Overdamped case, $x_0 = 0$, $\dot{x}_0 = 1$, $\zeta = 1$, $b_{new} = 2b_{critical} = 5.66 \times 10^3$ [Ns/m].

(f) A pothole acts as an impulse, so the forced response is the impulse response.

(g) Sports car: Want a fast response and tight handling.
Grandma's Caddy: Want a soft, cushy ride, could accept slower response.
Pickup Truck: The load (mass) may increase, so pick a damping ratio that still gives a good response if the mass is larger. Could do some calculations...

19.3 Recitation 3 Problem

(a) The second-order differential equation for this system is:

$$J\ddot{\theta} + b\dot{\theta} + k\theta = T$$

The characteristic equation is the unforced part:

$$J\ddot{\theta} + b\dot{\theta} + k\theta = 0$$

This can also be written in the standard form (from class):

$$\frac{1}{\omega_n^2}\ddot{\theta} + \frac{2\zeta}{\omega_n} + \theta = f(t) = \frac{T(t)}{k}$$

(b)

$$\zeta = \frac{b}{2\sqrt{kJ}}$$
$$\omega_n = \sqrt{\frac{k}{J}}$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

(c) Shown below is the MATLAB plot, using $T_0 = 10$ [Nm]

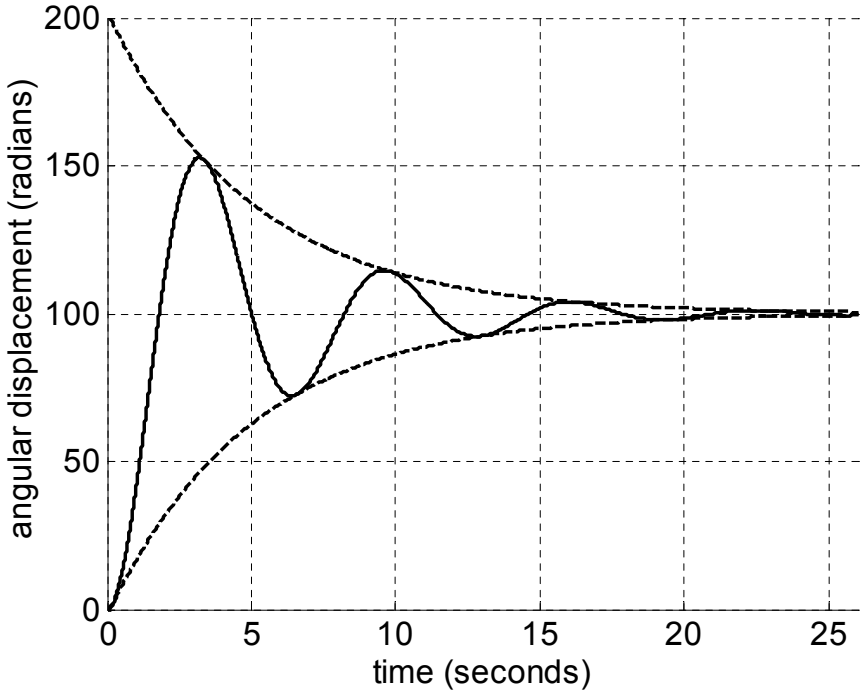


Figure 19.3.1: Angular displacement vs time for the swinging door.

(d) For this class, M_p is overshoot (not percent overshoot as elsewhere). Could calculate by finding derivative and looking for first zero, but too complicated for everyday use.

$$M_p = e^{-\sigma\pi/\omega_d} = e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}} = .53$$

Compare to M_p approximation (valid because $0 < \zeta < 0.6$)

$$M_p \approx 1 - \frac{\zeta}{0.6} = 0.67$$
$$t_p = \frac{\pi}{\omega_d} = 3.2 \text{ [s]}$$

(e) Rise time is the time it takes for the response to go from 10% to 90% of the final value.

$$t_r = \frac{1.8}{\omega_n} = 1.8 \text{ [s]}$$

(f) For this figure, the 1% settling time is

$$t_s \cong \frac{4.6}{\zeta\omega_n} = 23 \text{ [s]}$$

(g)

$$\theta(t) = \theta_p(t) + \theta_h(t)$$
$$\theta_p(t) = \frac{T_0}{k}u_s(t)$$
$$\theta_h(t) = -\frac{T_0}{k}e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right) = -\frac{T_0}{k}e^{-\sigma t}\left(\frac{\omega_n}{\omega_d}\sin\omega_d t + \varphi\right)$$

19.4 Recitation 4 Problem

(a) Just one. We say this system has one degree-of-freedom (DOF). There are multiple choices for the independent variable such as h_1, h_2 or the pressure at the bottom of tank 1 P_1.

(b) Answers will vary.

(c) The volume of fluid V is constant, so $h_1 A + h_2 A = V = 2Ah_{eq}$ where h_{eq} is the equilibrium position of the fluid. We can rewrite this as $h_2 = 2h_{eq} - h_1$. Note also that $\dot{h}_1 + \dot{h}_2 = 0$.

(d) The pressure at the bottom of the tank 1 is $P_1 - P_a = \rho g h_1$. Again for tank 2 $P_2 - P_a = \rho g h_2$. Note that atmospheric pressure P_a has a slightly different notation than Pascals, the SI units for pressure are [N/m^2].

(e) Define q positive inward, q_1 flowing into tank 1, q_2 flowing into tank 2. Flow into and out of tube is conserved $q_1 + q_2 = 0$. Resistance in the tube is modeled as

$$q_1 = \frac{1}{R}(P_1 - P_2) = \frac{\rho g}{R}(h_1 - h_2)$$

(f) Change in volume in tank 1 $V_1 = h_1 A$ equals flow $q_1 = \dot{h}_1 A$. The same is true for tank 2 $q_2 = \dot{h}_2 A$. Is the sign correct? Yes, if q_1 goes up, so does h_1. Combining:

$$\frac{RA}{2\rho g}\dot{h}_1 + h_1 = h_{eq}$$

Could also define $h_1 = h_{eq} + h_\Delta$ which leads to $h_\Delta = h_1 - h_{eq}$ and $\dot{h}_\Delta = \dot{h}_1$ Or

$$\frac{RA}{2\rho g}\dot{h}_\Delta + h_\Delta = 0$$

(g) First-order system so no oscillations.

(h) Time constant $\tau = RA/2\rho g$ doesn't depend on height, so initial amount of fluid doesn't change how quickly the system responds.

To double τ, double resistance in tube or double area of tanks, or change the fluid to one that has half the density (or move experiment to another planet and reduce g).

(i) This system has a dissipative element R and a spring-like element due to the potential energy, but even though some mass moves quickly through the tube the bulk of the mass moves very slowly, kinetic energy is small, and we essentially have a spring-damper first-order system.

All the liquid in the tube in class moved together, so that system was an equivalent spring-mass-damper.

19.5 Recitation 5 Problem

(a) $x = R\theta$

(b) The results are summarized below.

Spring	$PE = \frac{1}{2}k_t x^2$	$F_k = k_t x$
Mass	$KE = \frac{1}{2}m\dot{x}^2$	$F = m\ddot{x}$
Inertia	$KE = \frac{1}{2}J\dot{\theta}^2$	$T = J\ddot{\theta}$
Translational Damper	$P = b_t\dot{x}^2$	$F_b = b_t\dot{x}$
Rotational Damper	$P = b_\omega\dot{\theta}^2$	$T_b = b_\omega\dot{\theta}$

(c)

$$KE = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}J\dot{\theta}^2 = \frac{1}{2}\ \ mR^2 + J\ \ \dot{\theta}^2 = \frac{1}{2}J_{eq}\dot{\theta}^2$$
$$PE = \frac{1}{2}k_t x^2 = \frac{1}{2}k_t R^2\theta^2 = \frac{1}{2}\ \ k_t R^2\ \ \theta^2 = \frac{1}{2}k_{eq}\theta^2$$
$$P = b_t\dot{x}^2 + b_\omega\dot{\theta}^2 = b_t R^2\dot{\theta}^2 + b_\omega\dot{\theta}^2 = (b_\omega + b_t R^2)\dot{\theta}^2 = b_{eq}\dot{\theta}^2$$

(d)

$$J_{eq}\ddot{\theta} = -b_{eq}\dot{\theta} - k_{eq}\theta + T$$
$$(J + mR^2)\ddot{\theta} + (b_\omega + b_t R^2)\dot{\theta} + (k_t R^2)\theta = T$$

(e)

$$J_{eq} = J + mR^2$$
$$b_{eq} = b_\omega + b_t R^2$$
$$k_{eq} = k_t R^2$$

(f)

$$\omega_n = \sqrt{k_{eq}/J_{eq}} = \sqrt{\frac{k_t R^2}{J + mR^2}}$$
$$\frac{2\zeta}{\omega_n} = \frac{b_{eq}}{k_{eq}}$$
$$\zeta = \frac{b_{eq}}{2\sqrt{k_{eq}J_{eq}}} = \frac{b_\omega + b_t R^2}{2\sqrt{k_t R^2(J + mR^2)}}$$

(g) For large R, translational terms dominate

$$\omega_n \to \sqrt{\frac{k_t}{m}} \qquad \text{and} \qquad \zeta \to \frac{b_t}{2\sqrt{k_t m}}$$

(h)

$$\frac{J_{eq}}{R}\ddot{x} + \frac{b_{eq}}{R}\dot{x} + \frac{k_{eq}}{R}x = T$$

ω_n and ζ remain the same.

(i) The response is shown in Figure 19.5.1.

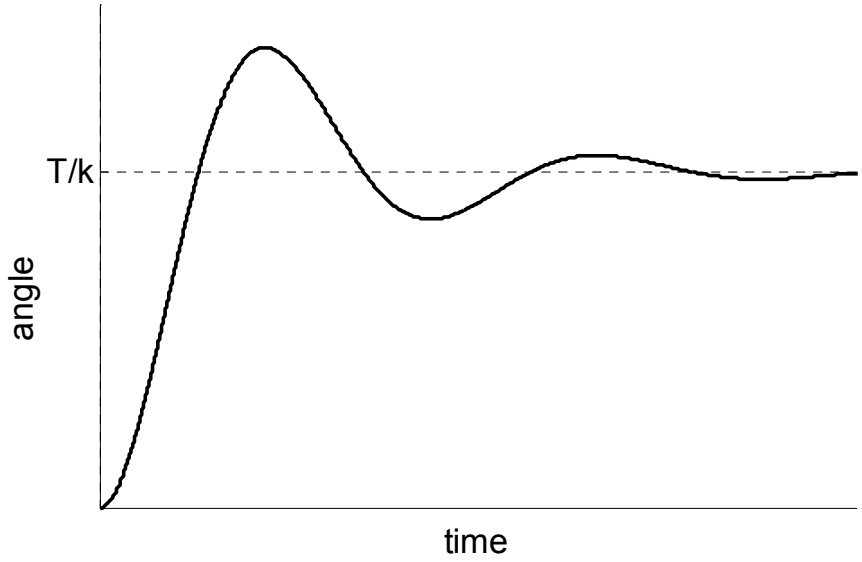


Figure 19.5.1: Underdamped Step Response

19.6 Recitation 6 Problem

(a) Start by redrawing the circuit with current definitions and node names.

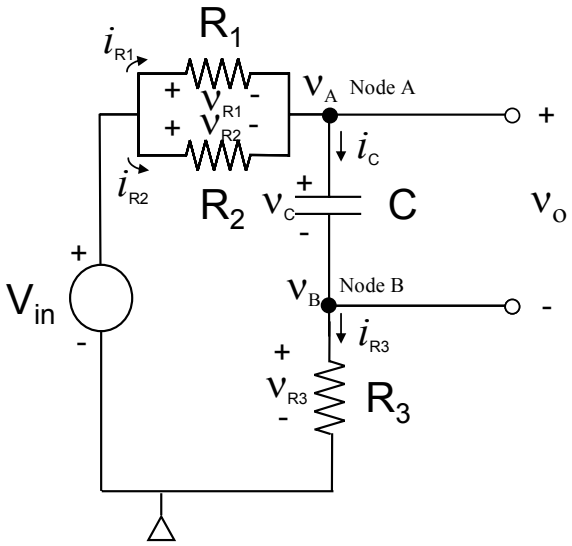


Figure 19.6.1: RC Circuit with node names

Elemental Equations:

$$i_{R1} = \frac{v_{R1}}{R_1} \qquad i_C = C\frac{dv_C}{dt}$$
$$i_{R2} = \frac{v_{R2}}{R_2} \qquad i_{R3} = \frac{v_{R3}}{R_3}$$

Voltages:

$$v_o = v_C \qquad v_B = v_{R3}$$
$$v_C = v_A - v_B \qquad v_{R1} = v_{R2} = V_{in} - v_A$$

Node Equations:

$$i_{R1} + i_{R2} - i_C = 0 \quad \rightarrow \quad \frac{v_{R1}}{R_1} + \frac{v_{R2}}{R_2} - C\frac{dv_C}{dt} = 0$$
$$i_C - i_{R3} = 0 \quad \rightarrow \quad C\frac{dv_C}{dt} - \frac{v_{R3}}{R_3} = 0$$

After the algebra:

$$\left(R_3 + \frac{R_1 R_2}{R_1 + R_2}\right) C\frac{dv_o}{dt} + v_o = V_{in}$$

(b)

$$v_o(t) = V_{in} + Ae^{-\frac{t}{R_{eq}C}}$$
$$v_o(0) = V_{in} + A = 0 \quad \rightarrow \quad A = -V_{in}$$
$$v_o(t) = V_{in}\left(1 + e^{-\frac{t}{R_{eq}C}}\right)$$

(c)

$$R_{eq} = R_3 + \frac{R_1 R_2}{R_1 + R_2}$$

19.7 Recitation 7 Problem

19.8 Recitation 8 Problem

19.9 Recitation 9 Problem

19.10 Recitation 10 Problem

19.11 Recitation 10A Problem

(a) We have the following closed-loop transfer function.

$$\frac{x(s)}{x_{ref}(s)} = \frac{\left(\frac{A}{s}+B\right)\left(\frac{1}{s+1}\right)}{1+\left(\frac{A}{s}+B\right)\left(\frac{1}{s+1}\right)} = \frac{A+Bs}{s^2+(B+1)s+A}$$

In order to have the poles at $s = -10 \pm 100j$, we must have

$$\begin{aligned} s^2+(B+1)s+A &= (-10+100j)(-10-100j) \\ &= s^2+20s+1.01\times 10^4 \end{aligned}$$

therefore

$$\begin{aligned} B &= 19 \\ A &= 1.01\times 10^4 \end{aligned}$$

(b) We have the following closed-loop transfer function.

$$\frac{x(s)}{x_{ref}(s)} = \frac{\left(\frac{A}{s}+B\right)\left(\frac{1}{(s+1)(10^{-4}s+1)}\right)}{1+\left(\frac{A}{s}+B\right)\left(\frac{1}{(s+1)(10^{-4}s+1)}\right)} = \frac{1.01\times 10^4+19s}{10^{-4}s^3+1.0001s^2+20s+1.01\times 10^4}$$

Its roots are

$$\begin{aligned} s &= -9.982\times 10^3 \\ s &= -9.51 \pm 100.1j \end{aligned}$$

Note that the dominate poles in the second system are very close to the poles in the first system, and we expect them to exhibit similar behavior.

(c) For the second transfer function, note that

$$\frac{x(s)}{x_{ref}(s)} = \frac{-18.05}{s+9.982\times 10^3} + \frac{18.05s+1.0137\times 10^4}{s^2+19s+2.0118\times 10^4}$$

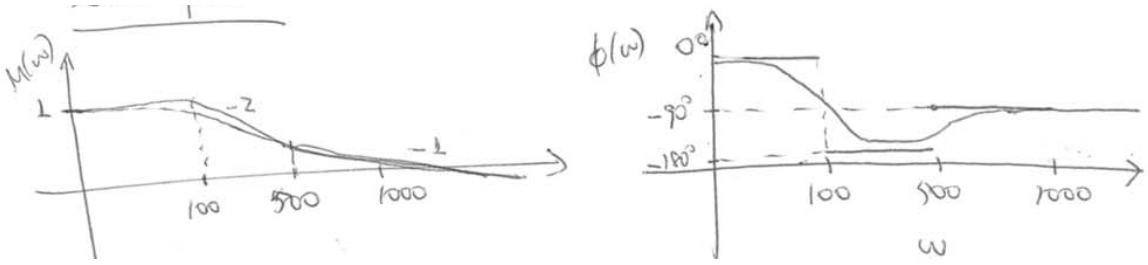


Figure 19.11.1: System 1 Bode Plot

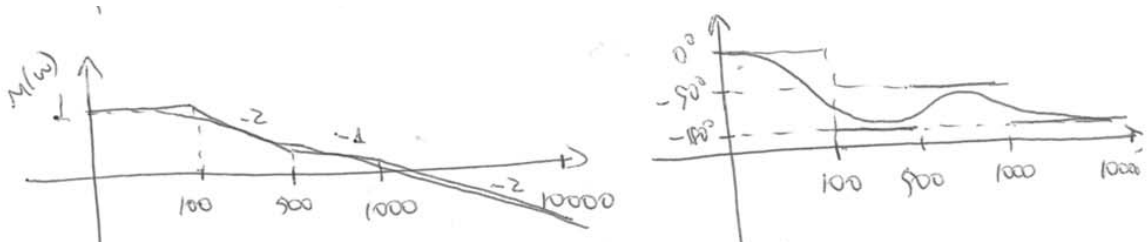


Figure 19.11.2: System 2 Bode Plot

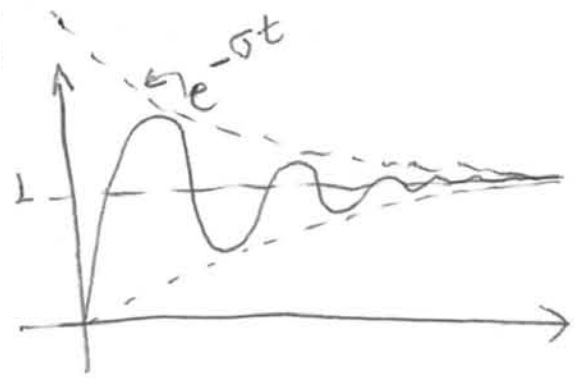


Figure 19.11.3: Step Response

Note that the step responses are virtually identical and the only differences are with high frequency input.

19.12 Recitation 11 Problem

19.13 Recitation 12 Problem

19.14 Recitation 13 Problem

20 Recitation Quizzes

20.1 Recitation 1 Quiz

(a) Combining and rearranging the 4 given equations, we get

$$\frac{AR}{\rho g}\dot{h} + h = 0$$

(b) The form of the equation is $\tau\dot{h} + h = 0$. Therefore

$$\tau = \frac{AR}{\rho g}$$

(c) The form of the solution is $h(t) = h_0 e^{-t/\tau}$. Remembering that $e^{-1} = .37$ and $e^{-2.3} = .1$, after one time constant $h(t = \tau) = .37h_0$, after 2.3 time constants $h(t = 2.3\tau) = .1h_0$.

20.2 Recitation 2 Quiz

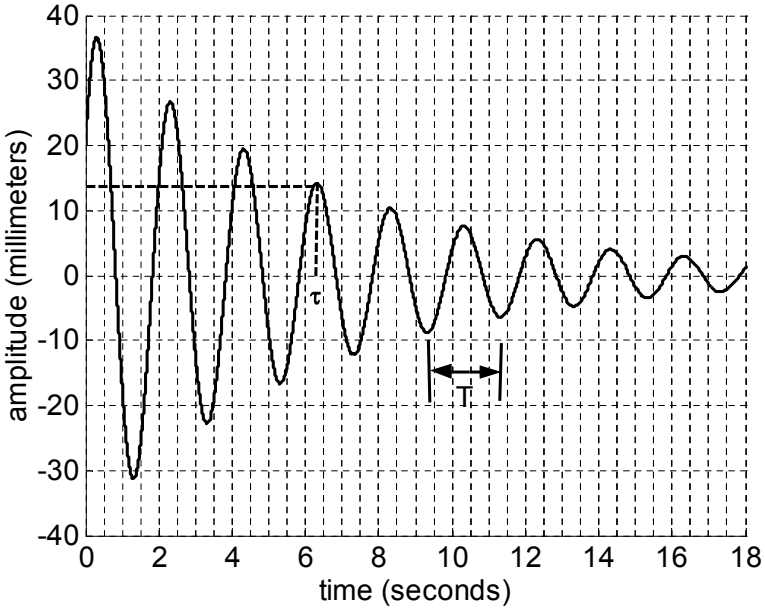


Figure 20.2.1: Second Order Response

The period $T \approx 2$ seconds, so $\omega_d = \frac{2\pi}{T} = \pi$. The natural frequency $\omega_n \approx \omega_d$, particularly for this system which is underdamped, so $\omega_n = \pi$. The damping ratio is found from the time constant of the exponential envelope, $\tau = \frac{1}{\sigma} = \frac{1}{\zeta\omega_n}$, where $\tau \approx 6.25$ seconds taken from the graph, leading to $\zeta = \frac{1}{\omega_n \tau} = \frac{1}{6.25\pi} = .05$. The initial position is read directly from the graph, $x(0) = 20$ mm.

The system is underdamped.

20.3 Recitation 3 Quiz

(a) The sketch is shown below.

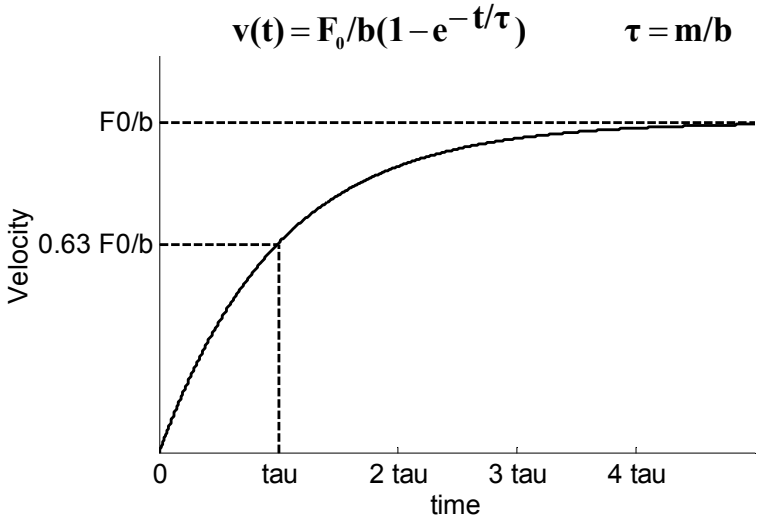


Figure 20.3.1: Plot of velocity vs time

(b) $v(\infty) = F_0/b$

20.4 Recitation 4 Quiz

(a) The free body diagram for the pulley is shown below (along with the free body diagrams for the rest of the system). Note that as drawn, $F_k = -kx$ and $F_b = -b\dot{x}$.

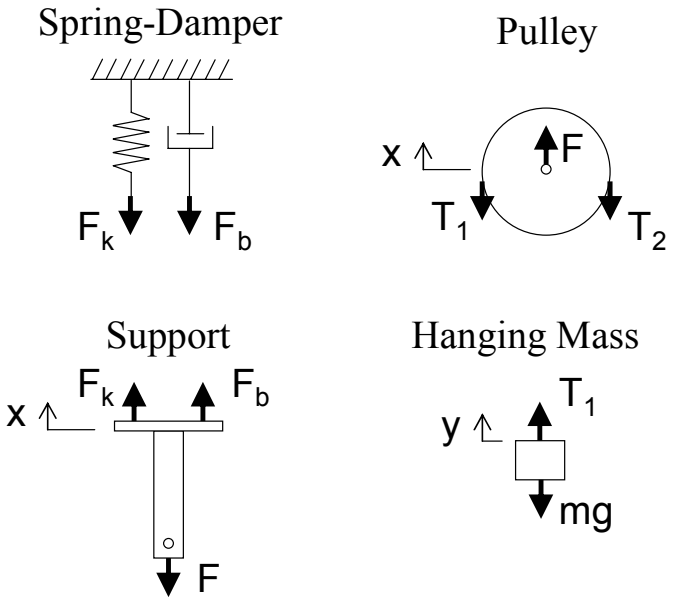


Figure 20.4.1: Free body diagrams

(b) $y = 2x$

20.5 Recitation 5 Quiz

(a) The free body diagrams are shown below.

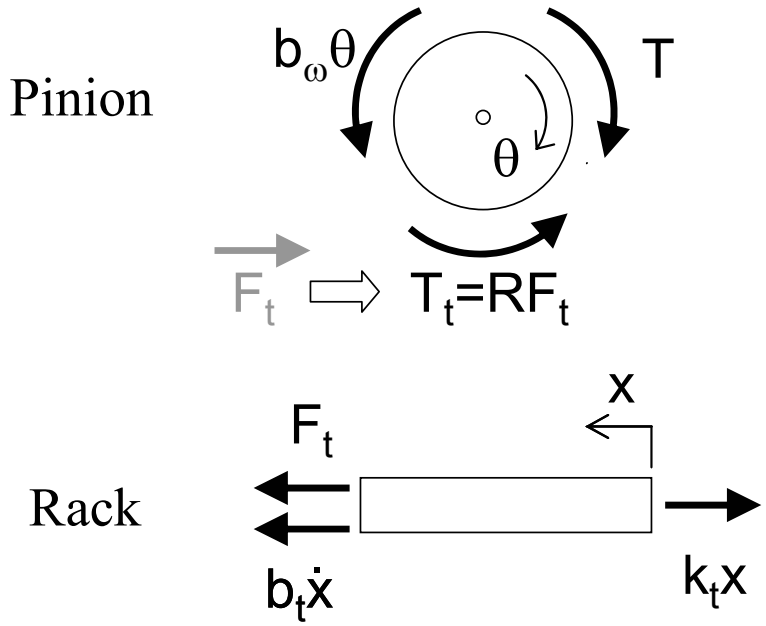


Figure 20.5.1: Free body diagrams

(b) $x = R\theta$

(c) Start with the standard force balancing equations.

$$J\ddot{\theta} = T - b_\omega\dot{\theta} - RF_t$$

and

$$m\ddot{x} + b_t\dot{x} + k_tx = F_t$$

use part (b) to modify this equation

$$mR\ddot{\theta} + b_tR\dot{\theta} + k_tR\theta = F_t$$

Combine this and the first equation to obtain

$$(J + mR^2)\ddot{\theta} + (b_\omega + b_tR^2)\dot{\theta} + (k_tR^2)\theta = T$$

20.6 Recitation 6 Quiz

Elemental Equations:

$$i_{R1} = \frac{v_{R1}}{R_1} \qquad\qquad i_C = C\frac{dv_C}{dt}$$
$$i_{R2} = \frac{v_{R2}}{R_2}$$

Voltages:

$$v_o = v_C$$
$$v_{R1} = v_{R2} = V_{in} - v_o$$

Node Equations:

$$i_{R1} + i_{R2} - i_C = 0 \quad \rightarrow \quad \frac{v_{R1}}{R_1} + \frac{v_{R2}}{R_2} - C\frac{dv_C}{dt} = 0$$

After the algebra:

$$\left(\frac{R_1 R_2}{R_1 + R_2}\right) C\frac{dv_o}{dt} + v_o = V_{in}$$

20.7 Recitation 7 Quiz

20.8 Recitation 8 Quiz

20.9 Recitation 9 Quiz

20.10 Recitation 10 Quiz

20.11 Recitation 10A Quiz

20.12 Recitation 11 Quiz

20.13 Recitation 12

20.14 Recitation 12

2.14 / 2.140 Analysis and Design of Feedback Control Systems

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